

A Lichnerowicz-Type Vanishing Theorem for the Mathai–Zhang Index

Ziran Liu

Chern Institute of Mathematical Sciences

Nankai University

A thesis submitted in partial fulfillment of the requirements

for the degree of

Master of Science

June 2014

(Updated March 2026)

Abstract

This thesis is organized in four parts. The first part reviews the classical background from spin geometry that underlies the Lichnerowicz vanishing theorem. The second and third parts develop the proper cocompact analogue considered in this thesis: we recall the Mathai–Zhang index [1], derive an explicit formula for the projection operator appearing in its definition, and prove a Lichnerowicz-type vanishing theorem for this index in the unimodular setting. The fourth part is forward-looking: it surveys the developments that grew out of this line of work, including higher indices, equivariant Atiyah–Patodi–Singer theory, higher localized genera, delocalized eta and rho invariants, and equivariant spectral flow, and it formulates a collection of focused open problems that appear especially promising for further research.

Keywords. Dirac operator; proper cocompact actions; Lichnerowicz-type vanishing theorem; Mathai–Zhang index.

Contents

Abstract	i
Part 1. Classical Background	1
Chapter 1. Introduction	3
1. Dirac operators and the \widehat{A} genus	3
2. Classical vanishing and index theorems	4
3. Aim, scope, and organization	4
Part 2. The Mathai–Zhang Index and Proper Cocompact Actions	5
Chapter 2. The Mathai–Zhang index on spin manifolds	7
1. Geometric setting	7
2. Sobolev spaces and cut-off sections	7
3. The Mathai–Zhang index	8
Chapter 3. Explicit formula for the projection operator P_f	11
1. Equivariant bundle conventions	11
2. The normalizing function	11
3. The explicit projection formula	12
Part 3. A Lichnerowicz-Type Vanishing Theorem	15
Chapter 4. A Lichnerowicz-type vanishing theorem for the Mathai–Zhang index	17
1. The twisted Lichnerowicz formula	17
2. The main vanishing theorem	17
3. The untwisted corollary	19
Part 4. Current Frontier and Open Directions	21
Chapter 5. Frontier developments after the original vanishing theorem	23
1. From the original theorem to the modern picture	23
1.1. The first completion of the vanishing theorem	23
1.2. Embedding the theorem into a broader equivariant PSC framework	23
2. Higher indices and higher genera for proper actions	23
2.1. Higher genera on closed proper manifolds	23
2.2. The boundary case	23
2.3. Higher localised \widehat{A} genera for groups with torsion	24
3. Boundary problems and equivariant APS theory	24

3.1. Numerical APS indices for proper actions	24
3.2. The K -theoretic APS index	24
4. From numerical indices to secondary invariants	24
4.1. Higher orbital integrals	24
4.2. Delocalized eta invariants and rho numbers	25
4.3. Equivariant spectral flow	25
5. New analytic containers and nearby frontier developments	25
5.1. Callias-type obstructions beyond the cocompact quotient case	25
5.2. Rapidly decaying kernels and finite-volume spaces	25
5.3. Motion groups and localization methods	25
5.4. An extension in 2026	26
6. A working description of the present frontier	26
Chapter 6. Focused open problems and possible strategies	27
1. How the problems below were selected	27
2. Problem A: a proper-action APS vanishing theorem	27
Problem statement	27
Why it is promising	27
The key observation	28
Possible first attack	28
Why the scope is controlled	28
3. Problem B: a cut-off model for equivariant spectral flow	28
Problem statement	28
Why it is promising	28
The key observation	28
Possible first attack	28
Why the scope is controlled	29
4. Problem C: explicit higher rho and orbital calculations on model spaces	29
Problem statement	29
Why it is promising	29
The key observation	29
Possible first attack	29
Why the scope is controlled	29
5. Problem D: higher localised \widehat{A} obstructions beyond degree two	29
Problem statement	29
Why it is promising	29
The key observation	30
Possible first attack	30

Why the scope is controlled	30
6. A suggested order of work	30
Bibliography	31

Part 1

Classical Background

CHAPTER 1

Introduction**1. Dirac operators and the \widehat{A} genus**

Let M be a closed even-dimensional spin manifold with Riemannian metric g^{TM} on the tangent bundle TM . Let ∇^{TM} be the Levi–Civita connection and let

$$R^{TM} = (\nabla^{TM})^2$$

be the curvature of ∇^{TM} .

Let

$$S(TM) = S^+(TM) \oplus S^-(TM)$$

be the complex spinor bundle associated to (TM, g^{TM}) , and let

$$\nabla^{S(TM)} = \nabla^{S^+(TM)} \oplus \nabla^{S^-(TM)}$$

be the induced Hermitian connection on $S(TM)$. For any vector $X \in TM$, we use $c(X)$ to denote Clifford multiplication by X on $S(TM)$.

The Dirac operator on M (cf. [2]) is defined by

$$(1.1) \quad D = \sum_{i=1}^{\dim M} c(e_i) \nabla_{e_i}^{S(TM)} : \Gamma(S(TM)) \longrightarrow \Gamma(S(TM)),$$

where $\{e_1, \dots, e_{\dim M}\}$ is a local oriented orthonormal frame of TM . Its chiral part is

$$D^+ = D|_{\Gamma(S^+(TM))} : \Gamma(S^+(TM)) \longrightarrow \Gamma(S^-(TM)).$$

Since M is closed, the operator D^+ is elliptic and Fredholm, and its formal adjoint is $D^- = (D^+)^*$. Therefore,

$$(1.2) \quad \text{ind}(D^+) = \dim(\text{Ker } D^+) - \dim(\text{Ker } D^-).$$

Let $\widehat{A}(TM, \nabla^{TM})$ be the Hirzebruch characteristic form defined by

$$\widehat{A}(TM, \nabla^{TM}) = \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh \left(\frac{\sqrt{-1}}{4\pi} R^{TM} \right)} \right).$$

The \widehat{A} -genus of M is

$$\widehat{A}(M) = \int_M \widehat{A}(TM, \nabla^{TM}).$$

2. Classical vanishing and index theorems

In 1963, A. Lichnerowicz established the classical vanishing theorem in [5]. We recall it in the following form.

