JOINT MULTIFRACTAL SPECTRA FOR CORRELATED GAUSSIAN MULTIPLICATIVE CHAOS AND AN APPLICATION TO CIRCULAR β-ENSEMBLES

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ABSTRACT. Under standard subcritical moment and regularity inputs, we prove a joint Hausdorff multifractal spectrum for a finite family of Gaussian multiplicative chaos measures built from a correlated vector-valued log-correlated Gaussian field. The resulting dimension formula is expressed through an explicit quadratic minimization problem governed by the Gram matrix of the chaos parameters. As an application, we combine recent subcritical chaos convergence results for the circular β-ensemble with our general theory on the circle. For the Poisson regularization of the full two-component boundary field, we verify the circle-specific convergence, local moment bounds, and the oscillation input **(R1)**. We then close route B by reducing the lower bound to one-dimensional weighted insertion integrals on the interval, comparing the resulting microscopic interval GMC family to the canonical limiting insertion model, and invoking the interval insertion moment bounds. This yields an unconditional sharp description of the joint thick-point spectrum for the limiting characteristic-polynomial and counting-field chaos measures on the unit circle: an exact formula in the strictly subcritical regime, emptiness beyond the critical threshold, and the corresponding upper bound on the critical boundary.

CONTENTS

1. Introduction

Log-correlated random fields and their associated Gaussian multiplicative chaos (GMC) measures appear throughout probability, statistical mechanics, random geometry, and number theory. Given a log-correlated Gaussian field X on a domain $D \subset \mathbb{R}^d$ and a parameter in the subcritical range, one can make rigorous sense of the formal random measure

$$e^{\langle t, X(x)\rangle - \frac{1}{2}\mathrm{Var}(\langle t, X(x)\rangle)}\,dx$$

through a regularization procedure going back to Kahane [4]; see the survey of Rhodes–Vargas [9] and the general uniqueness framework of Shamov [10].

For a Radon measure η, one studies its local dimensions through the asymptotic behavior of $\eta(B(x,r))$ as $r \downarrow 0$. In the single-measure setting, the Hausdorff multifractal spectrum of subcritical GMC is by now well understood; in particular, a general multifractal formalism has been established by Bertacco [1].

The point of the present paper is that many natural problems involve not one but several log-correlated observables at once. This leads to the following geometric question: given several chaos measures built from a common vector-valued log-correlated Gaussian field, what is the Hausdorff dimension of the set of points where they simultaneously exhibit prescribed local scaling exponents? Under standard subcritical moment and regularity inputs, our main theorem answers this question in closed form.

1.1. From single spectra to joint spectra.

For a Radon measure η on a metric space and a point x in its support, define (when the limit exists)

$$d_\eta(x) := \lim_{r\downarrow 0} \frac{\log \eta(B(x,r))}{\log r}.$$

Given $h \geq 0$, define the corresponding thick-point set

$$E_\eta(h) := \{x : \ d_\eta(x) = h\}.$$

If M_t denotes the GMC measure associated with a parameter vector t, then the single-spectrum problem is to understand $\dim_{\mathrm{H}} E_{M_t}(h)$. Our joint problem is the determination of

$$\dim_{\mathrm{H}} \bigcap_{i=1}^{k} E_{M_{t_i}}(h_i)$$

for a finite family of parameters $t_1, \ldots, t_k$.

The key observation is that rooting the probability space by one chaos measure rigidly fixes the local scaling of any other chaos measure at typical points. Once this cross local dimension theorem is established, the joint spectrum becomes an explicit finite-dimensional quadratic optimization problem.

1.2. Random matrix motivation.

A particularly important source of log-correlated fields is random matrix theory. For the characteristic polynomial of the circular unitary ensemble, subcritical chaos convergence in the L^1 phase was established by Nikula–Saksman–Webb [6]. Junnila–Lambert–Webb then showed that chaos measures can be reconstructed from thick points of log-correlated fields, yielding in particular a proof of the Fyodorov–Keating thick-point picture for the CUE characteristic polynomial [3]. Recently, Lambert–Najnudel proved subcritical chaos convergence for both the characteristic polynomial and the counting field throughout circular $\boldsymbol{\beta}$-ensembles [5], complementing the exact matching result of Chhaibi–Najnudel [2]. The extremal landscape and the associated derivative martingale were identified by Paquette–Zeitouni [7].

These results suggest a natural next step: what is the size of the set of points on the circle where *two distinct random matrix observables are simultaneously thick*? We answer this question at the level of the limiting chaos measures and obtain an explicit joint Hausdorff spectrum. More precisely, after isolating the circle-specific inputs, we show that for the Poisson regularization of the full two-component boundary field the subcritical convergence, local moment bounds, and the oscillation input **(R1)** can be handled directly on the circle. For the random-matrix application below, the upper bound is proved directly from independence and local moment estimates, so it does not rely on any spatially uniform cut-off stability statement. For the lower bound, we first reduce the needed circle-rooted input to finitely many one-dimensional weighted local-mass estimates, then pass to a microscopic interval GMC family with uniformly bounded covariance perturbations, and finally compare this family directly to the canonical limiting one-point insertion model on $[0,1]$. Combined with the interval insertion moment bounds, this closes route B in the strictly subcritical regime.

1.3. Main contributions.

- We prove a *cross local dimension theorem* describing the local exponent of one chaos measure M_t at points sampled from another chaos measure M_u (Theorem 4.1).
- We prove a *joint Hausdorff multifractal spectrum* theorem for finitely many chaos measures built from a common vector-valued log-correlated Gaussian field (Theorem 5.2). The dimension is given by a quadratic rate function controlled by the Gram matrix of the parameters.
- As an application, using the subcritical CβE convergence results of Lambert–Najnudel together with the circle-specific estimates proved in Section 6 and the route B comparison arguments on microscopic interval GMCs, we obtain an unconditional sharp description for the set of points on the unit circle where the limiting chaos measures for the characteristic polynomial and the eigenvalue counting field have prescribed local exponents simultaneously (Theorem 6.31).

1.4. **Organization.** Section 2 introduces the vector log-correlated framework and notation. Section 4 proves the cross local dimension theorem. Section 5 proves the joint multifractal spectrum. Section 6 contains the application to circular β-ensembles, including the closure of route B through weighted local masses and interval insertion integrals. Section 7 discusses the near-critical regime and derivative martingales. Appendix A collects the dyadic and oscillation estimates used in the covering argument.

2. Setup and notation

Let $D \subset \mathbb{R}^d$ be a bounded open set and let $m \geq 1$. Fix a symmetric positive definite matrix $\Sigma \in \mathbb{R}^{m\times m}$. We write

$$\langle a, b\rangle_\Sigma := a^\top \Sigma b, \qquad \|a\|_\Sigma^2 := \langle a, a\rangle_\Sigma .$$

Definition 2.1 (Vector log-correlated approximation)**.** A family $(X_\varepsilon(x))_{\varepsilon\in(0,1],\, x\in D}$ of centered continuous Gaussian $\mathbb{R}^m$-valued fields is a *log-correlated approximation with covariance matrix* Σ if there exists $C < \infty$ such that for all $\varepsilon \in (0, 1]$ and all $x, y \in D$,

$$\left\| \mathbb{E}\big[X_\varepsilon(x) X_\varepsilon(y)^\top\big] - \Sigma \log \frac{1}{|x-y|+\varepsilon} \right\| \leq C. \tag{2.1}$$

For $t \in \mathbb{R}^m$ define the scalar field

$$X_\varepsilon^{(t)}(x) := \langle t, X_\varepsilon(x)\rangle.$$

Then

$$\mathbb{E}\big[X_\varepsilon^{(t)}(x) X_\varepsilon^{(t)}(y)\big] = \|t\|_\Sigma^2 \log \frac{1}{|x-y|+\varepsilon} + O(1), \tag{2.2}$$

and for $u, t \in \mathbb{R}^m$,

$$\mathbb{E}\big[X_\varepsilon^{(t)}(x) X_{\varepsilon'}^{(u)}(y)\big] = \langle t, u\rangle_\Sigma \log \frac{1}{|x-y|+\varepsilon \vee \varepsilon'} + O(1). \tag{2.3}$$

Definition 2.2 (Subcritical chaos)**.** Fix $t \in \mathbb{R}^m$ with $\|t\|_\Sigma^2 < 2d$. For $\varepsilon \in (0, 1]$ define

$$M_{t,\varepsilon}(dx) := \exp\Big(X_\varepsilon^{(t)}(x) - \tfrac{1}{2}\mathbb{E}[X_\varepsilon^{(t)}(x)^2]\Big)\, dx. \tag{2.4}$$

We assume throughout that, for every parameter vector t under consideration, the measures $M_{t,\varepsilon}$ converge in probability in the topology of weak convergence of Radon measures to a non-trivial limit measure M_t as $\varepsilon \downarrow 0$.

Remark 2.3 (Standing assumptions)**.** All results below are proved under three inputs: the covariance control (2.1)–(2.3), the subcritical convergence in Definition 2.2, and the moment bounds of Proposition 3.1. We do not attempt to reprove these standard GMC facts in the present paper; they may be taken from the usual subcritical GMC theory, e.g. [9, 10, 1].

2.1. **Notation.** Throughout the paper, the symbol h denotes a local mass exponent. We reserve the notation $\boldsymbol{\beta}$ for the Dyson parameter in circular $\boldsymbol{\beta}$-ensembles.

For a Radon measure η on a metric space and x in its support, we write

$$d_\eta(x) := \lim_{r \downarrow 0} \frac{\log \eta(B(x,r))}{\log r},$$

whenever the limit exists, and define the thick-point set

$$E_\eta(h) := \{x : \ d_\eta(x) = h\}.$$

When $\eta = M_t$, we write $E_t(h) := E_{M_t}(h)$. For $\boldsymbol{h} = (h_1, \ldots, h_k)$ we set

$$E(\boldsymbol{h}) := \bigcap_{i=1}^{k} E_{t_i}(h_i).$$

3. Moment bounds

For t subcritical, define the structure function

$$\xi_t(p) := \Big(d + \frac{1}{2}\, \|t\|_\Sigma^2 \Big)p - \frac{1}{2}\, \|t\|_\Sigma^2\, p^2. \tag{3.1}$$

Proposition 3.1 (Power-law moment bounds)**.** *Fix $t \in \mathbb{R}^m$ with $\|t\|_\Sigma^2 < 2d$. For any $p \in (-\infty, 2d/\, \|t\|_\Sigma^2)$ there exist constants $C < \infty$ and $r_0 > 0$ such that for all $x \in D$ and all $r \in (0, r_0)$ with $B(x, 2r) \subset D$,*

$$\mathbb{E}\big[M_t(B(x,r))^p\big] \le C\, r^{\xi_t(p)}.$$

Moreover, the same bound holds (up to changing C) with M_t replaced by $M_{t,\varepsilon}$ uniformly over $\varepsilon \in (0, r]$.

Remark 3.2. These bounds are standard in the GMC literature; see [9, 1]. We use only the existence range of positive and negative moments and the scale-uniform upper bounds.

Remark 3.3 (Standing assumptions on the approximation)**.** Throughout the remainder of the paper we work under the following two inputs for every subcritical parameter t that appears:

(i) the convergence in probability $M_{t,\varepsilon} \Rightarrow M_t$ from Definition 2.2;
(ii) a uniform oscillation estimate at the cut-off scale: for every $A, q > 0$ and every compact cube $Q \Subset D$ there exists $C_{A,q,Q,t} < \infty$ such that

$$\sup_{n \ge 1} \sup_{x \in Q} \mathbb{E}\Big[\exp\Big(q \sup_{y \in B(x, A2^{-n})} |X^{(t)}_{2^{-n}}(y) - X^{(t)}_{2^{-n}}(x)|\Big)\Big] \le C_{A,q,Q,t}.$$

These hypotheses are compatible with the standard smooth regularizations considered in the GMC literature, but we do not verify them here.

3.1. **Standing regularity inputs.** From this point on we fix the dyadic sequence $r_n := 2^{-n}$. In addition to the convergence in Definition 2.2 and the moment input of Proposition 3.1, we assume the following two standard regularity properties of the chosen regularization.

(R1) Uniform local oscillation. For every subcritical $t \in \mathbb{R}^m$, every cube $Q \Subset D$, and every fixed $\kappa > 0$,

$$\omega^{(t)}_{n,\kappa}(Q) := \sup_{\substack{x,y \in Q \\ |x-y| \le \kappa r_n}} \big|X^{(t)}_{r_n}(x) - X^{(t)}_{r_n}(y)\big| = o(\log(1/r_n)) \qquad \text{almost surely.}$$

(R2) Local cut-off stability. For every subcritical $t \in \mathbb{R}^m$ and every cube $Q \Subset D$,

$$\sup_{x \in Q} |\log M_t(B(x, r_n)) - \log M_{t, r_n}(B(x, r_n))| = o(\log(1/r_n)) \qquad \text{almost surely.}$$

These two inputs are the only regularization-specific properties used in the proofs below. For the purposes of the present paper, we take them as standing assumptions. They are expected to hold for the standard smooth regularizations of GMC, but this verification is not part of the present paper.

4. Rooting and cross local dimensions

Theorem 4.1 (Cross local dimension / rooted typicality). *Let $u, t \in \mathbb{R}^m$ satisfy the subcritical conditions $\|u\|_\Sigma^2 < 2d$ and $\|t\|_\Sigma^2 < 2d$. Define the cross exponent*

$$h(t \mid u) := d + \frac{1}{2}\|t\|_\Sigma^2 - \langle t, u\rangle_\Sigma. \tag{4.1}$$

Then almost surely, for M_u-a.e. $x \in D$,

$$\lim_{r\downarrow 0} \frac{\log M_t(B(x,r))}{\log r} = h(t \mid u).$$

Equivalently, $M_u\big(D \setminus E_t(h(t \mid u))\big) = 0$ almost surely.