THEOREM 1.1 (Lichnerowicz). *Let M be a closed spin manifold. If M admits a Riemannian metric of positive scalar curvature, then every harmonic spinor vanishes. In particular,*

$$\text{ind}(D^+) = 0.$$

If moreover $\dim M \equiv 0 \pmod{4}$, then

$$\widehat{A}(M) = 0.$$

Also in 1963, Atiyah and Singer established their index theorem; see [6].

THEOREM 1.2 (Atiyah–Singer). *Let M be a closed even-dimensional spin manifold, and let D^+ be the chiral Dirac operator on M . Then*

$$\text{ind}(D^+) = \widehat{A}(M).$$

Combining Theorems 1.1 and 1.2, one sees that positive scalar curvature forces the index of the Dirac operator to vanish.

3. Aim, scope, and organization

The purpose of this thesis is to establish an extension of this classical vanishing phenomenon to the case of noncompact manifolds carrying proper cocompact group actions. More precisely, we work in the unimodular setting and prove a vanishing theorem for the Mathai–Zhang index introduced in [1]. The main theorem is stated as Theorem 4.1.

The exposition is organized in four parts. Chapter 2 recalls the definition and basic Fredholm property of the Mathai–Zhang index. Chapter 3 gives an explicit analytic formula for the orthogonal projection operator appearing in that construction. Chapter 4 uses the Lichnerowicz formula together with a maximum-principle argument to prove the desired vanishing theorem. The final part, consisting of Chapters 5 and 6, reviews the developments that followed this line of work and formulates several focused open problems intended as realistic starting points for further research.

REMARK 1.3. From a later bibliographical point of view, the unimodular vanishing theorem proved in Chapter 4 was subsequently generalized beyond the unimodular case; see [8, 9]. This observation is revisited in the review chapter, where we place the main theorem of the thesis inside the later development of the subject.

Part 2

The Mathai–Zhang Index and Proper Cocompact Actions

CHAPTER 2

The Mathai–Zhang index on spin manifolds

Throughout this chapter, the action of G on M is a left action, and all integrals over G are taken with respect to a fixed left Haar measure dg .

1. Geometric setting

Let M be a noncompact even-dimensional spin manifold, and let G be a locally compact group acting on M . We assume that the action is *proper* and *cocompact*, where by proper we mean that the map

$$G \times M \longrightarrow M \times M, \quad (g, x) \longmapsto (x, gx),$$

is proper, while by cocompact we mean that the quotient space M/G is compact. We also assume that the G -action preserves the spin structure on M .

Let g^{TM} be a Riemannian metric on TM . By averaging over the group action, one may assume that g^{TM} is G -invariant (cf. [1, (2.3)]). Let

$$S(TM) = S^+(TM) \oplus S^-(TM)$$

be the complex spinor bundle associated to (TM, g^{TM}) . The G -action on M lifts to an action on $S(TM)$ preserving the canonically induced Hermitian metric $g^{S(TM)}$ and Hermitian connection $\nabla^{S(TM)}$.

Let E be a Hermitian vector bundle over M equipped with a lifted G -action, a G -invariant Hermitian metric g^E , and a G -invariant Hermitian connection ∇^E . The tensor product $S(TM) \otimes E$ then carries the induced G -equivariant Hermitian metric and connection, still denoted by $g^{S(TM) \otimes E}$ and $\nabla^{S(TM) \otimes E}$.

The twisted Dirac operator is defined by

$$(2.1) \quad D^E = \sum_{i=1}^{\dim M} c(e_i) \nabla_{e_i}^{S(TM) \otimes E} : \Gamma(S(TM) \otimes E) \longrightarrow \Gamma(S(TM) \otimes E),$$

where $\{e_1, \dots, e_{\dim M}\}$ is a local oriented orthonormal frame of TM . Its chiral parts are

$$D_{\pm}^E : \Gamma(S^{\pm}(TM) \otimes E) \longrightarrow \Gamma(S^{\mp}(TM) \otimes E).$$

2. Sobolev spaces and cut-off sections

For compactly supported smooth sections $s_1, s_2 \in \Gamma_c^{\infty}(M, S(TM) \otimes E)$, define the L^2 inner product by

$$(2.2) \quad (s_1, s_2) = \int_M \langle s_1, s_2 \rangle_{S(TM) \otimes E} dv_{g^{TM}}.$$

Let $\|\cdot\|_0$ be the corresponding L^2 norm, and let $\|\cdot\|_1$ be a fixed G -invariant Sobolev 1-norm. Define

$$\mathbf{H}^0(M, S(TM) \otimes E) := \overline{\Gamma_c^\infty(M, S(TM) \otimes E)}^{\|\cdot\|_0},$$

and let $\mathbf{H}^1(M, S(TM) \otimes E)$ be the Sobolev-1 completion of the same space.

Denote the space of smooth G -invariant sections of $S(TM) \otimes E$ by

$$\Gamma(S(TM) \otimes E)^G.$$

Since M/G is compact, there exists a compact subset $Y \subset M$ such that $G(Y) = M$ (cf. [7, Lemma 2.3]). Choose open subsets $U, U' \subset M$ such that

$$Y \subset U, \quad \bar{U} \subset U', \quad \bar{U}' \text{ is compact.}$$

Following [1], choose a nonnegative cut-off function $f \in C_c^\infty(M)$ satisfying

$$f|_U = 1, \quad \text{Supp}(f) \subset U'.$$

Let $\mathbf{H}_f^0(M, S(TM) \otimes E)^G$ and $\mathbf{H}_f^1(M, S(TM) \otimes E)^G$ denote the closures of

$$\{fs : s \in \Gamma(S(TM) \otimes E)^G\}$$

in $\mathbf{H}^0(M, S(TM) \otimes E)$ and $\mathbf{H}^1(M, S(TM) \otimes E)$, respectively. Since f has compact support, every section of the form fs is compactly supported, so these norms are well defined. By Hilbert space theory, there exists a unique orthogonal projection

$$P_f : \mathbf{H}^0(M, S(TM) \otimes E) \longrightarrow \mathbf{H}_f^0(M, S(TM) \otimes E)^G.$$

Moreover,

$$P_f D^E : \mathbf{H}_f^1(M, S(TM) \otimes E)^G \longrightarrow \mathbf{H}_f^0(M, S(TM) \otimes E)^G$$

is a well-defined operator.