Corollary 4.2 (Exact dimensionality). *Let $u \in \mathbb{R}^m$ with $\|u\|_\Sigma^2 < 2d$. Then almost surely, for M_u-a.e. x,*

$$\lim_{r\downarrow 0} \frac{\log M_u(B(x,r))}{\log r} = d - \frac{1}{2}\|u\|_\Sigma^2.$$

Corollary 4.3 (Finite-family version). *Fix finitely many $t_1, \ldots, t_k \in \mathbb{R}^m$ with $\|t_i\|_\Sigma^2 < 2d$. Then almost surely, for M_u-a.e. x,*

$$\big(d_{M_{t_1}}(x), \ldots, d_{M_{t_k}}(x)\big) = \big(h(t_1 \mid u), \ldots, h(t_k \mid u)\big).$$

4.1. **Girsanov at a point.**

Lemma 4.4 (Path-space Girsanov at fixed scale). *Fix $\varepsilon, \varepsilon' \in (0,1]$, a compact cube $K \Subset D$, a point $x \in K$, and vectors $u, t \in \mathbb{R}^m$. Let $F : C(K) \to \mathbb{R}$ be bounded and Borel measurable, viewed as a functional of the continuous field $X_\varepsilon^{(t)}|_K \in C(K)$. Then*

$$\mathbb{E}\Big[e^{X_{\varepsilon'}^{(u)}(x) - \frac{1}{2}\mathbb{E}[X_{\varepsilon'}^{(u)}(x)^2]} F(X_\varepsilon^{(t)}|_K)\Big] = \mathbb{E}\Big[F\big(X_\varepsilon^{(t)}|_K + h_{x,\varepsilon,\varepsilon'}\big)\Big],$$

where $h_{x,\varepsilon,\varepsilon'} \in C(K)$ is the deterministic function

$$h_{x,\varepsilon,\varepsilon'}(y) := \mathbb{E}[X_\varepsilon^{(t)}(y) X_{\varepsilon'}^{(u)}(x)], \qquad y \in K.$$

Proof. For fixed ε, the pair $\big(X_\varepsilon^{(t)}|_K,\, X_{\varepsilon'}^{(u)}(x)\big)$ is a jointly Gaussian random element of $C(K) \times \mathbb{R}$. Tilting the law by

$$\exp\Big(X_{\varepsilon'}^{(u)}(x) - \frac{1}{2}\mathrm{Var}(X_{\varepsilon'}^{(u)}(x))\Big)$$

shifts the $C(K)$-valued marginal by its covariance with the scalar Gaussian variable $X_{\varepsilon'}^{(u)}(x)$, namely by the Cameron–Martin vector $h_{x,\varepsilon,\varepsilon'}$. This is the standard Cameron–Martin theorem for jointly Gaussian Banach-space-valued random variables. □

Lemma 4.5 (Global Gaussian tilt for bounded functionals). *Fix $\varepsilon' \in (0,1]$, a point $x \in D$, and a vector $u \in \mathbb{R}^m$. Let $\mathcal{G}$ be the σ-field generated by the full collection of regularized Gaussian variables*

$$\big\{X_\delta^{(s)}(y) : \ \delta \in (0,1],\ y \in D,\ s \in \mathbb{R}^m\big\}.$$

For every bounded $\mathcal{G}$-measurable nonnegative random variable F, one has

$$\mathbb{E}\Big[e^{X_{\varepsilon'}^{(u)}(x) - \frac{1}{2}\mathbb{E}[X_{\varepsilon'}^{(u)}(x)^2]}\, F\Big] = \mathbb{E}\big[F^{(x,\varepsilon',u)}\big],$$

where $F^{(x,\varepsilon',u)}$ denotes the same functional evaluated on the deterministically shifted family

$$X_\delta^{(s)}(y) \ \mapsto\ X_\delta^{(s)}(y) + \mathbb{E}[X_\delta^{(s)}(y) X_{\varepsilon'}^{(u)}(x)].$$

Proof. Let $\mathcal{A}$ be the class of bounded $\mathcal{G}$-measurable nonnegative random variables for which the stated identity holds. By Lemma 4.4, $\mathcal{A}$ contains every bounded cylinder functional depending on finitely many coordinates

$$\big(X_{\delta_1}^{(s_1)}(y_1), \ldots, X_{\delta_k}^{(s_k)}(y_k)\big).$$

The class $\mathcal{A}$ is stable under bounded monotone convergence: if $0 \le F_n \uparrow F$ and each $F_n \in \mathcal{A}$, then by monotone convergence the identity passes to the limit. Since the bounded cylinder functionals generate $\mathcal{G}$, the monotone class theorem implies that every bounded $\mathcal{G}$-measurable nonnegative random variable belongs to $\mathcal{A}$. □

Convention for the tail lemmas. Fix subcritical $u, t \in \mathbb{R}^m$ and a cube $Q \Subset D$. Choose once and for all a larger cube $Q^+ \Subset D$ such that $Q \Subset Q^+$. Set

$$r_0 := \frac{1}{2}\operatorname{dist}\big(Q, (Q^+)^c\big) > 0,$$

so that $B(x,r) \subset Q^+$ for every $x \in Q$ and every $r \in (0, r_0]$. Write

$$a := \langle t, u\rangle_\Sigma\,, \qquad \sigma^2 := \|t\|_\Sigma^2\,, \qquad h := h(t \mid u) = d + \frac{1}{2}\sigma^2 - a.$$

For $r > 0$ define the cut-off chaos at scale r by

$$M_{t,r}(dx) := \exp\Big(X_r^{(t)}(x) - \tfrac{1}{2}\mathbb{E}[X_r^{(t)}(x)^2]\Big)\,dx,$$

and similarly $M_{u,\varepsilon'}$ at scale $\varepsilon' > 0$.

4.2. **Tail bounds under a rooted measure.**

Lemma 4.6 (Upper tail under $M_{u,\varepsilon'}$, uniform in ε')**.** *Fix u, t subcritical and $\delta > 0$. There exist constants $C, \alpha > 0$ such that for all sufficiently small r and all $\varepsilon' \in (0, r^2]$,*

$$\mathbb{E}\left[\int_Q \mathbf{1}_{\{M_{t,r}(B(x,r)) > r^{h(t|u)-\delta}\}}\, M_{u,\varepsilon'}(dx)\right] \le C r^{\alpha}.$$

Proof. Fix $\delta > 0$. Let $r \in (0, r_0]$ and $\varepsilon' \in (0, r^2]$. Set $A_r(x) := \{M_{t,r}(B(x,r)) > r^{h-\delta}\}$ for $x \in Q$.

Step 1: Fubini + Girsanov. By definition of $M_{u,\varepsilon'}$ and Fubini,

$$\mathbb{E}\Big[\int_Q \mathbf{1}_{A_r(x)}\, M_{u,\varepsilon'}(dx)\Big] = \int_Q \mathbb{E}\Big[\mathbf{1}_{A_r(x)}\, e^{X_{\varepsilon'}^{(u)}(x) - \frac{1}{2}\mathbb{E}[X_{\varepsilon'}^{(u)}(x)^2]}\Big]\,dx.$$

Apply Lemma 4.4 on the larger cube Q^+ with the functional

$$F_x(f) := \mathbf{1}_{\{\int_{B(x,r)} e^{f(y) - \frac{1}{2}\mathbb{E}[X_r^{(t)}(y)^2]}dy > r^{h-\delta}\}}.$$

The inner expectation becomes

$$\mathbb{P}\Big(\widetilde{M}_{t,r}^{(x)}(B(x,r)) > r^{h-\delta}\Big),$$

where

$$\widetilde{M}_{t,r}^{(x)}(B(x,r)) = \int_{B(x,r)} \exp\Big(X_r^{(t)}(y) - \tfrac{1}{2}\mathbb{E}[X_r^{(t)}(y)^2] + m_{r,\varepsilon'}(x,y)\Big)dy$$

and $m_{r,\varepsilon'}(x,y) := \mathbb{E}[X_r^{(t)}(y) X_{\varepsilon'}^{(u)}(x)]$.

Step 2: uniform upper bound on the shift. By (2.3), there exists $C_0 < \infty$ such that

$$\Big|m_{r,\varepsilon'}(x,y) - a\log\frac{1}{|x-y|+r}\Big| \le C_0 \qquad (x \in Q,\ y \in B(x,r),\ \varepsilon' \le r).$$

Since $|x-y| + r \in [r, 2r]$, we have

$$m_{r,\varepsilon'}(x,y) \le a\log\frac{1}{r} + C_1, \qquad C_1 := C_0 + |a|\log 2.$$

Hence

$$\widetilde{M}_{t,r}^{(x)}(B(x,r)) \le e^{C_1} r^{-a} M_{t,r}(B(x,r)).$$

Therefore,

$$\mathbb{P}\Big(\widetilde{M}_{t,r}^{(x)}(B(x,r)) > r^{h-\delta}\Big) \le \mathbb{P}\Big(M_{t,r}(B(x,r)) > e^{-C_1} r^{h-\delta+a}\Big).$$

Step 3: Markov + moment bound. Fix $\eta \in (0,1)$. By Markov,

$$\mathbb{P}\Big(M_{t,r}(B(x,r)) > e^{-C_1} r^{h-\delta+a}\Big) \le e^{C_1\eta} r^{(\delta-h-a)\eta}\,\mathbb{E}\big[M_{t,r}(B(x,r))^\eta\big].$$

By Proposition 3.1,

$$\mathbb{E}\big[M_{t,r}(B(x,r))^\eta\big] \le C_2 r^{\xi_t(\eta)}, \qquad \xi_t(\eta) = \Big(d + \frac{1}{2}\sigma^2\Big)\eta - \frac{1}{2}\sigma^2\eta^2.$$

Since $h+a=d+\frac{1}{2}\sigma^2$, the exponent becomes

$$(\delta-h-a)\eta+\xi_t(\eta)=\delta\eta-\frac{1}{2}\sigma^2\eta^2.$$

Choose $\eta>0$ so small that

$$\alpha:=\delta\eta-\frac{1}{2}\sigma^2\eta^2>0.$$

Then

$$\mathbb{P}\Big(\widetilde{M}^{(x)}_{t,r}(B(x,r))>r^{h-\delta}\Big)\le C_3r^{\alpha}$$

uniformly in $x\in Q$, with $C_3:=e^{C_1\eta}C_2$.

Step 4: integrate over $x\in Q$. Therefore,

$$\mathbb{E}\Big[\int_Q \mathbf{1}_{A_r(x)}\,M_{u,\varepsilon'}(dx)\Big]\le |Q|C_3r^{\alpha},$$

as claimed. □

Lemma 4.7 (Lower tail under $M_{u,\varepsilon'}$, uniform in ε')**.** *Fix u,t subcritical and $\delta>0$. There exist constants $C,\alpha>0$ such that for all sufficiently small r and all $\varepsilon'\in(0,r^2]$,*

$$\mathbb{E}\left[\int_Q \mathbf{1}_{\{M_{t,r}(B(x,r))<r^{h(t|u)+\delta}\}}\,M_{u,\varepsilon'}(dx)\right]\le Cr^{\alpha}.$$

Proof. Fix $\delta>0$. Let $r\in(0,r_0]$ and $\varepsilon'\in(0,r^2]$. Set $A_r^-(x):=\{M_{t,r}(B(x,r))<r^{h+\delta}\}$.

Step 1: Fubini + Girsanov. As above,

$$\mathbb{E}\Big[\int_Q \mathbf{1}_{A_r^-(x)}\,M_{u,\varepsilon'}(dx)\Big]=\int_Q \mathbb{P}\Big(\widetilde{M}^{(x)}_{t,r}(B(x,r))<r^{h+\delta}\Big)dx.$$

Step 2: uniform lower bound on the shift. Again by (2.3),

$$\big|m_{r,\varepsilon'}(x,y)-a\log\tfrac{1}{|x-y|+r}\big|\le C_0\qquad (x\in Q,\ y\in B(x,r),\ \varepsilon'\le r),$$

so that

$$m_{r,\varepsilon'}(x,y)\ge a\log\tfrac{1}{r}-C_1,\qquad C_1:=C_0+|a|\log 2.$$

Hence

$$\widetilde{M}^{(x)}_{t,r}(B(x,r))\ge e^{-C_1}r^{-a}M_{t,r}(B(x,r)).$$

Consequently,

$$\{\widetilde{M}^{(x)}_{t,r}(B(x,r))<r^{h+\delta}\}\subset\{M_{t,r}(B(x,r))<e^{C_1}r^{h+\delta+a}\}.$$

Step 3: Markov on the inverse + negative moments. Fix $\eta\in(0,1)$. Then

$$\begin{aligned}\mathbb{P}\Big(M_{t,r}(B(x,r))<e^{C_1}r^{h+\delta+a}\Big)&=\mathbb{P}\Big(M_{t,r}(B(x,r))^{-\eta}>e^{-C_1\eta}r^{-(h+\delta+a)\eta}\Big)\\&\le e^{C_1\eta}r^{(h+\delta+a)\eta}\mathbb{E}\big[M_{t,r}(B(x,r))^{-\eta}\big].\end{aligned}$$

By Proposition 3.1,

$$\mathbb{E}\big[M_{t,r}(B(x,r))^{-\eta}\big]\le C_4r^{\xi_t(-\eta)},\qquad \xi_t(-\eta)=-\Big(d+\frac{1}{2}\sigma^2\Big)\eta-\frac{1}{2}\sigma^2\eta^2.$$

Since $h+a=d+\frac{1}{2}\sigma^2$, the exponent becomes

$$(h+\delta+a)\eta+\xi_t(-\eta)=\delta\eta-\frac{1}{2}\sigma^2\eta^2.$$

Choose $\eta>0$ small enough that

$$\alpha:=\delta\eta-\frac{1}{2}\sigma^2\eta^2>0.$$

Then

$$\mathbb{P}\Big(\widetilde{M}^{(x)}_{t,r}(B(x,r))<r^{h+\delta}\Big)\le C_5r^{\alpha}$$

uniformly in $x\in Q$, where $C_5:=e^{C_1\eta}C_4$.

Step 4: integrate over $x \in Q$. Therefore,

$$\mathbb{E}\Big[\int_Q \mathbf{1}_{A_r^-(x)}\, M_{u,\varepsilon'}(dx)\Big] \le |Q| C_5 r^{\alpha},$$

which proves the lemma. □

4.3. **Transfer and stability lemmas.**

Lemma 4.8 (Portmanteau transfer from $M_{u,\varepsilon}$ to M_u)**.** *Let $(\mu_\varepsilon)_{\varepsilon\downarrow 0}$ be random Radon measures on a compact metric space Q such that $\mu_\varepsilon \Rightarrow \mu$ in probability for the weak topology. Then there exists a deterministic sequence $\varepsilon_k \downarrow 0$ such that $\mu_{\varepsilon_k} \Rightarrow \mu$ almost surely.*

Moreover, on the event $\mu_{\varepsilon_k} \Rightarrow \mu$, for every random open set $O \subset Q$ such that the random variables $\mu(O)$ and $\mu_{\varepsilon_k}(O)$ are measurable,

$$\mu(O) \le \liminf_{k\to\infty} \mu_{\varepsilon_k}(O).$$

Consequently, for every such random open set O,

$$\mathbb{E}[\mu(O)] \le \liminf_{k\to\infty} \mathbb{E}[\mu_{\varepsilon_k}(O)].$$

Proof. Since the space of Radon measures on Q endowed with the weak topology is Polish, convergence in probability implies the existence of a deterministic subsequence converging almost surely. On the event of weak convergence, the Portmanteau theorem gives $\mu(O) \le \liminf_k \mu_{\varepsilon_k}(O)$ for every open O. Taking expectations and applying Fatou concludes the proof. □

Lemma 4.9 (Cut-off stability of local dimensions)**.** *Assume the standing regularity input **(R2)** of Section 3.1. Fix $t \in \mathbb{R}^m$ with $\|t\|_\Sigma^2 < 2d$ and a cube $Q \Subset D$. Then almost surely, for every $\eta > 0$ there exists $N(\eta)$ such that for all $n \ge N(\eta)$ and all $x \in Q$,*

$$r_n^{\eta} \le \frac{M_t(B(x,r_n))}{M_{t,r_n}(B(x,r_n))} \le r_n^{-\eta}. \tag{4.2}$$

Consequently,

$$\lim_{n\to\infty} \frac{\log M_{t,r_n}(B(x,r_n))}{\log r_n} = h \iff \lim_{n\to\infty} \frac{\log M_t(B(x,r_n))}{\log r_n} = h,$$

and either dyadic limit implies the full-radius limit $\lim_{r\downarrow 0} \frac{\log M_t(B(x,r))}{\log r} = h$.

Proof. By **(R2)**, almost surely,

$$\sup_{x\in Q} |\log M_t(B(x,r_n)) - \log M_{t,r_n}(B(x,r_n))| = o(\log(1/r_n)).$$

Hence for every $\eta > 0$ and all large n,

$$\sup_{x\in Q} \left|\log \frac{M_t(B(x,r_n))}{M_{t,r_n}(B(x,r_n))}\right| \le \eta \log(1/r_n),$$

which is exactly (4.2). The equivalence of the dyadic limits follows immediately from (4.2), and the passage from dyadic radii to all radii is standard by monotonicity of $r \mapsto M_t(B(x,r))$. □

Proof of Theorem 4.1. Let $\{Q_m\}_{m\ge1}$ be a countable family of cubes with $Q_m \Subset D$ and $D = \bigcup_{m\ge1} Q_m$. It is enough to prove the claim on each fixed cube $Q := Q_m$ and then intersect the corresponding full-probability events.

Fix such a cube $Q \Subset D$, fix $\delta > 0$, and let $r_n := 2^{-n}$. Define the random open sets

$$A_n^+ := \Big\{x \in Q :\ M_{t,r_n}(B(x,r_n)) > r_n^{h(t|u)-\delta}\Big\}, \qquad A_n^- := \Big\{x \in Q :\ M_{t,r_n}(B(x,r_n)) < r_n^{h(t|u)+\delta}\Big\}.$$

Because M_{t,r_n} has a continuous density, the map $x \mapsto M_{t,r_n}(B(x,r_n))$ is continuous, so $A_n^\pm$ are open.

By Lemmas 4.6 and 4.7, there exist $C, \alpha > 0$ such that for all large n and all $\varepsilon' \in (0, r_n^2]$,

$$\mathbb{E}\big[M_{u,\varepsilon'}(A_n^+)\big] \le C r_n^{\alpha}, \qquad \mathbb{E}\big[M_{u,\varepsilon'}(A_n^-)\big] \le C r_n^{\alpha}.$$

Choose a deterministic sequence $\varepsilon_k \downarrow 0$ such that $M_{u,\varepsilon_k} \Rightarrow M_u$ almost surely (by Lemma 4.8). Applying Lemma 4.8 to the random open sets $A_n^\pm$ yields

$$\mathbb{E}\big[M_u(A_n^\pm)\big] \le \liminf_{k\to\infty} \mathbb{E}\big[M_{u,\varepsilon_k}(A_n^\pm)\big] \le C r_n^{\alpha}.$$

Hence

$$\sum_{n\geq 1}\mathbb{E}\big[M_u(A_n^\pm)\big]<\infty.$$

By Tonelli,

$$\mathbb{E}\Big[\sum_{n\geq 1}M_u(A_n^\pm)\Big]<\infty,$$

so $\sum_n M_u(A_n^\pm)<\infty$ almost surely. In particular,

$$M_u\Big(\limsup_{n\to\infty}A_n^\pm\Big)=0\qquad\text{almost surely.}$$

Thus, for M_u-a.e. $x\in Q$, there exists $N(x)$ such that for all $n\geq N(x)$,

$$r_n^{h(t|u)+\delta}\leq M_{t,r_n}(B(x,r_n))\leq r_n^{h(t|u)-\delta}.$$

Dividing by $\log r_n<0$ and letting $n\to\infty$ gives

$$\lim_{n\to\infty}\frac{\log M_{t,r_n}(B(x,r_n))}{\log r_n}=h(t\mid u)\qquad\text{for } M_u\text{-a.e. } x\in Q.$$

Lemma 4.9 transfers this from M_{t,r_n} to M_t and from dyadic radii to all radii. Since $\delta>0$ was arbitrary, the theorem follows on Q, hence on all of D. □

Proof of Corollary 4.2. Apply Theorem 4.1 with $t=u$. Then

$$h(u\mid u)=d+\frac{1}{2}\|u\|_\Sigma^2-\|u\|_\Sigma^2=d-\frac{1}{2}\|u\|_\Sigma^2.$$

□

Proof of Corollary 4.3. Apply Theorem 4.1 to each t_i and intersect the corresponding full-M_u-measure events. □

5. Joint Hausdorff multifractal spectrum

5.1. **A quadratic rate function.** Fix $k\in\mathbb{N}$ and subcritical parameters $t_1,\dots,t_k\in\mathbb{R}^m$. Let $G\in\mathbb{R}^{k\times k}$ be the Gram matrix

$$G_{ij}:=\langle t_i,t_j\rangle_\Sigma,\qquad 1\leq i,j\leq k,$$

and let $G^\dagger$ be its Moore–Penrose pseudoinverse. Define $T:\mathbb{R}^m\to\mathbb{R}^k$ by

$$T(u):=\big(\langle t_1,u\rangle_\Sigma,\dots,\langle t_k,u\rangle_\Sigma\big).$$

For $\boldsymbol{c}\in\mathbb{R}^k$, define the quadratic rate function

$$I(\boldsymbol{c}):=\frac{1}{2}\inf\Big\{\|u\|_\Sigma^2:\ u\in\mathbb{R}^m,\ Tu=\boldsymbol{c}\Big\}\in[0,\infty],\tag{5.1}$$

with the convention $I(\boldsymbol{c})=+\infty$ if the constraint is infeasible.

Lemma 5.1 (Closed form of the rate function)**.** *For all $\boldsymbol{c}\in\mathbb{R}^k$,*

$$I(\boldsymbol{c})=\begin{cases}\frac{1}{2}\boldsymbol{c}^\mathsf{T}G^\dagger\boldsymbol{c}, & \boldsymbol{c}\in\operatorname{Ran}(G)=\operatorname{Ran}(T),\\ +\infty, & \boldsymbol{c}\notin\operatorname{Ran}(G).\end{cases}$$

Moreover, when $\boldsymbol{c}\in\operatorname{Ran}(G)$ the unique minimizer of minimal Σ-norm is

$$u_*=\sum_{i,j=1}^k(G^\dagger)_{ij}c_j\,t_i,\qquad \|u_*\|_\Sigma^2=\boldsymbol{c}^\mathsf{T}G^\dagger\boldsymbol{c}.\tag{5.2}$$

Proof. The minimization problem is the orthogonal projection of 0 onto the affine space $\{u:Tu=\boldsymbol{c}\}$ in the Hilbert space $(\mathbb{R}^m,\langle\cdot,\cdot\rangle_\Sigma)$. If $\boldsymbol{c}\notin\operatorname{Ran}(T)$, the feasible set is empty and $I(\boldsymbol{c})=+\infty$. If $\boldsymbol{c}\in\operatorname{Ran}(T)$, the minimum is attained at the unique element of minimal Σ-norm, given by the normal equations. The explicit formula (5.2) is the standard pseudoinverse solution, and the identity $\|u_*\|_\Sigma^2=\boldsymbol{c}^\mathsf{T}G^\dagger\boldsymbol{c}$ follows by direct computation. □

Theorem 5.2 (Joint Hausdorff multifractal spectrum)**.** *Let X be an $\mathbb{R}^m$-valued log-correlated Gaussian field on a bounded domain $D \subset \mathbb{R}^d$ with covariance matrix Σ, and let M_t denote the corresponding GMC measure for each $t \in \mathbb{R}^m$ with $\|t\|_\Sigma^2 < 2d$.*

Fix subcritical parameters $t_1, \dots, t_k \in \mathbb{R}^m$ and define

$$E(\boldsymbol{h}) := \bigcap_{i=1}^{k} E_{t_i}(h_i), \qquad \boldsymbol{h} = (h_1, \dots, h_k) \in [0,\infty)^k.$$

Set

$$c_i(\boldsymbol{h}) := d + \frac{1}{2}\,\|t_i\|_\Sigma^2 - h_i, \qquad 1 \le i \le k. \tag{5.3}$$

Then almost surely:

- *If $I(\boldsymbol{c}(\boldsymbol{h})) < d$, then*
$$\dim_{\mathrm{H}} E(\boldsymbol{h}) = d - I(\boldsymbol{c}(\boldsymbol{h})) = d - \frac{1}{2}\,\boldsymbol{c}(\boldsymbol{h})^{\mathsf{T}} G^{\dagger} \boldsymbol{c}(\boldsymbol{h}). \tag{5.4}$$
- *If $I(\boldsymbol{c}(\boldsymbol{h})) = d$, then $\dim_{\mathrm{H}} E(\boldsymbol{h}) \le 0$.*
- *If $I(\boldsymbol{c}(\boldsymbol{h})) > d$ or $I(\boldsymbol{c}(\boldsymbol{h})) = +\infty$ (equivalently, $\boldsymbol{c}(\boldsymbol{h}) \notin \mathrm{Ran}(G)$), then $E(\boldsymbol{h}) = \varnothing$.*

In the case $I(\boldsymbol{c}(\boldsymbol{h})) < \infty$, the minimizer u_* is given by (5.2); if moreover $I(\boldsymbol{c}(\boldsymbol{h})) < d$, then M_{u_*} gives full mass to $E(\boldsymbol{h})$.

Proposition 5.3 (Lower bound)**.** *If $I(\boldsymbol{c}(\boldsymbol{h})) < d$, then almost surely*

$$\dim_{\mathrm{H}} E(\boldsymbol{h}) \ge d - I(\boldsymbol{c}(\boldsymbol{h})).$$

Proof. Let u_* be the minimizer (5.2). Since $I(\boldsymbol{c}(\boldsymbol{h})) < d$, we have $\|u_*\|_\Sigma^2 = 2I(\boldsymbol{c}(\boldsymbol{h})) < 2d$, so M_{u_*} is subcritical and non-trivial. By Theorem 4.1, for M_{u_*}-a.e. x and every i,

$$d_{M_{t_i}}(x) = h(t_i \mid u_*) = d + \frac{1}{2}\,\|t_i\|_\Sigma^2 - \langle t_i, u_* \rangle_\Sigma = h_i,$$

because $\langle t_i, u_* \rangle_\Sigma = c_i(\boldsymbol{h})$ by construction. Hence $M_{u_*}(E(\boldsymbol{h})) = M_{u_*}(D)$ almost surely. By Corollary 4.2, M_{u_*} is exact-dimensional with dimension

$$d - \frac{1}{2}\,\|u_*\|_\Sigma^2 = d - I(\boldsymbol{c}(\boldsymbol{h})).$$

The mass distribution principle yields the lower bound. $\square$

Proposition 5.4 (Upper bound)**.** *Fix $\boldsymbol{h} \in [0,\infty)^k$. Then almost surely:*

- *if $I(\boldsymbol{c}(\boldsymbol{h})) < \infty$, one has*
$$\dim_{\mathrm{H}} E(\boldsymbol{h}) \le d - I(\boldsymbol{c}(\boldsymbol{h}));$$
- *if $I(\boldsymbol{c}(\boldsymbol{h})) = +\infty$, then $E(\boldsymbol{h}) = \varnothing$.*

Moreover, if $I(\boldsymbol{c}(\boldsymbol{h})) > d$, then $E(\boldsymbol{h}) = \varnothing$.

Proof. Let $\{Q_m\}_{m\ge1}$ be a countable family of cubes with $Q_m \Subset D$ and $D = \bigcup_{m\ge1} Q_m$. It suffices to prove the claimed estimate on each fixed cube $Q := Q_m$ and then take the supremum over m.

Fix such a cube $Q \Subset D$. Choose a larger cube $Q^+ \Subset D$ with $Q \Subset Q^+$ and fix an integer $L \ge 8\sqrt{d}$. Let $r_n := 2^{-n}$, set $\ell_n := r_n/L$, and let $\{I_j^n\}_{j\in\mathcal{J}_n}$ be the partition of Q into cubes of side length ℓ_n with centers x_j^n. Define

$$Y_n(x) := \big(X_{r_n}^{(t_1)}(x), \dots, X_{r_n}^{(t_k)}(x)\big) \in \mathbb{R}^k, \qquad L_n := \log(1/r_n).$$

By (2.3), uniformly in $x \in Q^+$,

$$\mathrm{Cov}(Y_n(x)) = L_n G + R_n(x),$$

where $\sup_{n, x\in Q^+} \|R_n(x)\| < \infty$.

Fix $\delta > 0$ and let $\boldsymbol{c} := \boldsymbol{c}(\boldsymbol{h})$. Here and below, $|\cdot|$ on $\mathbb{R}^k$ denotes the sup norm. For each n define the set of δ-good cubes

$$\mathcal{J}_n(\delta) := \Big\{ j \in \mathcal{J}_n : \ \big|Y_n(x_j^n)/L_n - \boldsymbol{c}\big| \le \delta \Big\}.$$

We claim that, almost surely,

$$E(\boldsymbol{h}) \cap Q \subset \limsup_{n\to\infty} \bigcup_{j\in\mathcal{J}_n(\delta)} I_j^n. \tag{5.5}$$

Indeed, fix $x \in E(\boldsymbol{h}) \cap Q$ and let $j(n)$ be such that $x \in I^n_{j(n)}$. By Corollary A.3, we have

$$\frac{Y_n(x)}{L_n} \longrightarrow \boldsymbol{c}.$$

Moreover, since $|x - x^n_{j(n)}| \leq \frac{\sqrt{d}}{2}\ell_n \leq r_n/8$, the oscillation input **(R1)** applied on Q^+ gives

$$\frac{|Y_n(x^n_{j(n)}) - Y_n(x)|}{L_n} \longrightarrow 0.$$

Hence $\big|Y_n(x^n_{j(n)})/L_n - \boldsymbol{c}\big| \leq \delta$ for all sufficiently large n, i.e. $j(n) \in \mathcal{J}_n(\delta)$ eventually. This proves (5.5).