3. The Mathai–Zhang index

We now recall the basic Fredholm property proved in [1, Proposition 2.1].

PROPOSITION 2.1 (Mathai–Zhang). *The operator*

$$P_f D^E : \mathbf{H}_f^1(M, S(TM) \otimes E)^G \longrightarrow \mathbf{H}_f^0(M, S(TM) \otimes E)^G$$

is Fredholm.

It was shown in [1] that the index of the chiral part

$$P_f D_+^E : \mathbf{H}_f^1(M, S^+(TM) \otimes E)^G \longrightarrow \mathbf{H}_f^0(M, S^-(TM) \otimes E)^G$$

is independent of the choice of the cut-off function f , as well as of the G -invariant metrics and connections involved. Following [1, Definition 2.4], we denote this index by

$$\mathrm{ind}_G(D_+^E).$$

This is the Mathai–Zhang index of the twisted Dirac operator.

CHAPTER 3

Explicit formula for the projection operator P_f

In this chapter, we give an explicit formula for the projection operator appearing in Chapter 2. Since the vanishing theorem proved later is stated for unimodular groups, we assume throughout this chapter that G is unimodular.

1. Equivariant bundle conventions

For the convenience of notation, we write $\mathbf{H}^0(M, S)$ for the L^2 -completion of $\Gamma_c^\infty(M, S)$, and we write $\mathbf{H}_f^0(M, S)^G$ for the closure in $\mathbf{H}^0(M, S)$ of the subspace

$$\{f\sigma : \sigma \in \Gamma(S)^G\}.$$

Thus the orthogonal projection

$$P_f : \mathbf{H}^0(M, S) \longrightarrow \mathbf{H}_f^0(M, S)^G$$

is defined exactly as in Chapter 2, but now for the general equivariant Hermitian bundle $S \rightarrow M$.

Let $S \rightarrow M$ be a G -equivariant Hermitian vector bundle. For $g \in G$ and $x \in M$, we denote by

$$g : S_{g^{-1}x} \longrightarrow S_x$$

the induced unitary map on fibres. Equivalently, the action of G on sections is given by

$$(g \cdot s)(x) := g(s(g^{-1}x)).$$

A smooth section s is G -invariant if and only if $g \cdot s = s$ for all $g \in G$.

2. The normalizing function

For $x \in M$, define

$$(3.1) \quad A(x)^2 := \int_G f(g^{-1}x)^2 dg.$$

To justify this definition, fix $x \in M$ and consider the set

$$K_x := \{g \in G : f(g^{-1}x) \neq 0\}.$$

Its inverse is

$$K_x^{-1} = \{h \in G : f(hx) \neq 0\} = \{h \in G : hx \in \text{Supp}(f)\}.$$

Because the action is proper, the map

$$G \longrightarrow M \times M, \quad h \longmapsto (x, hx)$$

is proper. Hence the inverse image of the compact set $\{x\} \times \text{Supp}(f)$ is compact, so K_x^{-1} is compact, and therefore K_x is compact as well. This proves that the integral in (3.1) is finite. Since $G(U) = M$ and $f|_U = 1$, for each $x \in M$ there exists $g_0 \in G$ such that $g_0^{-1}x \in U$. As U is open and the

action is continuous, one has $g^{-1}x \in U$ for all g sufficiently close to g_0 , and hence $f(g^{-1}x) = 1$ on a neighborhood of g_0 . It follows that the integral in (3.1) is strictly positive. Moreover, by left invariance of Haar measure,

$$(3.2) \quad A(hx) = A(x), \quad h \in G.$$

LEMMA 3.1. *The function $A : M \rightarrow (0, \infty)$ is smooth and G -invariant.*

PROOF. The integrand in (3.1) is smooth in x , and for each fixed x the set of $g \in G$ on which it is nonzero is contained in the compact set K_x . By properness of the action, if x' varies in a sufficiently small neighborhood of x , then all sets $K_{x'}$ are contained in a single compact subset of G . One may therefore differentiate under the integral sign, which shows that A^2 is smooth, and hence so is A because A is strictly positive. The G -invariance follows from a direct change of variables:

$$A(hx)^2 = \int_G f(g^{-1}hx)^2 dg = \int_G f(z^{-1}x)^2 dz = A(x)^2.$$

□

3. The explicit projection formula

For $s \in \Gamma_c^\infty(M, S)$, define

$$(3.3) \quad (Q_f s)(x) := \frac{f(x)}{A(x)^2} \int_G f(g^{-1}x) g(s(g^{-1}x)) dg.$$

The same compact-support argument as above shows that the integral is well defined for all $x \in M$ and produces a smooth section.

PROPOSITION 3.2. *For every $s \in \Gamma_c^\infty(M, S)$, the section $Q_f s$ belongs to*

$$\{f\sigma : \sigma \in \Gamma(S)^G\}.$$

Moreover, for every $\sigma \in \Gamma(S)^G$,

$$(3.4) \quad (s - Q_f s, f\sigma) = 0.$$

Consequently, $Q_f s = P_f s$ for all $s \in \Gamma_c^\infty(M, S)$.