Case 1: $\boldsymbol{c} \in \mathrm{Ran}(G)$. Let $\lambda_* := G^\dagger \boldsymbol{c} \in \mathbb{R}^k$. Then

$$\langle \lambda_*, \boldsymbol{c}\rangle - \frac{1}{2}\lambda_*^\mathsf{T} G \lambda_* = \frac{1}{2}\boldsymbol{c}^\mathsf{T} G^\dagger \boldsymbol{c} = I(\boldsymbol{c}).$$

For any $j \in \mathcal{J}_n(\delta)$ we have

$$\big\langle \lambda_*, Y_n(x^n_j)\big\rangle \geq L_n\big(\langle \lambda_*, \boldsymbol{c}\rangle - \delta\|\lambda_*\|_1\big).$$

Therefore, by exponential Chebyshev,

$$\begin{aligned}\mathbb{P}\big(j \in \mathcal{J}_n(\delta)\big) &\leq \exp\Big(-L_n\big(\langle \lambda_*, \boldsymbol{c}\rangle - \delta\|\lambda_*\|_1\big)\Big)\mathbb{E}\Big[e^{\langle \lambda_*, Y_n(x^n_j)\rangle}\Big] \\ &\leq C_{\lambda_*}\, r_n^{I(\boldsymbol{c}) - \delta\|\lambda_*\|_1},\end{aligned}$$

with a constant independent of n and j. Since $|\mathcal{J}_n| \asymp \ell_n^{-d} \asymp r_n^{-d}$, we obtain

$$\mathbb{E}\Big[\sum_{j \in \mathcal{J}_n(\delta)} (\mathrm{diam} I^n_j)^s\Big] \leq C_{\delta,s}\, r_n^{s-d+I(\boldsymbol{c})-\delta\|\lambda_*\|_1}.$$

Choose $s > d - I(\boldsymbol{c})$ and then choose $\delta > 0$ small enough that

$$s - d + I(\boldsymbol{c}) - \delta\|\lambda_*\|_1 > 0.$$

The series in n is summable. Hence, by Tonelli and Borel–Cantelli, the s-dimensional Hausdorff content of the limsup cover in (5.5) is finite almost surely. Thus

$$\dim_{\mathrm{H}}(E(\boldsymbol{h}) \cap Q) \leq s.$$

Since $s > d - I(\boldsymbol{c})$ was arbitrary, we get

$$\dim_{\mathrm{H}}(E(\boldsymbol{h}) \cap Q) \leq d - I(\boldsymbol{c}).$$

If moreover $I(\boldsymbol{c}) > d$, choose $s = 0$ and then choose $\delta > 0$ so small that

$$-d + I(\boldsymbol{c}) - \delta\|\lambda_*\|_1 > 0.$$

Then

$$\sum_n \mathbb{E}[\#\mathcal{J}_n(\delta)] < \infty,$$

so almost surely only finitely many δ-good cubes occur. By (5.5), this implies $E(\boldsymbol{h}) \cap Q = \varnothing$.

Case 2: $\boldsymbol{c} \notin \mathrm{Ran}(G)$. Since G is symmetric, $\mathrm{Ran}(G) = \mathrm{Ker}(G)^\perp$. Thus there exists $\lambda_0 \in \mathrm{Ker}(G)$ such that $\langle \lambda_0, \boldsymbol{c}\rangle > 0$. Fix $\delta > 0$ so small that

$$\langle \lambda_0, \boldsymbol{c}\rangle - \delta\|\lambda_0\|_1 > 0.$$

For any $m > 0$, applying the exponential Chebyshev bound with $m\lambda_0$ yields

$$\mathbb{P}\big(j \in \mathcal{J}_n(\delta)\big) \leq C_{m,\lambda_0}\, r_n^{m(\langle \lambda_0, \boldsymbol{c}\rangle - \delta\|\lambda_0\|_1)}.$$

Since the exponent can be made arbitrarily large by choosing m large, the expected number of δ-good cubes is summable in n. Hence only finitely many such cubes occur almost surely, and (5.5) implies $E(\boldsymbol{h}) \cap Q = \varnothing$. ☐

Proof of Theorem 5.2. If $I(\boldsymbol{c}(\boldsymbol{h})) < d$, combine Propositions 5.3 and 5.4. If $I(\boldsymbol{c}(\boldsymbol{h})) = d$, Proposition 5.4 gives $\dim_{\mathrm{H}} E(\boldsymbol{h}) \leq 0$. If $I(\boldsymbol{c}(\boldsymbol{h})) > d$ or $I(\boldsymbol{c}(\boldsymbol{h})) = +\infty$, Proposition 5.4 gives $E(\boldsymbol{h}) = \varnothing$. ☐

6. Application to Random Matrices: C$\boldsymbol{\beta}$E and a joint thick-point spectrum

This section connects Theorem 5.2 to random matrix theory. For the Poisson regularization of the full two-component boundary field on $\mathbb{T}$, the subcritical convergence, local moment bounds, and oscillation input **(R1)** can be treated directly on the circle. For the C$\boldsymbol{\beta}$E application below, the upper bound is derived directly from independence and local moment estimates, and therefore does not use any spatially uniform cut-off stability input. The only additional circle-specific input retained for the matching lower bound is a circle-cross local dimension statement along the minimizing chaos measure. We will use the local-chart extensions stated below to pass from bounded intervals in $\mathbb{R}$ to the circle $\mathbb{T}$.

6.1. **Circular $\boldsymbol{\beta}$-ensemble and two observables.** Let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. For $\boldsymbol{\beta} > 0$ and $n \in \mathbb{N}$, the circular β-ensemble C$\boldsymbol{\beta}$E$_n$ is the law on $\mathbb{T}^n$ with density proportional to

$$\prod_{1\le k<j\le n} |e^{i\theta_k} - e^{i\theta_j}|^{\boldsymbol{\beta}} \prod_{k=1}^{n} \frac{d\theta_k}{2\pi}.$$

Define the characteristic polynomial

$$X_n(z) := \prod_{j=1}^{n} (1 - ze^{-i\theta_j}), \qquad z \in \mathbb{D},$$

and let $Y_n(\theta)$ denote the centered counting/argument-type field from [5].

6.2. **Poisson regularization and microscopic scales.** Assuming the coupled GAF φ of Proposition 6.1, let $\rho \uparrow 1$ and set $\varepsilon := 1 - \rho$. For notational compatibility with the Euclidean sections, we occasionally write

$$X_\varepsilon^{(t)} := X_{1-\varepsilon}^{(t)}, \qquad M_{t,\varepsilon} := M_{t,1-\varepsilon},$$

for the Poisson-regularized field and chaos measure at boundary scale ε. Then the Poisson-regularized boundary fields

$$\chi_\rho(\theta) = \Re\varphi(\rho e^{i\theta}), \qquad \psi_\rho(\theta) = \Im\varphi(\rho e^{i\theta})$$

have correlation length of order ε along the boundary, and their covariance kernels satisfy

$$\mathrm{Cov}(\chi_\rho(\theta), \chi_\rho(\vartheta)) = \boldsymbol{\beta}^{-1} \log \frac{1}{|\theta - \vartheta|_{\mathbb{T}} + \varepsilon} + O(1), \qquad \mathrm{Cov}(\psi_\rho(\theta), \psi_\rho(\vartheta)) = \boldsymbol{\beta}^{-1} \log \frac{1}{|\theta - \vartheta|_{\mathbb{T}} + \varepsilon} + O(1),$$

uniformly for $\theta, \vartheta \in \mathbb{T}$ and $\varepsilon \in (0, 1/2]$. In particular, when studying local masses on arcs $B(\theta, r)$ with $r \downarrow 0$, it is natural to take $\varepsilon \asymp r$ (i.e. $\rho = 1 - \Theta(r)$), matching the cut-off-at-scale-r framework of Sections 4 and 5.

6.3. **A subcritical chaos limit theorem for CβE.**

Proposition 6.1 (Lambert–Najnudel: two subcritical chaos limits). *Fix $\boldsymbol{\beta} > 0$ and let $(\theta_1^{(n)}, \dots, \theta_n^{(n)})$ have the C$\boldsymbol{\beta}$E$_n$ law on $\mathbb{T}$. Then there exists a coupling and a complex Gaussian analytic function φ on $\mathbb{D}$ such that*

$$X_n(\cdot) \longrightarrow e^{\varphi(\cdot)} \qquad \textit{locally uniformly on } \mathbb{D}$$

almost surely. Moreover, the boundary fields

$$\chi(e^{i\theta}) = \lim_{\rho\uparrow 1} \Re\varphi(\rho e^{i\theta}), \qquad \psi(e^{i\theta}) = \lim_{\rho\uparrow 1} \Im\varphi(\rho e^{i\theta})$$

exist as generalized functions, are independent, and satisfy

$$\mathrm{Cov}(\chi(e^{i\theta}), \chi(e^{i\vartheta})) = \boldsymbol{\beta}^{-1} \log |e^{i\theta} - e^{i\vartheta}|^{-1} + O(1), \qquad \mathrm{Cov}(\psi(e^{i\theta}), \psi(e^{i\vartheta})) = \boldsymbol{\beta}^{-1} \log |e^{i\theta} - e^{i\vartheta}|^{-1} + O(1),$$

with zero cross-covariance.

Fix $\gamma \in \mathbb{R}$ and set $\widehat{\gamma} := \gamma/\sqrt{2\boldsymbol{\beta}}$. Assume $|\widehat{\gamma}| < 1$. Define

$$\mu_{n,\gamma}(d\theta) := \frac{|X_n(e^{i\theta})|^\gamma}{\mathbb{E}|X_n(1)|^\gamma} \frac{d\theta}{2\pi}, \qquad \nu_{n,\gamma}(d\theta) := \frac{e^{\gamma Y_n(\theta)}}{\mathbb{E}e^{\gamma Y_n(0)}} \frac{d\theta}{2\pi}.$$

Then, as $n \to \infty$, the following hold:

(a) If $\gamma > -1$, then $\mu_{n,\gamma} \Rightarrow \mu^{\widehat{\gamma}}$ in probability for weak convergence of Radon measures on $\mathbb{T}$.

(b) For every $\gamma \in \mathbb{R}$ with $|\widehat{\gamma}| < 1$, one has $\nu_{n,\gamma} \Rightarrow \nu^{\widehat{\gamma}}$ in probability for weak convergence of Radon measures on $\mathbb{T}$.

Here $\mu^{\widehat{\gamma}}$ and $\nu^{\widehat{\gamma}}$ are the subcritical GMC measures driven respectively by χ and ψ.

Proof. This is exactly Proposition 2, Proposition 3, and Theorem 1 of [5], rewritten in our notation. The condition $\gamma > -1$ appears only on the characteristic-polynomial side and reflects the local integrability of $|X_n(e^{i\theta})|^\gamma$ near eigenangles. $\square$

Remark 6.2 (On the condition $\gamma > -1$)**.** Near an eigenangle θ_j one has $|X_n(e^{i\theta})| \asymp |\theta - \theta_j|$, hence $|X_n(e^{i\theta})|^\gamma$ is locally integrable on $\mathbb{T}$ if and only if $\gamma > -1$.

Lemma 6.3 (Normalization dictionary for C$\boldsymbol{\beta}$E GMC parameters)**.** *Let $\boldsymbol{\beta} > 0$ and assume that the boundary fields χ, ψ satisfy the covariance structure of Proposition 6.1. Let $\gamma \in \mathbb{R}$ and set $\widehat{\gamma} := \gamma/\sqrt{2\boldsymbol{\beta}}$. Then, in the notation of Theorems 4.1 and 5.2:*

(i) For the 2-component field $X = (\chi, \psi)$ one has

$$\Sigma = \operatorname{diag}(\boldsymbol{\beta}^{-1}, \boldsymbol{\beta}^{-1}).$$

(ii) For the parameter vector $t^{(1)} = (\gamma, 0)$ corresponding to $\mu^{\widehat{\gamma}}$,

$$\left\|t^{(1)}\right\|_\Sigma^2 = \frac{\gamma^2}{\boldsymbol{\beta}} = 2\widehat{\gamma}^{\,2}.$$

(iii) The subcriticality condition $\left\|t^{(1)}\right\|_\Sigma^2 < 2d$ with $d = 1$ is equivalent to $|\widehat{\gamma}| < 1$.

(iv) For $q > 0$ with $q|\widehat{\gamma}| < 1$, the exponent level singled out by the q-parameterization is

$$h(q) = 1 + \left(\frac{1}{2} - q\right)\frac{\gamma^2}{\boldsymbol{\beta}} = 1 + (1 - 2q)\widehat{\gamma}^{\,2}.$$

Proof. Since $|e^{i\theta} - e^{i\vartheta}| \asymp |\theta - \vartheta|_{\mathbb{T}}$ for nearby points on the circle, the covariance coefficient in the logarithmic singularity is unchanged when passing from chord distance to angular distance. Thus Proposition 6.1 gives $\Sigma = \operatorname{diag}(\boldsymbol{\beta}^{-1}, \boldsymbol{\beta}^{-1})$. For $t^{(1)} = (\gamma, 0)$,

$$\left\|t^{(1)}\right\|_\Sigma^2 = (\gamma, 0)\Sigma(\gamma, 0)^\mathsf{T} = \gamma^2/\boldsymbol{\beta} = 2\widehat{\gamma}^{\,2}.$$

This immediately yields the subcriticality condition. The formula for $h(q)$ is the specialization of

$$h(q) = d + \left(\frac{1}{2} - q\right)\left\|t^{(1)}\right\|_\Sigma^2$$

to $d = 1$ and $\left\|t^{(1)}\right\|_\Sigma^2 = 2\widehat{\gamma}^{\,2}$. $\square$

Proposition 6.4 (Subcritical GMC on $\mathbb{T}$ under Poisson regularization)**.** *Let X be a centered Gaussian generalized field on $\mathbb{T}$ with covariance*

$$\operatorname{Cov}(X(\theta), X(\vartheta)) = \sigma^2 \log |e^{i\theta} - e^{i\vartheta}|^{-1} + g(\theta, \vartheta),$$

where g is bounded and continuous. Let X_ρ be the Poisson regularization of X. If $\sigma^2 < 2$, then the associated subcritical chaos measures

$$M_\rho(d\theta) := \exp\left(X_\rho(\theta) - \tfrac{1}{2}\operatorname{Var}(X_\rho(\theta))\right)\frac{d\theta}{2\pi}$$

converge in probability, for weak convergence of Radon measures on $\mathbb{T}$, to a non-trivial limit measure M.

Proof. This is standard subcritical GMC on the circle with Poisson kernel regularization; see, for example, [9, Sections 2–3], [10], and the circle-specific discussion in [2]. $\square$

Proposition 6.5 (Local moment bounds on $\mathbb{T}$)**.** *Under the assumptions of Proposition 6.4, for every*

$$p \in (-\infty, 2/\sigma^2)$$

there exist $C < \infty$ and $r_0 > 0$ such that for all $\theta \in \mathbb{T}$ and all $r \in (0, r_0)$,

$$\mathbb{E}[M(B(\theta, r))^p] \le C\, r^{\xi(p)}, \qquad \xi(p) = \left(1 + \frac{\sigma^2}{2}\right)p - \frac{\sigma^2}{2}p^2.$$

Moreover, the same estimate holds for the Poisson-regularized measures M_ρ uniformly whenever $1 - \rho \asymp r$.

Proof. By local charts, the circle is locally bi-Lipschitz equivalent to bounded intervals in $\mathbb{R}$, and the covariance kernel becomes $\sigma^2 \log |x - y|^{-1} + O(1)$ in local coordinates. The one-dimensional log-correlated GMC moment bounds (including both positive and negative moments in the full range $p < 2/\sigma^2$) therefore apply on each chart, and a finite chart cover transfers them back to $\mathbb{T}$. See [9, Section 4] and [1]. $\square$

For every subcritical parameter direction $t \in \mathbb{R}^2$, we denote by M_t the circle GMC measure associated with the scalar field $X^{(t)}$.

Proposition 6.6 (Poisson regularization satisfies the oscillation input)**.** *Let (χ, ψ) be as in Proposition 6.1. For $t = (a, b) \in \mathbb{R}^2$, set*

$$X^{(t)} := a\chi + b\psi, \qquad X^{(t)}_\rho := a\chi_\rho + b\psi_\rho, \qquad \sigma_t^2 := \|t\|_\Sigma^2 = \frac{a^2+b^2}{\boldsymbol{\beta}}.$$

Then for every $\kappa > 0$ and along the dyadic sequence $\rho_n := 1 - 2^{-n}$,

$$\omega^{(t)}_{n,\kappa} := \sup_{|\theta-\vartheta|_{\mathbb{T}} \le \kappa 2^{-n}} \left| X^{(t)}_{\rho_n}(\theta) - X^{(t)}_{\rho_n}(\vartheta) \right| = o(n) \qquad \textit{almost surely.}$$

*In particular, the oscillation input **(R1)** holds for the Poisson regularization of every subcritical parameter direction t.*

Proof. Fix $\kappa > 0$ and write $\varepsilon_n := 2^{-n} = 1 - \rho_n$ and $q_n := \rho_n^2$.

Step 1: explicit Fourier representation. By [5, Remark 8], the GAF from Proposition 6.1 admits the representation

$$\varphi(z) = \sum_{k \ge 1} \frac{N_k}{\sqrt{k}} z^k, \qquad z \in \mathbb{D},$$

where $(N_k)_{k \ge 1}$ are i.i.d. centered complex Gaussian random variables such that

$$\mathbb{E}[N_k^2] = 0, \qquad \mathbb{E}[|N_k|^2] = \frac{2}{\boldsymbol{\beta}}.$$

Write $N_k = A_k + iB_k$, where (A_k, B_k) are i.i.d. centered real Gaussian vectors with

$$\mathrm{Var}(A_k) = \mathrm{Var}(B_k) = \frac{1}{\boldsymbol{\beta}}, \qquad \mathrm{Cov}(A_k, B_k) = 0.$$

Then, for $\rho \in (0, 1)$ and $\theta \in \mathbb{T}$,

$$\chi_\rho(\theta) = \sum_{k \ge 1} \frac{\rho^k}{\sqrt{k}} \big(A_k \cos(k\theta) - B_k \sin(k\theta) \big),$$

$$\psi_\rho(\theta) = \sum_{k \ge 1} \frac{\rho^k}{\sqrt{k}} \big(A_k \sin(k\theta) + B_k \cos(k\theta) \big).$$

Hence

$$X^{(t)}_\rho(\theta) = \sum_{k \ge 1} \frac{\rho^k}{\sqrt{k}} \Big(U_k \cos(k\theta) + V_k \sin(k\theta) \Big),$$

where

$$U_k := aA_k + bB_k, \qquad V_k := bA_k - aB_k.$$

For each k, the pair (U_k, V_k) is centered Gaussian, independent of (U_ℓ, V_ℓ) for $\ell \neq k$, and satisfies

$$\mathrm{Var}(U_k) = \mathrm{Var}(V_k) = \frac{a^2+b^2}{\boldsymbol{\beta}} = \sigma_t^2, \qquad \mathrm{Cov}(U_k, V_k) = 0.$$

Step 2: the scaled derivative process. Define

$$Z_n(\phi) := \varepsilon_n \, \partial_\phi X^{(t)}_{\rho_n}(\phi), \qquad \phi \in \mathbb{T}.$$

Differentiating term-by-term gives

$$Z_n(\phi) = \varepsilon_n \sum_{k \ge 1} \rho_n^k \sqrt{k} \Big(-U_k \sin(k\phi) + V_k \cos(k\phi) \Big).$$

Thus Z_n is a centered continuous Gaussian process with covariance

$$\mathrm{Cov}\big(Z_n(\phi), Z_n(\psi)\big) = \sigma_t^2 \, \varepsilon_n^2 \sum_{k \ge 1} k q_n^k \cos\big(k(\phi - \psi)\big). \tag{6.1}$$

In particular,

$$\sup_{n\geq 1}\sup_{\phi\in\mathbb{T}}\mathrm{Var}(Z_n(\phi)) = \sigma_t^2\sup_{n\geq 1}\varepsilon_n^2\sum_{k\geq 1}kq_n^k \leq C_t,$$

since

$$\sum_{k\geq 1}kq_n^k = \frac{q_n}{(1-q_n)^2} \quad \text{and} \quad 1-q_n = 1-(1-\varepsilon_n)^2 \asymp \varepsilon_n.$$

Step 3: canonical metric estimate. Let $\Delta := |\phi-\psi|_{\mathbb{T}} \in [0,\pi]$. By (6.1),

$$d_n(\phi,\psi)^2 := \mathbb{E}[(Z_n(\phi)-Z_n(\psi))^2] = 2\sigma_t^2\varepsilon_n^2\sum_{k\geq 1}kq_n^k\big(1-\cos(k\Delta)\big).$$

If $\Delta \geq \varepsilon_n$, then using $1-\cos(k\Delta)\leq 2$ gives

$$d_n(\phi,\psi)^2 \leq 4\sigma_t^2\varepsilon_n^2\sum_{k\geq 1}kq_n^k \leq C_t.$$

If $\Delta \leq \varepsilon_n$, then $1-\cos(k\Delta)\leq \frac{1}{2}k^2\Delta^2$, hence

$$d_n(\phi,\psi)^2 \leq \sigma_t^2\varepsilon_n^2\Delta^2\sum_{k\geq 1}k^3q_n^k.$$

Since

$$\sum_{k\geq 1}k^3q_n^k = \frac{q_n(1+4q_n+q_n^2)}{(1-q_n)^4} \leq C\varepsilon_n^{-4},$$

we get

$$d_n(\phi,\psi)^2 \leq C_t\frac{\Delta^2}{\varepsilon_n^2}.$$

Combining the two cases,

$$d_n(\phi,\psi)^2 \leq C_t\min\Big\{1,\frac{\Delta^2}{\varepsilon_n^2}\Big\}. \tag{6.2}$$

Step 4: entropy bound and Borell–TIS. By (6.2), a d_n-ball of radius $\eta\in(0,1]$ contains an angular arc of length at least $c_t\varepsilon_n\eta$, so

$$N(\mathbb{T},d_n,\eta) \leq \frac{C_t}{\varepsilon_n\eta} \qquad (\eta\in(0,1]).$$

Dudley's entropy bound yields

$$\mathbb{E}\Big[\sup_{\phi\in\mathbb{T}}|Z_n(\phi)-Z_n(0)|\Big] \leq C_t\int_0^1\sqrt{\log\Big(\frac{C_t}{\varepsilon_n\eta}\Big)}\,d\eta \leq C_t\sqrt{\log(1/\varepsilon_n)}.$$

Since $Z_n(0)$ is a centered Gaussian variable with variance uniformly bounded in n, we also have

$$\mathbb{E}|Z_n(0)| \leq C_t.$$

Hence

$$\mathbb{E}\Big[\sup_{\phi\in\mathbb{T}}|Z_n(\phi)|\Big] \leq \mathbb{E}|Z_n(0)| + \mathbb{E}\Big[\sup_{\phi\in\mathbb{T}}|Z_n(\phi)-Z_n(0)|\Big] \leq C_t\sqrt{n}.$$

Together with the uniform variance bound, Borell–TIS implies that for every $u>0$,

$$\mathbb{P}\Big(\sup_{\phi\in\mathbb{T}}|Z_n(\phi)| > C_t\sqrt{n}+u\Big) \leq 2e^{-u^2/C_t}.$$

Choosing $u=\sqrt{3C_tn\log 2}$ and summing in n, Borel–Cantelli gives

$$\sup_{\phi\in\mathbb{T}}|Z_n(\phi)| = O(\sqrt{n}) \qquad \text{almost surely.}$$

Step 5: conclude. Since $X_{\rho_n}^{(t)}$ is smooth, for $|\theta-\vartheta|_{\mathbb{T}}\leq\kappa\varepsilon_n$ we have

$$|X_{\rho_n}^{(t)}(\theta)-X_{\rho_n}^{(t)}(\vartheta)| \leq \kappa\varepsilon_n\sup_{\phi\in\mathbb{T}}|\partial_\phi X_{\rho_n}^{(t)}(\phi)| = \kappa\sup_{\phi\in\mathbb{T}}|Z_n(\phi)|.$$

Therefore

$$\omega_{n,\kappa}^{(t)} \leq \kappa\sup_{\phi\in\mathbb{T}}|Z_n(\phi)| = O(\sqrt{n}) = o(n) \qquad \text{almost surely,}$$

which is exactly **(R1)** along the dyadic scales. ☐

Remark 6.7 (A pointwise cut-off stability problem on $\mathbb{T}$)**.** For a fixed deterministic point $\theta \in \mathbb{T}$ and a subcritical parameter direction $t \in \mathbb{R}^2$, it is natural to expect a pointwise cut-off stability statement of the form

$$|\log M_t(B(\theta, r_n)) - \log M_{t,\rho_n}(B(\theta, r_n))| = o(n) \qquad \text{almost surely.}$$

This would follow from subpolynomial control of the normalized local masses

$$W_n(\theta) := r_n^{-1} \exp\Big(- X_{\rho_n}^{(t)}(\theta) + \tfrac{1}{2}\mathrm{Var}(X_{\rho_n}^{(t)}(\theta))\Big)\, M_t(B(\theta, r_n)).$$

Although such a statement is plausible and can be approached by the same Girsanov-moment mechanism as in Appendix A, we do not need it for the results of the present paper and therefore do not pursue it here.

Remark 6.8 (What remains unresolved on the circle)**.** The present paper does not attempt to establish a spatially uniform cut-off stability input on $\mathbb{T}$. For the random-matrix application below, this is harmless on the upper-bound side, since we bypass the slope reduction entirely by a direct mass-cover argument. The only additional circle-specific input retained for the matching lower bound is the rooted local ball factorization introduced below.

Remark 6.9 (Why the spatially uniform input is genuinely strong)**.** A naive attempt to prove a spatially uniform cut-off stability statement by taking an r_n-net of cardinality $\asymp r_n^{-1} = e^{L_n}$ and applying a union bound runs into the limited moment range of GMC. Indeed, even if one had a uniform estimate

$$\sup_{n\geq 1} \sup_{\theta \in \mathbb{T}} \mathbb{E}[W_n(\theta)^p] < \infty \qquad \text{for every } p < 2/\sigma_t^2,$$

one would only get

$$\mathbb{P}\Big(\max_j W_n(\theta_j^n) > e^{\eta L_n} \Big) \lesssim e^{(1-p\eta)L_n},$$

which is summable in n only if $p\eta > 1$. Since the available positive moments stop at $p < 2/\sigma_t^2$, this argument cannot prove the required subpolynomial control for arbitrarily small $\eta > 0$. In this precise sense, the spatially uniform version of **(R2)** is a genuinely nontrivial strengthening and should not be viewed as a routine consequence of pointwise arguments.

Corollary 6.10 (Local-chart extension to the circle for joint spectra)**.** *Assume that on $\mathbb{T}$, for the relevant subcritical parameter directions, the covariance kernels admit logarithmic asymptotics up to bounded error in local angular charts, with coefficients prescribed by Σ. Assume moreover that the analogues on $\mathbb{T}$ of the subcritical convergence in Definition 2.2, the moment input in Proposition 3.1, and the oscillation input **(R1)** are available. Assume in addition that the following two circle-specific inputs hold:*

(a) the circle-cross local dimension conclusion of Theorem 4.1 for the rooted/observed parameter pairs needed in the lower-bound argument;
(b) the analogue on $\mathbb{T}$ of Corollary A.3 for the parameter family entering the upper-bound argument.