PROOF. Define a smooth section σ_s by

$$(3.5) \quad \sigma_s(x) := \frac{1}{A(x)^2} \int_G f(g^{-1}x) g(s(g^{-1}x)) dg.$$

Then $Q_f s = f \sigma_s$ by definition. We first show that σ_s is G -invariant. Let $h \in G$. Using (3.2) and the change of variables $g = hz$, one gets

$$\begin{aligned}\sigma_s(hx) &= \frac{1}{A(hx)^2} \int_G f(g^{-1}hx) g(s(g^{-1}hx)) dg \\ &= \frac{1}{A(x)^2} \int_G f(z^{-1}x) hz(s(z^{-1}x)) dz \\ &= h \left(\frac{1}{A(x)^2} \int_G f(z^{-1}x) z(s(z^{-1}x)) dz \right) \\ &= h \sigma_s(x).\end{aligned}$$

Hence $\sigma_s \in \Gamma(S)^G$, and therefore

$$Q_f s = f \sigma_s \in \{f \sigma : \sigma \in \Gamma(S)^G\}.$$

Next let $\sigma \in \Gamma(S)^G$. Since s is compactly supported and f is compactly supported, the integrand below is supported in a compact subset of $G \times M$; thus Fubini's theorem applies. Using (3.3), the G -invariance of the Hermitian metric, and the change of variables $x = gy$, we obtain

$$\begin{aligned}(Q_f s, f \sigma) &= \int_M \left\langle \frac{f(x)}{A(x)^2} \int_G f(g^{-1}x) g(s(g^{-1}x)) dg, f(x) \sigma(x) \right\rangle dv_{gTM}(x) \\ &= \int_G \int_M \frac{f(gy)^2 f(y)}{A(gy)^2} \langle g(s(y)), \sigma(gy) \rangle dv_{gTM}(y) dg \\ &= \int_G \int_M \frac{f(gy)^2 f(y)}{A(y)^2} \langle s(y), \sigma(y) \rangle dv_{gTM}(y) dg.\end{aligned}$$

Since G is unimodular, inversion preserves Haar measure, and hence

$$\int_G f(gy)^2 dg = \int_G f(g^{-1}y)^2 dg = A(y)^2.$$

Therefore,

$$(Q_f s, f \sigma) = \int_M f(y) \langle s(y), \sigma(y) \rangle dv_{gTM}(y) = (s, f \sigma).$$

This proves (3.4).

Finally, the subspace

$$\{f \sigma : \sigma \in \Gamma(S)^G\}$$

is dense in $\mathbf{H}_f^0(M, S)^G$ by definition. Since $Q_f s$ lies in this dense subspace and $s - Q_f s$ is orthogonal to it, $Q_f s$ is exactly the orthogonal projection of s onto $\mathbf{H}_f^0(M, S)^G$. Hence $Q_f s = P_f s$ for every $s \in \Gamma_c^\infty(M, S)$. \square

COROLLARY 3.3. *The operator defined by the integral formula (3.3) on the dense subspace $\Gamma_c^\infty(M, S) \subset \mathbf{H}^0(M, S)$ agrees there with the orthogonal projection P_f . Hence it determines P_f uniquely, and P_f is the unique bounded extension of this operator to all of $\mathbf{H}^0(M, S)$.*

PROOF. By Proposition 3.2, the operator defined by (3.3) agrees with P_f on the dense subspace $\Gamma_c^\infty(M, S)$. Since orthogonal projections are bounded operators on Hilbert spaces, P_f extends uniquely from this dense subspace to all of $\mathbf{H}^0(M, S)$, and this extension is again P_f . \square

REMARK 3.4. For non-unimodular groups, an analogous explicit formula involves the modular character. Since the vanishing theorem in this thesis is proved in the unimodular case, the above formula is sufficient for our purposes.

Part 3

A Lichnerowicz-Type Vanishing Theorem

CHAPTER 4

A Lichnerowicz-type vanishing theorem for the Mathai–Zhang index

In this chapter, we adapt the classical Bochner–Lichnerowicz argument to the Mathai–Zhang index in the unimodular proper cocompact setting.

1. The twisted Lichnerowicz formula

Let k^{TM} be the scalar curvature of the G -invariant metric g^{TM} , and let R^E be the curvature of ∇^E . If $\{e_i\}$ is a local orthonormal frame of TM , define the Hermitian endomorphism $c(R^E)$ of $S(TM) \otimes E$ by

$$(4.1) \quad c(R^E) = \frac{1}{2} \sum_{i,j=1}^{\dim M} c(e_i)c(e_j)R^E(e_i, e_j).$$

Recall the classical Lichnerowicz formula (cf. [2, Chapter II, Theorem 8.17]) for the twisted Dirac operator:

$$(4.2) \quad (D^E)^2 = -\Delta^E + \frac{k^{TM}}{4} + c(R^E),$$

where the Bochner Laplacian is given by

$$\Delta^E = \sum_{i=1}^{\dim M} \left((\nabla_{e_i}^{S(TM) \otimes E})^2 - \nabla_{\nabla_{e_i}^{TM} e_i}^{S(TM) \otimes E} \right).$$

2. The main vanishing theorem

We can now state the main result.

THEOREM 4.1. *Let M be a noncompact even-dimensional spin manifold carrying a proper cocompact action by a locally compact unimodular group G , and assume that the action preserves the spin structure on M . Let E be a G -equivariant Hermitian vector bundle over M with a G -invariant Hermitian metric and a G -invariant Hermitian connection ∇^E . Assume that the Hermitian endomorphism*

$$(4.3) \quad R := \frac{k^{TM}}{4} + c(R^E)$$

is pointwise positive definite on $S(TM) \otimes E$, that is,

$$\langle R(x)u, u \rangle > 0 \quad \text{for all } x \in M \text{ and all } u \in (S(TM) \otimes E)_x \setminus \{0\}.$$

Then

$$\text{ind}_G(D_+^E) = 0.$$

PROOF. Since G is unimodular, Mathai–Zhang proved in [1, Theorem 2.7] that

$$(4.4) \quad \text{ind}_G(D_+^E) = \dim((\text{Ker } D_+^E)^G) - \dim((\text{Ker } D_-^E)^G).$$

Therefore it is enough to prove that

$$(\text{Ker } D_+^E)^G = (\text{Ker } D_-^E)^G = \{0\}.$$

Let $s \in (\text{Ker } D_\pm^E)^G$. By elliptic regularity, s is smooth. Since $D^E s = 0$, the Lichnerowicz formula (4.2) gives

$$(D^E)^2 s = 0.$$

Fix any $\varphi \in C_c^\infty(M)$. Pairing $(D^E)^2 s$ with φs and integrating, we get

$$\begin{aligned} 0 &= \int_M \langle (D^E)^2 s, \varphi s \rangle \, dv_{g^{TM}} \\ &= \int_M \left(\langle -\Delta^E s, \varphi s \rangle + \varphi \langle Rs, s \rangle \right) \, dv_{g^{TM}} \\ &= \int_M \left(\sum_{i=1}^{\dim M} \langle \nabla_{e_i} s, \nabla_{e_i} (\varphi s) \rangle + \varphi \langle Rs, s \rangle \right) \, dv_{g^{TM}} \\ (4.5) \quad &= \int_M \left(\frac{1}{2} \sum_{i=1}^{\dim M} e_i(\varphi) e_i(|s|^2) + \varphi |\nabla s|^2 + \varphi \langle Rs, s \rangle \right) \, dv_{g^{TM}}. \end{aligned}$$

Since the left-hand side vanishes for every compactly supported test function φ , it follows that the smooth function

$$\frac{1}{2} \Delta(|s|^2) - |\nabla s|^2 - \langle Rs, s \rangle$$

vanishes identically on M . Hence

$$(4.6) \quad \frac{1}{2} \Delta(|s|^2) = |\nabla s|^2 + \langle Rs, s \rangle \geq \langle Rs, s \rangle.$$

Choose a compact subset $Y \subset M$ such that $G(Y) = M$; such a subset exists because M/G is compact. Since s is G -invariant, the function $|s|^2$ is constant along G -orbits. Hence any maximum of $|s|^2$ on the compact set Y is automatically a global maximum on all of M . Let $p \in Y$ be such a point. By the maximum principle,

$$\Delta(|s|^2)(p) \leq 0.$$

Evaluating (4.6) at p , we obtain

$$0 \geq \frac{1}{2} \Delta(|s|^2)(p) = |\nabla s|^2(p) + \langle R(p)s(p), s(p) \rangle \geq \langle R(p)s(p), s(p) \rangle \geq 0.$$

Since $R(p)$ is positive definite, it follows that $s(p) = 0$. As p is a maximum point of the nonnegative function $|s|^2$, we conclude that $|s|^2 \equiv 0$ on M , hence $s \equiv 0$.

Thus both $(\text{Ker } D_+^E)^G$ and $(\text{Ker } D_-^E)^G$ are trivial. By (4.4), this implies

$$\text{ind}_G(D_+^E) = 0.$$

The proof is complete. □

3. The untwisted corollary

COROLLARY 4.2. *Let M be a noncompact even-dimensional spin manifold carrying a proper cocompact action by a locally compact unimodular group G , and assume that the action preserves the spin structure on M . If M admits a G -invariant Riemannian metric of positive scalar curvature, then*

$$\text{ind}_G(D_+) = 0.$$

PROOF. This is the special case of Theorem 4.1 obtained by taking E to be the trivial line bundle with its flat connection, so that $c(R^E) = 0$. \square

Part 4

Current Frontier and Open Directions

CHAPTER 5

Frontier developments after the original vanishing theorem

This chapter is forward-looking. Its purpose is to record the developments that grew most directly out of the line considered in Chapters 2–4 and to identify the themes that now seem to form the genuine frontier of the subject. The review is intentionally selective rather than encyclopedic: emphasis is placed on the developments that are closest in spirit to the Mathai–Zhang index, to the Bochner–Lichnerowicz argument, and to the positive scalar curvature questions that motivated this thesis.

1. From the original theorem to the modern picture

1.1. The first completion of the vanishing theorem. The unimodular proper cocompact vanishing theorem proved in this thesis is closely related to the short note [8]. Shortly afterwards, Zhang observed that the unimodularity assumption can be removed and obtained the corresponding theorem for proper cocompact actions in general [9]. From the point of view of the present thesis, this means that the basic numerical vanishing statement is now part of the established background of the subject rather than its current frontier.

1.2. Embedding the theorem into a broader equivariant PSC framework. A second step was to place the vanishing statement inside a broader equivariant index framework for proper cocompact actions. Guo, Mathai and Wang studied positive scalar curvature and Poincaré duality for proper actions, proving obstruction and existence results for G -invariant positive scalar curvature metrics, together with equivariant Poincaré duality results for almost connected Lie groups and for discrete groups [10]. In particular, the original vanishing theorem becomes one visible shadow of a richer higher-index picture.

2. Higher indices and higher genera for proper actions

2.1. Higher genera on closed proper manifolds. Piazza and Posthuma initiated a systematic study of higher genera for proper actions of Lie groups [11]. Under hypotheses such as rapid decay for G and nonpositive sectional curvature on G/K , they established index formulae for C^* -higher indices of G -equivariant Dirac-type operators on cocompact G -proper manifolds. One consequence is a vanishing statement for the resulting \widehat{A} -type higher genera on G -spin manifolds admitting G -invariant positive scalar curvature metrics.

2.2. The boundary case. This theory was extended to manifolds with boundary in the second part of the same program [13]. There, higher Atiyah–Patodi–Singer C^* -indices were constructed for cocompact G -proper manifolds with boundary and product-type metrics near the boundary, and higher index formulae were proved under an L^2 -invertibility hypothesis on the boundary operator. This is the first clear sign that the natural continuation of the thesis is not merely another vanishing

theorem on a closed manifold, but the interaction between positivity, proper actions, and boundary index theory.