*A sufficient (but possibly stronger than necessary) hypothesis for both (a) and (b) is the spatially uniform circle input **(R2)**. Then the proof of Theorem 5.2 carries over verbatim to $\mathbb{T}$ (with Euclidean balls replaced by arcs). If, moreover, the circle-cross local dimension conclusion in* (a) *holds for every subcritical pair (u, t), then the conclusion of Theorem 4.1 also holds on $\mathbb{T}$.*

Proof. The proof of Theorem 5.2 uses only local logarithmic covariance asymptotics, the subcritical convergence and moment inputs, the oscillation input **(R1)**, and the two circle-specific consequences listed above. In local angular charts on $\mathbb{T}$, chord distance and angular distance are bi-Lipschitz equivalent, so the same covering and rooted-measure arguments apply. The final sentence is immediate, since once the circle-cross local dimension conclusion is available for all subcritical pairs, the statement of Theorem 4.1 is exactly the same on $\mathbb{T}$. ☐

Proposition 6.11 (Rooted field slopes on the circle)**.** *Let $u, t \in \mathbb{R}^2$ be subcritical parameter vectors for the full two-component field on $\mathbb{T}$. Assume that for the pair (u, t) the Poisson-regularized covariances satisfy the logarithmic asymptotics*

$$\mathrm{Cov}\big(X_\varepsilon^{(t)}(\theta), X_{\varepsilon'}^{(u)}(\vartheta)\big) = \langle t, u\rangle_\Sigma \, \log \frac{1}{|\theta - \vartheta|_{\mathbb{T}} + \varepsilon \vee \varepsilon'} + O(1),$$

uniformly in $\theta, \vartheta \in \mathbb{T}$ and $\varepsilon, \varepsilon' \in (0, 1/2]$. Assume also that the analogue on $\mathbb{T}$ of the subcritical convergence in Definition 2.2 is available for the parameter u. Let

$$a := \langle t, u\rangle_\Sigma, \qquad \sigma_t^2 := \|t\|_\Sigma^2, \qquad L_n := \log(1/r_n), \qquad r_n := 2^{-n}, \qquad \rho_n := 1 - r_n.$$

Then almost surely, for M_u-a.e. $\theta \in \mathbb{T}$,

$$\lim_{n\to\infty} \frac{X^{(t)}_{\rho_n}(\theta)}{L_n} = a.$$

Proof. Fix $\delta > 0$ and define the random open sets

$$A_n^+ := \Big\{\theta \in \mathbb{T} : \ X^{(t)}_{\rho_n}(\theta) > (a + \delta)L_n\Big\}, \qquad A_n^- := \Big\{\theta \in \mathbb{T} : \ X^{(t)}_{\rho_n}(\theta) < (a - \delta)L_n\Big\}.$$

The openness follows from the continuity of $\theta \mapsto X^{(t)}_{\rho_n}(\theta)$.

We claim that there exist constants $c_\delta, C_\delta > 0$ such that for all sufficiently large n and all $\varepsilon' \in (0, r_n^2]$,

$$\mathbb{E}\Big[\int_{\mathbb{T}} \mathbf{1}_{A_n^\pm}(\theta)\, M_{u,\varepsilon'}(d\theta)\Big] \le C_\delta e^{-c_\delta L_n}. \tag{6.3}$$

Indeed, by Fubini and the definition of $M_{u,\varepsilon'}$,

$$\mathbb{E}\Big[\int_{\mathbb{T}} \mathbf{1}_{A_n^+}(\theta)\, M_{u,\varepsilon'}(d\theta)\Big] = \int_{\mathbb{T}} \mathbb{E}\Big[\mathbf{1}_{\{X^{(t)}_{\rho_n}(\theta) > (a+\delta)L_n\}} e^{X^{(u)}_{\varepsilon'}(\theta) - \frac{1}{2}\mathbb{E}[X^{(u)}_{\varepsilon'}(\theta)^2]}\Big]\, \frac{d\theta}{2\pi}.$$

The path-space Girsanov formula of Lemma 4.4 extends verbatim from compact cubes to the compact space $\mathbb{T}$ (the proof is identical with $C(K)$ replaced by $C(\mathbb{T})$), so the inner expectation equals

$$\mathbb{P}\Big(X^{(t)}_{\rho_n}(\theta) + m_{n,\varepsilon'}(\theta) > (a + \delta)L_n\Big),$$

where

$$m_{n,\varepsilon'}(\theta) := \mathrm{Cov}\big(X^{(t)}_{\rho_n}(\theta), X^{(u)}_{\varepsilon'}(\theta)\big) = aL_n + O(1)$$

uniformly in θ and $\varepsilon' \le r_n^2$, by the assumed covariance asymptotics. Likewise,

$$\mathrm{Var}\big(X^{(t)}_{\rho_n}(\theta)\big) = \sigma_t^2 L_n + O(1)$$

uniformly in θ. Gaussian tail estimates therefore imply

$$\mathbb{P}\Big(X^{(t)}_{\rho_n}(\theta) + m_{n,\varepsilon'}(\theta) > (a + \delta)L_n\Big) \le C_\delta e^{-c_\delta L_n},$$

uniformly in θ and $\varepsilon' \le r_n^2$, yielding (6.3) for A_n^+. The same argument gives the bound for A_n^-.

Now apply Lemma 4.8 on $\mathbb{T}$: choose a deterministic sequence $\varepsilon_k \downarrow 0$ such that $M_{u,\varepsilon_k} \Rightarrow M_u$ almost surely. Since $A_n^\pm$ are open,

$$\mathbb{E}[M_u(A_n^\pm)] \le \liminf_{k\to\infty} \mathbb{E}[M_{u,\varepsilon_k}(A_n^\pm)] \le C_\delta e^{-c_\delta L_n}.$$

Summing in n and using Borel–Cantelli yields

$$M_u\Big(\limsup_{n\to\infty}(A_n^+ \cup A_n^-)\Big) = 0 \qquad \text{almost surely.}$$

Equivalently, for M_u-a.e. θ and all large n,

$$(a - \delta)L_n \le X^{(t)}_{\rho_n}(\theta) \le (a + \delta)L_n.$$

Applying the argument with $\delta = 1/m$ for each $m \in \mathbb{N}$ and intersecting the resulting full-probability events, the conclusion follows. $\square$

Definition 6.12 (Rooted local ball factorization on the circle)**.** Let $u, t \in \mathbb{R}^2$ be subcritical parameter vectors. We say that *rooted local ball factorization* holds on $\mathbb{T}$ for the pair (u, t) if, with $r_n = 2^{-n}$, $\rho_n = 1 - r_n$, and $L_n = \log(1/r_n)$,

$$\log M_t(B(\theta, r_n)) = -L_n + X^{(t)}_{\rho_n}(\theta) - \frac{1}{2}\mathrm{Var}\big(X^{(t)}_{\rho_n}(\theta)\big) + o(L_n)$$

for M_u-a.e. $\theta \in \mathbb{T}$ almost surely.

Proposition 6.13 (Regularized rooted moment criterion for rooted local ball factorization)**.** *Let $u,t\in\mathbb{R}^2$ be subcritical parameter vectors and define, for $n\ge 1$ and $\theta\in\mathbb{T}$,*

$$W_n^{(t)}(\theta) := r_n^{-1}\exp\Big(-X_{\rho_n}^{(t)}(\theta)+\tfrac12\mathrm{Var}(X_{\rho_n}^{(t)}(\theta))\Big)\,M_t(B(\theta,r_n)).$$

Assume that there exist $p_+,p_->0$ and $C<\infty$ such that for all $n\ge1$ and all $\varepsilon'\in(0,r_n^2]$,

$$\mathbb{E}\Big[\int_{\mathbb{T}}\big(W_n^{(t)}(\theta)\big)^{p_+}\,M_{u,\varepsilon'}(d\theta)\Big]\le C,\qquad \mathbb{E}\Big[\int_{\mathbb{T}}\big(W_n^{(t)}(\theta)\big)^{-p_-}\,M_{u,\varepsilon'}(d\theta)\Big]\le C. \tag{6.4}$$

Then rooted local ball factorization holds on $\mathbb{T}$ for the pair (u,t) in the sense of Definition 6.12.

Proof. Fix $\eta>0$ and define the random open subsets of $\mathbb{T}$

$$A_n^+(\eta) := \Big\{\theta\in\mathbb{T}:\ W_n^{(t)}(\theta)>e^{\eta L_n}\Big\},\qquad A_n^-(\eta) := \Big\{\theta\in\mathbb{T}:\ W_n^{(t)}(\theta)<e^{-\eta L_n}\Big\}.$$

By Markov's inequality and (6.4), for all $\varepsilon'\in(0,r_n^2]$,

$$\mathbb{E}\big[M_{u,\varepsilon'}(A_n^+(\eta))\big]\le e^{-p_+\eta L_n}\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(t)}(\theta))^{p_+}M_{u,\varepsilon'}(d\theta)\Big]\le Ce^{-p_+\eta L_n},$$

and similarly

$$\mathbb{E}\big[M_{u,\varepsilon'}(A_n^-(\eta))\big]\le e^{-p_-\eta L_n}\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(t)}(\theta))^{-p_-}M_{u,\varepsilon'}(d\theta)\Big]\le Ce^{-p_-\eta L_n}.$$

Choose a deterministic sequence $\varepsilon_k\downarrow0$ such that $M_{u,\varepsilon_k}\Rightarrow M_u$ almost surely on $\mathbb{T}$. Since $A_n^{\pm}(\eta)$ are open, Lemma 4.8 gives

$$\mathbb{E}\big[M_u(A_n^{\pm}(\eta))\big]\le\liminf_{k\to\infty}\mathbb{E}\big[M_{u,\varepsilon_k}(A_n^{\pm}(\eta))\big]\le Ce^{-p_\pm\eta L_n}.$$

Because $L_n=n\log2$, both series $\sum_n e^{-p_+\eta L_n}$ and $\sum_n e^{-p_-\eta L_n}$ converge, hence

$$\sum_{n\ge1}\mathbb{E}\big[M_u(A_n^+(\eta))+M_u(A_n^-(\eta))\big]<\infty.$$

Therefore

$$\sum_{n\ge1}M_u(A_n^+(\eta))+M_u(A_n^-(\eta))<\infty\qquad\text{almost surely,}$$

so for M_u-a.e. θ and all sufficiently large n,

$$e^{-\eta L_n}\le W_n^{(t)}(\theta)\le e^{\eta L_n}.$$

Taking logarithms yields

$$\log M_t(B(\theta,r_n)) = -L_n+X_{\rho_n}^{(t)}(\theta)-\frac12\mathrm{Var}\big(X_{\rho_n}^{(t)}(\theta)\big)+O(\eta L_n)$$

for M_u-a.e. θ and all large n, almost surely. Applying this with $\eta=1/m$ for each $m\in\mathbb{N}$ and intersecting the resulting full-probability events yields rooted local ball factorization. ☐

Corollary 6.14 (Circle cross theorem from regularized rooted moments)**.** *Under the assumptions of Proposition 6.11, assume moreover that the analogues on $\mathbb{T}$ of the subcritical convergence in Definition 2.2, the moment input in Proposition 3.1, and the oscillation input **(R1)** are available for the relevant subcritical parameter directions. If the regularized rooted moment criterion* (6.4) *holds for the pair (u,t), then the circle-cross local dimension conclusion of Theorem 4.1 holds on $\mathbb{T}$ for this pair.*

Proof. Proposition 6.13 gives rooted local ball factorization, and Corollary 6.15 then yields the claimed circle-cross local dimension conclusion. ☐

Corollary 6.15 (Circle cross theorem under rooted local ball factorization)**.** *Assume that for the pair (u,t) on $\mathbb{T}$ the Poisson-regularized covariances satisfy the same logarithmic asymptotics up to bounded error as in Proposition 6.11. Assume moreover that the analogues on $\mathbb{T}$ of the subcritical convergence in Definition 2.2, the moment input in Proposition 3.1, and the oscillation input **(R1)** are available for the relevant subcritical parameter directions. If rooted local ball factorization holds on $\mathbb{T}$ for the pair (u,t) in the sense of Definition 6.12, then the circle-cross local dimension conclusion of Theorem 4.1 holds on $\mathbb{T}$ for this pair. In particular, if rooted local ball factorization holds for (u,u), then M_u is exact-dimensional on $\mathbb{T}$ with dimension $1-\frac12\|u\|_\Sigma^2$.*

Proof. By Proposition 6.11, for M_u-a.e. θ,

$$\frac{X^{(t)}_{\rho_n}(\theta)}{L_n} \to \langle t, u\rangle_\Sigma .$$

Also, the assumed covariance asymptotics imply

$$\frac{1}{L_n}\mathrm{Var}(X^{(t)}_{\rho_n}(\theta)) \to \|t\|^2_\Sigma$$

uniformly in θ. Dividing the rooted factorization identity by $\log r_n = -L_n$ yields, for M_u-a.e. θ,

$$\frac{\log M_t(B(\theta, r_n))}{\log r_n} = 1 - \frac{X^{(t)}_{\rho_n}(\theta)}{L_n} + \frac{1}{2}\frac{\mathrm{Var}(X^{(t)}_{\rho_n}(\theta))}{L_n} + o(1) \longrightarrow 1 + \frac{1}{2}\|t\|^2_\Sigma - \langle t, u\rangle_\Sigma .$$

Thus the desired exponent is obtained along the dyadic sequence $r_n = 2^{-n}$. To pass from dyadic radii to arbitrary radii, fix θ in the full-measure set where the above convergence holds and let $r \downarrow 0$. Choose $n = n(r)$ so that

$$r_{n+1} < r \le r_n.$$

Since

$$B(\theta, r_{n+1}) \subset B(\theta, r) \subset B(\theta, r_n),$$

monotonicity of the map $r \mapsto M_t(B(\theta, r))$ gives

$$M_t(B(\theta, r_{n+1})) \le M_t(B(\theta, r)) \le M_t(B(\theta, r_n)).$$

Because $\log r_n$ and $\log r_{n+1}$ differ by the constant $\log 2$, dividing by $\log r < 0$ and using the dyadic convergence yields

$$\frac{\log M_t(B(\theta, r))}{\log r} \longrightarrow 1 + \frac{1}{2}\|t\|^2_\Sigma - \langle t, u\rangle_\Sigma .$$

This is exactly the circle-cross local dimension conclusion. The case $t = u$ gives exact-dimensionality of M_u. □

Remark 6.16 (Minimal circle input for lower bounds)**.** For lower bounds on joint thick-point intersections on $\mathbb{T}$, one does not need a spatially uniform cut-off stability statement. The combination of Proposition 6.11 and Corollary 6.14 shows that the genuinely nontrivial input is a rooted moment control for the normalized local masses. In the independent two-component setting relevant to C$\boldsymbol{\beta}$E, Proposition 6.23 further reduces the lower bound to two one-component rooted moment estimates, together with exact-dimensionality of the auxiliary rooted measure M_{u_*}.

Remark 6.17 (A scalar rooted moment problem on the circle)**.** Let X be a stationary scalar log-correlated field on $\mathbb{T}$ with covariance coefficient $c_X > 0$, let M_λ be the corresponding subcritical chaos measure, and define

$$W^{(\lambda)}_n(\theta) := r_n^{-1}\exp\Big(-\lambda X_{\rho_n}(\theta) + \tfrac{1}{2}\lambda^2\mathrm{Var}(X_{\rho_n}(\theta))\Big) M_\lambda(B(\theta, r_n)).$$

A natural route to the circle lower bound is to establish the regularized rooted moment criterion of Proposition 6.13 for the scalar pairs needed in the C$\boldsymbol{\beta}$E application. The most naive Girsanov argument proceeds by rooting with $M_{u,\varepsilon'}$ and comparing the rooted expectation to the unrooted moments of $W^{(\lambda)}_n(0)$. The obstruction is that the covariance profile

$$h_{n,\varepsilon'}(\vartheta) := \mathrm{Cov}(X(\vartheta), X_{1-\varepsilon'}(0))$$

behaves like $c_X \log \frac{1}{|\vartheta|_{\mathbb{T}}+\varepsilon'}$, so when $\varepsilon' \ll r_n$ it is *not* uniformly equal to $c_X \log(1/r_n) + O(1)$ on $B(0, r_n)$. Hence the leading power cancellation between the rooted point factor and the shifted ball mass fails to be uniform in $\varepsilon' \le r_n^2$. The remaining scalar input can therefore be reformulated as a weighted local-mass problem, recorded next.

Proposition 6.18 (Weighted reduction of the scalar rooted input)**.** *Let X be a stationary scalar log-correlated field on $\mathbb{T}$ with covariance coefficient $c_X > 0$, let M_λ be the corresponding subcritical chaos measure, and fix a rooted parameter $u \in \mathbb{R}$. Set*

$$a := c_X \lambda u.$$

For $n \geq 1$ and $\kappa \in (0, r_n]$, define the weighted normalized local mass

$$\mathcal{I}_{n,\kappa}^{(u,\lambda)} := r_n^{-1+a} \exp\Big(- \lambda X_{\rho_n}(0) + \tfrac{1}{2}\lambda^2 \mathrm{Var}(X_{\rho_n}(0))\Big) \int_{B(0,r_n)} (|\vartheta|_{\mathbb{T}} + \kappa r_n)^{-a} \, M_\lambda(d\vartheta). \tag{6.5}$$

Assume that there exist $p_+, p_- > 0$ such that

$$\sup_{n\geq 1} \sup_{\kappa \in (0,r_n]} \mathbb{E}\big[(\mathcal{I}_{n,\kappa}^{(u,\lambda)})^{p_+}\big] < \infty, \qquad \sup_{n\geq 1} \sup_{\kappa \in (0,r_n]} \mathbb{E}\big[(\mathcal{I}_{n,\kappa}^{(u,\lambda)})^{-p_-}\big] < \infty. \tag{6.6}$$

Then the regularized rooted moment criterion of Proposition 6.13 holds for the pair (u, λ).

Proof. By stationarity, it suffices to root at the point $0 \in \mathbb{T}$. Fix $n \geq 1$ and $\varepsilon' \in (0, r_n^2]$, and write

$$\kappa := \varepsilon'/r_n \in (0, r_n].$$

For every bounded Borel function $F : \mathbb{R}_+ \to \mathbb{R}_+$, stationarity and Fubini give

$$\mathbb{E}\Big[\int_{\mathbb{T}} F(W_n^{(\lambda)}(\theta)) \, M_{u,\varepsilon'}(d\theta)\Big] = \mathbb{E}\Big[e^{uX_{\varepsilon'}(0) - \frac{1}{2}u^2 \mathrm{Var}(X_{\varepsilon'}(0))} F(W_n^{(\lambda)}(0))\Big].$$

We now apply Lemma 4.5 to the bounded functional $F(W_n^{(\lambda)}(0))$. Under the corresponding deterministic shift, the regularized field at scale r_n is shifted by

$$u \,\mathrm{Cov}(X_{\rho_n}(0), X_{\varepsilon'}(0)) = u \, c_X \log \frac{1}{r_n} + O(1),$$

uniformly in $\varepsilon' \leq r_n^2$, while the limiting chaos measure M_λ is multiplied by the deterministic weight

$$\vartheta \longmapsto \exp\big(\lambda u \, h_{n,\varepsilon'}(\vartheta)\big), \qquad h_{n,\varepsilon'}(\vartheta) := \lim_{\delta \downarrow 0} \mathrm{Cov}(X_\delta(\vartheta), X_{\varepsilon'}(0)).$$

By the logarithmic covariance asymptotics, for $|\vartheta|_{\mathbb{T}} < r_n$ we have

$$h_{n,\varepsilon'}(\vartheta) = c_X \log \frac{1}{|\vartheta|_{\mathbb{T}} + \varepsilon'} + O(1) = c_X \log \frac{1}{r_n} + c_X \log \frac{1}{|\vartheta|_{\mathbb{T}}/r_n + \kappa} + O(1),$$

with constants uniform in n and $\varepsilon' \leq r_n^2$. Therefore there exist deterministic constants $0 < c < C < \infty$, depending only on (u, λ, c_X), such that almost surely,

$$c\,\mathcal{I}_{n,\kappa}^{(u,\lambda)} \leq \widetilde{W}_{n,\varepsilon'}^{(u,\lambda)} \leq C\,\mathcal{I}_{n,\kappa}^{(u,\lambda)},$$

where $\widetilde{W}_{n,\varepsilon'}^{(u,\lambda)}$ denotes the shifted version of $W_n^{(\lambda)}(0)$.

To bound positive moments, fix $p > 0$ and $M > 0$, and define

$$F_{p,M}(x) := x^p \wedge M.$$

Then $F_{p,M}$ is bounded and Borel, so by the above identity and the pointwise comparison,

$$\mathbb{E}\Big[\int_{\mathbb{T}} F_{p,M}(W_n^{(\lambda)}(\theta)) \, M_{u,\varepsilon'}(d\theta)\Big] \leq C^p \, \mathbb{E}\big[(\mathcal{I}_{n,\kappa}^{(u,\lambda)})^p\big].$$

Letting $M \uparrow \infty$ and using monotone convergence yields

$$\mathbb{E}\Big[\int_{\mathbb{T}} (W_n^{(\lambda)}(\theta))^p \, M_{u,\varepsilon'}(d\theta)\Big] \leq C^p \, \mathbb{E}\big[(\mathcal{I}_{n,\kappa}^{(u,\lambda)})^p\big].$$

For negative moments, fix $p > 0$ and $M > 0$, and define

$$G_{p,M}(x) := x^{-p} \wedge M.$$

Again this is bounded and Borel. Since $\widetilde{W}_{n,\varepsilon'}^{(u,\lambda)} \geq c\mathcal{I}_{n,\kappa}^{(u,\lambda)}$, we have pointwise

$$G_{p,M}(\widetilde{W}_{n,\varepsilon'}^{(u,\lambda)}) \leq G_{p,M}(c\mathcal{I}_{n,\kappa}^{(u,\lambda)}) \leq c^{-p} (\mathcal{I}_{n,\kappa}^{(u,\lambda)})^{-p}.$$

Therefore

$$\mathbb{E}\Big[\int_{\mathbb{T}} G_{p,M}(W_n^{(\lambda)}(\theta)) \, M_{u,\varepsilon'}(d\theta)\Big] \leq c^{-p} \, \mathbb{E}\big[(\mathcal{I}_{n,\kappa}^{(u,\lambda)})^{-p}\big].$$

Letting $M \uparrow \infty$ and using monotone convergence gives

$$\mathbb{E}\Big[\int_{\mathbb{T}} (W_n^{(\lambda)}(\theta))^{-p} \, M_{u,\varepsilon'}(d\theta)\Big] \leq c^{-p} \, \mathbb{E}\big[(\mathcal{I}_{n,\kappa}^{(u,\lambda)})^{-p}\big].$$

Taking the supremum over n and $\varepsilon' \le r_n^2$, and using (6.6), proves the rooted moment criterion of Proposition 6.13. $\square$

Definition 6.19 (Microscopic rescaled rooted field family)**.** Let X be a stationary scalar log-correlated field on $\mathbb{T}$ with logarithmic covariance coefficient $c_X > 0$ and regularization $(X_\varepsilon)_{\varepsilon>0}$. For $n \ge 1$ and $\delta \in (0,1]$, define the rescaled centered field on $I := [-1,1]$ by

$$Y_{n,\delta}(x) := X_{\delta r_n}(r_n x) - X_{r_n}(0), \qquad x \in I.$$

For a fixed chaos parameter λ with $c_X\lambda^2 < 2$, let $\widehat{M}_{n,\delta}^{(\lambda)}$ denote the regularized GMC measure on I built from $Y_{n,\delta}$, and let $\widehat{M}_n^{(\lambda)}$ denote the corresponding subcritical GMC limit as $\delta \downarrow 0$.

Proposition 6.20 (Microscopic interval reduction)**.** *Let X, M_λ, u, $a = c_X\lambda u$, and $\mathcal{I}_{n,\kappa}^{(u,\lambda)}$ be as in Proposition 6.18. Then the following hold.*

(i) The rescaled centered family $(Y_{n,\delta})$ from Definition 6.19 is uniformly log-correlated on $I = [-1,1]$: there exists $C < \infty$ such that for all $n \ge 1$, all $\delta, \delta' \in (0,1]$, and all $x, y \in I$,

$$\operatorname{Cov}(Y_{n,\delta}(x), Y_{n,\delta'}(y)) = c_X \log \frac{1}{|x-y| + \delta \vee \delta'} + R_{n,\delta,\delta'}(x,y),$$

with $|R_{n,\delta,\delta'}(x,y)| \le C$.

(ii) For every $n \ge 1$ and every $\kappa \in (0, r_n]$, there exist deterministic constants $0 < c < C < \infty$, depending only on (c_X, λ, u) and the uniform covariance bounds of X, such that

$$c\int_{-1}^{1} (|x|+\kappa)^{-a}\, \widehat{M}_n^{(\lambda)}(dx) \le \mathcal{I}_{n,\kappa}^{(u,\lambda)} \le C\int_{-1}^{1} (|x|+\kappa)^{-a}\, \widehat{M}_n^{(\lambda)}(dx).$$

Proof. For (i), write

$$\begin{aligned}\operatorname{Cov}(Y_{n,\delta}(x), Y_{n,\delta'}(y)) &= \operatorname{Cov}(X_{\delta r_n}(r_n x), X_{\delta' r_n}(r_n y)) - \operatorname{Cov}(X_{\delta r_n}(r_n x), X_{r_n}(0)) \\ &\quad - \operatorname{Cov}(X_{r_n}(0), X_{\delta' r_n}(r_n y)) + \operatorname{Var}(X_{r_n}(0)).\end{aligned}$$

Using the logarithmic covariance bound on $\mathbb{T}$ and stationarity, we obtain

$$\begin{aligned}\operatorname{Cov}(X_{\delta r_n}(r_n x), X_{\delta' r_n}(r_n y)) &= c_X \log \frac{1}{r_n(|x-y| + \delta \vee \delta')} + O(1), \\ \operatorname{Cov}(X_{\delta r_n}(r_n x), X_{r_n}(0)) &= c_X \log \frac{1}{r_n(|x|+1)} + O(1), \\ \operatorname{Cov}(X_{r_n}(0), X_{\delta' r_n}(r_n y)) &= c_X \log \frac{1}{r_n(|y|+1)} + O(1), \\ \operatorname{Var}(X_{r_n}(0)) &= c_X \log \frac{1}{r_n} + O(1),\end{aligned}$$

with constants uniform in n, δ, δ', x, y. The $\log(1/r_n)$ terms cancel, and therefore

$$\operatorname{Cov}(Y_{n,\delta}(x), Y_{n,\delta'}(y)) = c_X \log \frac{1}{|x-y| + \delta \vee \delta'} + c_X \log(|x|+1) + c_X \log(|y|+1) + O(1).$$

Since $x, y \in [-1,1]$, the two extra logarithmic terms are uniformly bounded, proving (i).

For (ii), work first at the regularized level. Let $\widetilde{M}_{n,\delta}^{(\lambda)}$ be the GMC measure on I associated with the field $Y_{n,\delta}$, and let $\mu_{n,\delta}^{(\lambda)}$ denote the pushforward to I of the rescaled rooted circle chaos appearing in (6.5). A direct comparison of exponential weights gives, for $x \in I$,

$$\begin{aligned}&\lambda X_{\delta r_n}(r_n x) - \frac{\lambda^2}{2}\operatorname{Var}(X_{\delta r_n}(r_n x)) - \lambda X_{r_n}(0) + \frac{\lambda^2}{2}\operatorname{Var}(X_{r_n}(0)) \\ &\quad = \lambda Y_{n,\delta}(x) - \frac{\lambda^2}{2}\operatorname{Var}(Y_{n,\delta}(x)) + \lambda^2\big(\operatorname{Var}(X_{r_n}(0)) - \operatorname{Cov}(X_{\delta r_n}(r_n x), X_{r_n}(0))\big).\end{aligned}$$

By the same covariance estimate as above,

$$\operatorname{Var}(X_{r_n}(0)) - \operatorname{Cov}(X_{\delta r_n}(r_n x), X_{r_n}(0)) = c_X \log(|x|+1) + O(1),$$

uniformly in $n, \delta, x \in I$, hence the correction term is bounded uniformly. Therefore there exist deterministic constants $0 < c < C < \infty$, independent of (n, δ), such that for every nonnegative bounded Borel function φ on I,

$$c \int_I \varphi(x)\, \widetilde{M}^{(\lambda)}_{n,\delta}(dx) \le \int_I \varphi(x)\, \mu^{(\lambda)}_{n,\delta}(dx) \le C \int_I \varphi(x)\, \widetilde{M}^{(\lambda)}_{n,\delta}(dx).$$

Now fix n. By subcritical GMC convergence, there exists a deterministic sequence $\delta_m \downarrow 0$ such that both regularized measures converge weakly almost surely to their respective limiting measures $\widehat{M}^{(\lambda)}_n$ and $\mu^{(\lambda)}_n$. Since the above inequalities hold for every bounded continuous nonnegative φ, passing to the limit along this sequence yields

$$c \int_I \varphi(x)\, \widehat{M}^{(\lambda)}_n(dx) \le \int_I \varphi(x)\, \mu^{(\lambda)}_n(dx) \le C \int_I \varphi(x)\, \widehat{M}^{(\lambda)}_n(dx),$$

where $\mu^{(\lambda)}_n$ is the limiting pushforward of the rooted circle chaos. Taking

$$\varphi_\kappa(x) := (|x| + \kappa)^{-a}, \qquad \kappa \in (0, r_n],$$

which is bounded and continuous on I, yields (ii). $\square$

6.4. **Joint thick-point spectrum.** For $\theta \in \mathbb{T}$ and $r \in (0, \pi)$ define the arc

$$B(\theta, r) := \{\vartheta \in \mathbb{T} :\ |\vartheta - \theta|_{\mathbb{T}} < r\}.$$

For a Radon measure η on $\mathbb{T}$ and $h \ge 0$, define

$$E_\eta(h) := \Big\{\theta \in \mathbb{T} :\ \lim_{r \downarrow 0} \frac{\log \eta(B(\theta, r))}{\log r} = h\Big\}.$$

Proposition 6.21 (Direct upper bound for the C$\boldsymbol{\beta}$E limit pair)**.** *Fix $\boldsymbol{\beta} > 0$ and let $\gamma_1, \gamma_2 \in \mathbb{R}$ satisfy $|\widehat{\gamma}_i| < 1$, where $\widehat{\gamma}_i := \gamma_i / \sqrt{2\boldsymbol{\beta}}$. Let $\mu^{\widehat{\gamma}_1}$ and $\nu^{\widehat{\gamma}_2}$ be the independent limiting GMC measures from Proposition 6.1. Let $q_1, q_2 > 0$ satisfy $q_i |\widehat{\gamma}_i| < 1$ and set*

$$h_i := 1 + \Big(\frac{1}{2} - q_i\Big) \frac{\gamma_i^2}{\boldsymbol{\beta}} = 1 + (1 - 2q_i)\widehat{\gamma}_i^{\,2}, \qquad i = 1, 2.$$

Then almost surely,

$$\dim_{\mathrm{H}}\big(E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2)\big) \le \Big(1 - q_1^2 \widehat{\gamma}_1^{\,2} - q_2^2 \widehat{\gamma}_2^{\,2}\Big)_+.$$

Moreover, if $q_1^2 \widehat{\gamma}_1^2 + q_2^2 \widehat{\gamma}_2^2 > 1$, then

$$E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2) = \varnothing \qquad \textit{almost surely.}$$

Proof. Fix $\delta > 0$ and let $r_n := 2^{-n}$. Partition $\mathbb{T}$ into arcs I^n_j of length $r_n/8$ with centers θ^n_j, and define the enlarged arcs

$$A^n_j := B(\theta^n_j, r_n).$$

Since the number of such arcs is $N_n \asymp r_n^{-1}$, it suffices to estimate the expected s-dimensional cost of the random cover by those A^n_j for which both chaos masses are large.