2.3. Higher localised \widehat{A} genera for groups with torsion. Guo and Mathai introduced higher localised \widehat{A} -genera for proper actions of discrete groups with torsion [17]. Their work formulates new topological obstructions to Γ -invariant positive scalar curvature that take into account the cohomology of the classifying space $B\Gamma$ for proper actions. In the cocompact case, this gives a natural generalisation of higher \widehat{A} -genera to groups with torsion. The paper also formulates a clear conjectural obstruction statement and proves it for classes arising from the subring of $H^*(B\Gamma; \mathbb{R})$ generated by elements of degree at most 2. This is one of the most concrete open ends in the current literature.

3. Boundary problems and equivariant APS theory

3.1. Numerical APS indices for proper actions. Hochs, Bai-Ling Wang and Hang Wang proved an equivariant Atiyah–Patodi–Singer index theorem for proper actions in two stages. In Part I they considered a proper, isometric action by a unimodular locally compact group on a Riemannian manifold with boundary and compact quotient, defined numerical indices $\text{index}_g(D)$, and established an equivariant APS formula for these indices [16]. The theorem applies in particular for the identity element, for suitable polynomially growing conjugacy classes in finitely generated discrete groups, and for semisimple elements in connected linear real semisimple Lie groups.

3.2. The K -theoretic APS index. In Part II the same authors introduced an equivariant APS index $\text{index}_G(D)$ in $K_*(C_r^*G)$ and showed that it recovers the numerical indices of Part I after applying the orbital-integral traces τ_g under suitable hypotheses [15]. Conceptually, this provides a common generalisation of the Baum–Connes analytic assembly map and the equivariant APS index. For the line of this thesis, this is highly significant: it means that a Bochner–Lichnerowicz-type vanishing statement on a manifold with boundary can reasonably be expected to have both numerical and K -theoretic formulations.

4. From numerical indices to secondary invariants

4.1. Higher orbital integrals. A crucial development in the study of proper actions of reductive Lie groups is the appearance of higher orbital integrals. Hochs, Song and Tang proved an index theorem for pairings of equivariant indices with higher cyclic cocycles on Harish-Chandra’s Schwartz algebra [14]. Their theorem gives a complete topological description of these pairings for linear reductive groups, in the sense that the family of higher orbital cocycles detects all information in the relevant K -theory classes. This is a major shift away from purely numerical indices: one now has a mechanism for extracting far more refined invariants.

4.2. Delocalized eta invariants and rho numbers. Piazza, Posthuma, Song and Tang developed this direction further in [19]. For connected linear real reductive groups, they proved a higher index theorem in which delocalized eta invariants appear as boundary correction terms, and they introduced higher rho numbers associated to G -invariant positive scalar curvature metrics. Their companion work on delocalized eta invariants of the signature operator on G -proper manifolds [18] clarified convergence issues, bordism properties, and the role of the signature operator in the same circle of ideas. From the perspective of positive scalar curvature, this is precisely the stage where the theory begins to retain information even after the primary index has vanished.

4.3. Equivariant spectral flow. The next step is spectral flow. Hochs and Yanes constructed equivariant spectral flow for paths of Dirac-type operators in the setting of proper cocompact actions by unimodular locally compact groups [20]. The resulting invariant takes values in the K -theory of the group C^* -algebra and satisfies index-equals-spectral-flow type results. It also relates naturally to delocalized eta invariants and rho invariants. This is one of the clearest indications of the present frontier: once vanishing of the primary index is understood, one is naturally led to path-wise and secondary invariants.

5. New analytic containers and nearby frontier developments

5.1. Callias-type obstructions beyond the cocompact quotient case. The cocompact quotient assumption itself has also been relaxed in nearby work. Guo, Hochs and Mathai proved an equivariant Callias-type index theorem for proper actions and used it to obtain obstructions to complete G -invariant metrics of uniformly positive scalar curvature in non-cocompact settings [12]. This shows that the proper-action positive scalar curvature story does not stop at the cocompact case considered in this thesis.

5.2. Rapidly decaying kernels and finite-volume spaces. Very recent work by Guo, Hochs and Wang constructed higher indices in a Fréchet algebra of smooth kernels with faster than exponential off-diagonal decay [22]. The point of this construction is that the rapid decay improves convergence of pairings with cyclic cocycles while the heat-kernel representation of the index makes those pairings computable. The same circle of ideas has already been applied to a higher index theorem on finite-volume locally symmetric spaces [21], where the resulting traces yield nonzero and computable results even in cases where more classical numerical formulas are too coarse.

5.3. Motion groups and localization methods. Two additional 2025 developments are particularly relevant for explicit calculations. First, Rom'an, Song and Tang constructed explicit higher orbital cocycles on Cartan motion groups and related them to the reductive-group side through Mackey deformation [23]. Second, Liu, Wang, Wang and Xiang introduced an equivariant localization algebra for almost connected Lie groups and used it to localize the K -homological Euler class of the de Rham operator [24]. Together, these works suggest that the subject is moving toward

analytic frameworks that are both flexible enough for noncompact proper actions and explicit enough for actual computations.

5.4. An extension in 2026. A very recent neighboring development is the Lorentzian equivariant index theorem of Islam and Ronge [25], which proves an equivariant APS-type fixed-point formula in a Lorentzian setting. This is not a direct continuation of the present thesis, but it shows that proper-action index methods are now spreading beyond the original Riemannian framework.

6. A working description of the present frontier

The discussion above suggests a useful working summary. The original vanishing theorem is no longer the frontier; it is now background. The present frontier lies where four themes meet:

- (i) boundary and APS theory for proper actions;
- (ii) higher and localized index classes rather than purely numerical indices;
- (iii) secondary invariants such as delocalized eta, rho, and spectral flow;
- (iv) analytic frameworks that make higher pairings explicit and computable.

The most promising problems are therefore not those that merely reprove vanishing in a slightly different form, but those that connect the very concrete operator-theoretic picture of the Mathai–Zhang construction with these newer and more refined invariants.

CHAPTER 6

Focused open problems and possible strategies

This chapter proposes a small number of open problems that are intentionally chosen to be focused and high-yield. The goal is not to formulate the largest possible conjectures, but to isolate problems that are:

- (a) directly continuous with the material of this thesis;
- (b) visibly connected to the current frontier described in Chapter 5;
- (c) sharp enough to make a good paper title;
- (d) and still controlled enough that one can attack them without launching a very long technical campaign.

1. How the problems below were selected

The main strategic observation is that the strongest reusable asset of the present thesis is not only the vanishing theorem itself, but the *operator-level cut-off picture* of the Mathai–Zhang index developed in Chapters 2 and 3. The modern literature supplies several richer containers — APS index classes, higher orbital pairings, delocalized eta invariants, rho numbers, and equivariant spectral flow — but it often works at a more abstract level. This creates natural bridge problems whose novelty lies in making those new invariants concrete, computable, or visibly connected to the classical cut-off model.

2. Problem A: a proper-action APS vanishing theorem

Problem statement. Let M be an even-dimensional G -spin manifold with boundary, carrying a proper cocompact action by a unimodular locally compact group G . Assume that the metric is of product type near the boundary, that the boundary Dirac operator is L^2 -invertible, and that the endomorphism

$$R = \frac{k^{TM}}{4} + c(R^E)$$

is pointwise positive definite on $S(TM) \otimes E$. Prove that the equivariant APS index class vanishes:

$$\text{index}_G(D_{APS}^{E,+}) = 0 \in K_0(C_r^*G),$$

and, under the trace or cocycle hypotheses appearing in the proper-action APS literature, show that the corresponding numerical or higher pairings vanish as well.

Why it is promising. This problem is the cleanest direct extension of the theorem proved in Chapter 4. It is close enough to the thesis that the geometric core of the proof should remain visible, while the result itself sits squarely inside the modern proper-action APS framework of [16, 15, 13].

The key observation. The key observation is that the Bochner–Lichnerowicz argument is fundamentally an interior positivity argument, while the post-2020 literature already provides the correct boundary index containers. In other words, the hard boundary-theoretic infrastructure now exists; what seems to be missing is a thesis-style positivity argument phrased directly in that language.

Possible first attack. A practical first attack is to begin with the identity trace and the K -theoretic APS index of [15]. One would first prove vanishing of the invariant kernel under APS boundary conditions by combining the usual Bochner argument with the product structure near the boundary. After that, one can promote the statement from the identity trace to the full K -theoretic index class, and only afterwards study higher or orbital pairings.

Why the scope is controlled. This is a focused project because it does not ask for a full new theory; it asks for one sharp theorem inside an already available APS framework. Among the problems listed here, it is probably the most natural candidate for a relatively short first paper.

3. Problem B: a cut-off model for equivariant spectral flow

Problem statement. Given a continuous path D_t of G -equivariant Dirac-type operators on a proper cocompact G -manifold, construct a spectral-flow invariant directly on the cut-off Hilbert space of Chapters 2 and 3, using the operators obtained from the explicit projection formula for P_f . Then compare this invariant with the equivariant spectral flow of Hochs and Yanes [20], at least after applying the identity trace and, if possible, suitable orbital or higher traces.

Why it is promising. This problem uses the most distinctive technical feature of the thesis: the explicit projection operator. It therefore has unusually strong continuity with the existing work. At the same time, spectral flow is a genuinely modern invariant, so a successful result would connect an older and concrete operator model to one of the newest developments in the field.

The key observation. The good observation here is that the Mathai–Zhang cut-off construction already packages noncompact equivariant geometry into a concrete Hilbert-space operator. The current spectral-flow theory lives one level higher, in K -theory and Kasparov theory. A bridge between these two levels would be conceptually valuable even if the first theorem only covers unimodular groups and only identifies the identity-trace image.

Possible first attack. Start with a path of Dirac operators whose domains can be described uniformly on the cut-off Sobolev space. Establish self-adjointness or regular self-adjointness for an appropriate realization of $P_f D_t$, prove norm-resolvent continuity in t , and define a Fredholm spectral flow in the cut-off model. The first comparison theorem should probably be an identity-trace formula. Only after that should one try to compare the full K -theoretic classes.

Why the scope is controlled. This is still a concentrated problem because one can stop as soon as one proves a convincing operator-level comparison theorem. A full theory of all traces, all groups, and all perturbations is not needed for a first paper.

4. Problem C: explicit higher rho and orbital calculations on model spaces

Problem statement. Choose a tractable family of proper cocompact model spaces — for example Cartan motion groups, homogeneous spaces G/K , or manifolds of the form $G \times_K N$ — and carry out explicit computations of higher orbital pairings, delocalized eta invariants, or rho numbers for natural Dirac operators or natural G -invariant positive scalar curvature metrics.

Why it is promising. A large part of the current literature proves the existence of refined invariants and the corresponding index formulae, but comparatively few papers work out concrete examples in detail. Explicit examples are valuable because they test the general theory, reveal which invariants are actually nonzero, and often suggest better conjectures than abstract arguments alone.

The key observation. The 2025 motion-group work of Rom'an, Song and Tang [23] supplies explicit higher cocycles in a setting that is considerably more computable than the full reductive-group side. At the same time, the higher-orbital-integral and rho-number theory of [14, 19] already tells us what kind of quantities should be extracted. This makes model-space calculations much more realistic than they would have been even a few years ago.

Possible first attack. The safest attack is to be extremely conservative in the choice of examples. One could fix a single family of cocycles and one class of model spaces, and only ask whether the relevant pairings vanish or not, postponing closed formulas to a later paper. A good first target would be rank-one or motion-group examples where the fixed-point geometry is simple and Mackey deformation can be exploited.

Why the scope is controlled. This is probably the shortest of the proposed projects if one keeps the model class very small. It is also likely to produce a paper with a very clear title and a very visible message: namely, that the modern invariants can be computed in genuine nontrivial examples.

5. Problem D: higher localised \widehat{A} obstructions beyond degree two

Problem statement. Push the obstruction theorem of [17] beyond the presently proved range coming from the subring of $H^*(B\Gamma; \mathbb{R})$ generated by classes of degree at most 2. The aim is not to solve the full conjectural obstruction statement at once, but to identify one genuinely new family — for example classes of degree 4, or a concrete class of groups with torsion — for which the vanishing of the corresponding higher localised \widehat{A} -genus can be proved under positive scalar curvature.

Why it is promising. This problem is especially attractive because the literature already isolates it as a real frontier. The target is neither artificial nor ad hoc: it is exactly the first natural region beyond the currently established theorem.

The key observation. The main observation is that the obstruction statement is already known to be true in a nontrivial range, so the issue is no longer whether the idea works at all, but how to extend the analytic input. The newer analytic containers of [22, 24] suggest that improved decay and localization methods may be the right tools for controlling more complicated pairings.

Possible first attack. One should resist the temptation to attack the full conjecture. A more realistic strategy is to choose a class of examples where both the cohomology and the isotropy geometry are explicit — for instance crystallographic groups with torsion, or a carefully chosen real-rank-one family. The first goal should be one new theorem beyond degree 2, not a general classification.

Why the scope is controlled. Among the problems listed here, this is the deepest and potentially the longest. Even so, it can still be made paper-sized if one fixes a very narrow target. It is best viewed as a second project after one of Problems A–C, rather than the first attack.

6. A suggested order of work

If the immediate goal is to obtain a new and focused paper with the shortest path from the current thesis, then the problems above should be prioritised as follows:

- (1) **Problem A:** proper-action APS vanishing;
- (2) **Problem B:** cut-off spectral flow;
- (3) **Problem C:** explicit higher rho/orbital calculations;
- (4) **Problem D:** higher localised \widehat{A} -obstructions beyond degree 2.

The first two problems are the most direct continuations of the mathematics developed in this thesis. The third is the most likely to produce a short and striking paper. The fourth is the most ambitious, but it is also the one that touches one of the clearest open conjectural frontiers currently visible in the literature.

Bibliography

- [1] V. Mathai and W. Zhang, *Geometric quantization for proper actions* (with an appendix by U. Bunke), *Adv. Math.* **225** (2010), 1224–1247.
- [2] H. B. Lawson and M. L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series, Vol. 38, Princeton University Press, Princeton, NJ, 1989.
- [3] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.
- [4] W. P. Zhang, *Lectures on Chern–Weil Theory and Witten Deformations*, Nankai Tracts in Mathematics, Vol. 4, World Scientific, Singapore, 2001.
- [5] A. Lichnerowicz, *Spineurs harmoniques*, *C. R. Acad. Sci. Paris Sér. A* **257** (1963), 7–9.
- [6] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: III*, *Ann. of Math.* **87** (1968), 546–604.
- [7] N. C. Phillips, *Equivariant K-Theory for Proper Actions*, Longman Scientific & Technical, 1989.
- [8] Z. Liu, *A Lichnerowicz vanishing theorem for proper cocompact actions*, arXiv:1310.4903.
- [9] W. Zhang, *A note on the Lichnerowicz vanishing theorem for proper actions*, *J. Noncommut. Geom.* **11** (2017), no. 3, 823–826.
- [10] H. Guo, V. Mathai and H. Wang, *Positive scalar curvature and Poincaré duality for proper actions*, *J. Noncommut. Geom.* **13** (2019), 1381–1433.
- [11] P. Piazza and H. Posthuma, *Higher genera for proper actions of Lie groups*, *Ann. K-Theory* **4** (2019), no. 3, 473–504.
- [12] H. Guo, P. Hochs and V. Mathai, *Positive scalar curvature and an equivariant Callias-type index theorem for proper actions*, *Ann. K-Theory* **6** (2021), no. 2, 319–356.
- [13] P. Piazza and H. Posthuma, *Higher genera for proper actions of Lie groups, II: The case of manifolds with boundary*, *Ann. K-Theory* **6** (2021), no. 4, 713–782.
- [14] P. Hochs, Y. Song and X. Tang, *An index theorem for higher orbital integrals*, *Math. Ann.* **382** (2022), no. 1–2, 503–553.
- [15] P. Hochs, B.-L. Wang and H. Wang, *An equivariant Atiyah–Patodi–Singer index theorem for proper actions II: the K-theoretic index*, *Math. Z.* **301** (2022), 1333–1367.
- [16] P. Hochs, B.-L. Wang and H. Wang, *An equivariant Atiyah–Patodi–Singer index theorem for proper actions I: the index formula*, *Int. Math. Res. Not. IMRN* 2023, no. 4, 3138–3193.
- [17] H. Guo and V. Mathai, *Higher localised \widehat{A} -genera for proper actions and applications*, *J. Funct. Anal.* **283** (2022), no. 12, 109695.
- [18] P. Piazza, H. Posthuma, Y. Song and X. Tang, *Delocalized eta invariants of the signature operator on G-proper manifolds*, arXiv:2307.09252.
- [19] P. Piazza, H. Posthuma, Y. Song and X. Tang, *Higher orbital integrals, rho numbers and index theory*, *Math. Ann.* **391** (2025), no. 3, 3687–3763.
- [20] P. Hochs and A. Yanes, *Equivariant spectral flow for families of Dirac-type operators*, *Ann. K-Theory* **10** (2025), no. 2, 189–235.
- [21] H. Guo, P. Hochs and H. Wang, *A higher index theorem on finite-volume locally symmetric spaces*, *Münster J. Math.* **18** (2025), 85–118.
- [22] H. Guo, P. Hochs and H. Wang, *A higher index and rapidly decaying kernels*, *J. Noncommut. Geom.* (2026), published online first.
- [23] A. Román, Y. Song and X. Tang, *Higher orbital integrals on motion groups and Mackey deformation*, arXiv:2507.21008.
- [24] H. Liu, H. Wang, Z. Wang and S. Xiang, *Equivariant localization of K-homological Euler class for almost connected Lie groups*, arXiv:2507.21415.

- [25] O. Islam and L. Ronge, *A Lorentzian equivariant index theorem*, arXiv:2602.16547.