Let

$$\mathcal{J}_n(\delta) := \Big\{j :\ \mu^{\widehat{\gamma}_1}(A^n_j) \ge r_n^{h_1 + \delta} \text{ and } \nu^{\widehat{\gamma}_2}(A^n_j) \ge r_n^{h_2 + \delta}\Big\}.$$

We claim that almost surely,

$$E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2) \subset \limsup_{n \to \infty} \bigcup_{j \in \mathcal{J}_n(\delta)} I^n_j. \tag{6.7}$$

Indeed, fix $\theta \in E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2)$ and let $j(n)$ be such that $\theta \in I^n_{j(n)}$. Then $|\theta - \theta^n_{j(n)}|_{\mathbb{T}} \le r_n/16$, so

$$B(\theta, r_n/2) \subset A^n_{j(n)}.$$

Since $\theta \in E_{\mu^{\widehat{\gamma}_1}}(h_1)$, for all large n,

$$\mu^{\widehat{\gamma}_1}(B(\theta, r_n/2)) \ge (r_n/2)^{h_1 + \delta/2} \ge r_n^{h_1 + \delta},$$

and similarly

$$\nu^{\widehat{\gamma}_2}(B(\theta, r_n/2)) \ge (r_n/2)^{h_2 + \delta/2} \ge r_n^{h_2 + \delta},$$

for all large n; hence $j(n) \in \mathcal{J}_n(\delta)$ eventually. This proves (6.7).

Now fix $s \geq 0$. By independence of $\mu^{\widehat{\gamma}_1}$ and $\nu^{\widehat{\gamma}_2}$ and Markov's inequality with exponents q_1, q_2,

$$\begin{aligned}\mathbb{P}\big(j \in \mathcal{J}_n(\delta)\big) &\leq r_n^{-q_1(h_1+\delta)-q_2(h_2+\delta)}\mathbb{E}\big[(\mu^{\widehat{\gamma}_1}(A_j^n))^{q_1}(\nu^{\widehat{\gamma}_2}(A_j^n))^{q_2}\big] \\ &= r_n^{-q_1(h_1+\delta)-q_2(h_2+\delta)}\mathbb{E}\big[(\mu^{\widehat{\gamma}_1}(A_j^n))^{q_1}\big]\mathbb{E}\big[(\nu^{\widehat{\gamma}_2}(A_j^n))^{q_2}\big].\end{aligned}$$

By Proposition 6.5 and Lemma 6.3,

$$\mathbb{E}\big[(\mu^{\widehat{\gamma}_i}(A_j^n))^{q_i}\big] \leq C_i r_n^{\xi_i(q_i)}, \qquad \xi_i(p) = (1+\widehat{\gamma}_i^{\,2})p - \widehat{\gamma}_i^{\,2}p^2.$$

Hence

$$\mathbb{P}\big(j \in \mathcal{J}_n(\delta)\big) \leq C r_n^{-q_1(h_1+\delta)-q_2(h_2+\delta)+\xi_1(q_1)+\xi_2(q_2)}.$$

Using the definition of h_i one computes

$$-q_i(h_i+\delta) + \xi_i(q_i) = q_i^2\widehat{\gamma}_i^{\,2} - q_i\delta,$$

so

$$\mathbb{P}\big(j \in \mathcal{J}_n(\delta)\big) \leq C r_n^{q_1^2\widehat{\gamma}_1^{\,2}+q_2^2\widehat{\gamma}_2^{\,2}-(q_1+q_2)\delta}.$$

Therefore,

$$\begin{aligned}\mathbb{E}\Big[\sum_{j\in\mathcal{J}_n(\delta)}(\mathrm{diam} I_j^n)^s\Big] &\leq C' r_n^s N_n r_n^{q_1^2\widehat{\gamma}_1^{\,2}+q_2^2\widehat{\gamma}_2^{\,2}-(q_1+q_2)\delta} \\ &\leq C'' r_n^{s-1+q_1^2\widehat{\gamma}_1^{\,2}+q_2^2\widehat{\gamma}_2^{\,2}-(q_1+q_2)\delta}.\end{aligned}$$

If

$$s > 1 - q_1^2\widehat{\gamma}_1^2 - q_2^2\widehat{\gamma}_2^2,$$

then for sufficiently small $\delta > 0$ the exponent is positive, and the series in n is summable. By Tonelli and Borel–Cantelli, the s-dimensional Hausdorff content of the limsup cover in (6.7) is finite almost surely. Hence

$$\dim_{\mathrm{H}}\big(E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2)\big) \leq 1 - q_1^2\widehat{\gamma}_1^{\,2} - q_2^2\widehat{\gamma}_2^{\,2}.$$

If $q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} > 1$, choose $s = 0$ and then $\delta > 0$ small enough that

$$-1 + q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} - (q_1+q_2)\delta > 0.$$

Then $\sum_n \mathbb{E}[\#\mathcal{J}_n(\delta)] < \infty$, so only finitely many good arcs occur almost surely. By (6.7), the intersection set is empty. ☐

Lemma 6.22 (Lifting scalar rooted moment criteria under an independent factor)**.** *Let χ and ψ be independent scalar log-correlated Gaussian fields on $\mathbb{T}$. Fix $\lambda \in \mathbb{R}$ and rooted parameters $u_1, u_2 \in \mathbb{R}$ in the subcritical range. Let η be the chaos measure built from χ with parameter λ, and define*

$$W_n^{(\lambda,\chi)}(\theta) := r_n^{-1}\exp\Big(-\lambda\chi_{\rho_n}(\theta) + \tfrac{1}{2}\lambda^2\mathrm{Var}(\chi_{\rho_n}(\theta))\Big)\,\eta(B(\theta, r_n)).$$

Let $M^{\chi}_{u_1,\varepsilon'}$ denote the regularized rooted chaos measure of χ with parameter u_1, and let

$$M_{(u_1,u_2),\varepsilon'}(d\theta) := \exp\Big(u_1\chi_{\varepsilon'}(\theta) - \tfrac{1}{2}u_1^2\mathrm{Var}(\chi_{\varepsilon'}(\theta)) + u_2\psi_{\varepsilon'}(\theta) - \tfrac{1}{2}u_2^2\mathrm{Var}(\psi_{\varepsilon'}(\theta))\Big)\,\frac{d\theta}{2\pi}.$$

Then for every $p > 0$, every $n \geq 1$, and every $\varepsilon' \in (0, r_n^2]$,

$$\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^p\,M_{(u_1,u_2),\varepsilon'}(d\theta)\Big] = \mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^p\,M^{\chi}_{u_1,\varepsilon'}(d\theta)\Big],$$

and similarly

$$\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^{-p}\,M_{(u_1,u_2),\varepsilon'}(d\theta)\Big] = \mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^{-p}\,M^{\chi}_{u_1,\varepsilon'}(d\theta)\Big].$$

Consequently, whenever the regularized rooted moment criterion (6.4) holds for the scalar pair $(\chi; u_1, \lambda)$, it also holds for the two-component rooted pair $((u_1,u_2), t)$, where $t = (\lambda, 0)$ in the notation of the full vector field. The analogous statement holds with ψ in place of χ.

Proof. Because $W_n^{(\lambda,\chi)}(\theta)$ depends only on (χ,θ), conditioning on (χ,θ) and using independence of ψ gives

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^{\pm p}\, M_{(u_1,u_2),\varepsilon'}(d\theta)\Big]\\
&\quad=\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^{\pm p}\exp\Big(u_1\chi_{\varepsilon'}(\theta)-\tfrac12 u_1^2\mathrm{Var}(\chi_{\varepsilon'}(\theta))\Big)\mathbb{E}\big[e^{u_2\psi_{\varepsilon'}(\theta)-\frac12 u_2^2\mathrm{Var}(\psi_{\varepsilon'}(\theta))}\big]\,\frac{d\theta}{2\pi}\Big]\\
&\quad=\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^{\pm p}\exp\Big(u_1\chi_{\varepsilon'}(\theta)-\tfrac12 u_1^2\mathrm{Var}(\chi_{\varepsilon'}(\theta))\Big)\,\frac{d\theta}{2\pi}\Big]\\
&\quad=\mathbb{E}\Big[\int_{\mathbb{T}}(W_n^{(\lambda,\chi)}(\theta))^{\pm p}\, M^{\chi}_{u_1,\varepsilon'}(d\theta)\Big],
\end{aligned}$$

using the normalization $\mathbb{E}[e^{u_2\psi_{\varepsilon'}(\theta)-\frac12 u_2^2\mathrm{Var}(\psi_{\varepsilon'}(\theta))}]=1$. The final claim follows immediately. $\square$

Proposition 6.23 (Conditional lower bound for the C$\boldsymbol{\beta}$E limit pair)**.** *Fix $\boldsymbol{\beta}>0$ and let $\gamma_1,\gamma_2\in\mathbb{R}$ satisfy $|\widehat{\gamma}_i|<1$, where $\widehat{\gamma}_i:=\gamma_i/\sqrt{2\boldsymbol{\beta}}$. Let $q_1,q_2>0$ satisfy $q_i|\widehat{\gamma}_i|<1$, and set*

$$t^{(1)}=(\gamma_1,0),\qquad t^{(2)}=(0,\gamma_2),\qquad u_*=(u_1,u_2):=(q_1\gamma_1,q_2\gamma_2)\in\mathbb{R}^2.$$

Assume

$$q_1^2\widehat{\gamma}_1^{\,2}+q_2^2\widehat{\gamma}_2^{\,2}\in(0,1).$$

Assume moreover the following scalar weighted inputs:

(a) the weighted moment input (6.6) *of Proposition 6.18 holds for the stationary scalar field $Z:=u_1\chi+u_2\psi$ with chaos parameter $\lambda=1$ and rooted parameter $u=1$;*
(b) the same weighted moment input holds for the scalar field χ with chaos parameter $\lambda=\gamma_1$ and rooted parameter $u=u_1$;
(c) the same weighted moment input holds for the scalar field ψ with chaos parameter $\lambda=\gamma_2$ and rooted parameter $u=u_2$.

Then almost surely,

$$\dim_{\mathrm{H}}\big(E_{\mu^{\widehat{\gamma}_1}}(h_1)\cap E_{\nu^{\widehat{\gamma}_2}}(h_2)\big)\ge 1-q_1^2\widehat{\gamma}_1^{\,2}-q_2^2\widehat{\gamma}_2^{\,2},$$

where $h_i=1+(1-2q_i)\widehat{\gamma}_i^{\,2}$.

Proof. By Proposition 6.1, the measures $\mu^{\widehat{\gamma}_1}$ and $\nu^{\widehat{\gamma}_2}$ are subcritical GMC measures on $\mathbb{T}$ driven by the independent boundary fields χ and ψ.

Step 1: exact-dimensionality of the auxiliary rooted measure. Set $Z:=u_1\chi+u_2\psi$. Then Z is a stationary scalar log-correlated Gaussian field on $\mathbb{T}$ with covariance coefficient

$$c_Z=\boldsymbol{\beta}^{-1}(u_1^2+u_2^2)=2\big(q_1^2\widehat{\gamma}_1^{\,2}+q_2^2\widehat{\gamma}_2^{\,2}\big)<2.$$

The chaos measure associated with Z and parameter $\lambda=1$ is exactly M_{u_*}. Assumption (a) and Proposition 6.18 imply the regularized rooted moment criterion for the scalar pair $(u,\lambda)=(1,1)$. Corollary 6.14 therefore yields the circle-cross local dimension conclusion for this scalar pair, hence M_{u_*} is exact-dimensional on $\mathbb{T}$ with dimension

$$1-\frac12 c_Z=1-q_1^2\widehat{\gamma}_1^{\,2}-q_2^2\widehat{\gamma}_2^{\,2}.$$

Step 2: rooted local dimensions for the two observables. Assumption (b) and Proposition 6.18 imply the regularized rooted moment criterion for the scalar pair $(\chi;u_1,\gamma_1)$. By Lemma 6.22, the same criterion holds for the two-component rooted pair $(u_*,t^{(1)})$, where $t^{(1)}=(\gamma_1,0)$. Corollary 6.14 therefore gives, for M_{u_*}-a.e. θ,

$$d_{\mu^{\widehat{\gamma}_1}}(\theta)=1+\frac{\gamma_1^2}{2\boldsymbol{\beta}}-\frac{u_1\gamma_1}{\boldsymbol{\beta}}=h_1.$$

Likewise, assumption (c) and Proposition 6.18 imply the regularized rooted moment criterion for the scalar pair $(\psi;u_2,\gamma_2)$. By Lemma 6.22, the same criterion holds for the two-component rooted pair $(u_*,t^{(2)})$, where $t^{(2)}=(0,\gamma_2)$. Corollary 6.14 therefore gives

$$d_{\nu^{\widehat{\gamma}_2}}(\theta)=1+\frac{\gamma_2^2}{2\boldsymbol{\beta}}-\frac{u_2\gamma_2}{\boldsymbol{\beta}}=h_2$$

for M_{u_*}-a.e. θ. Therefore

$$M_{u_*}\big(E_{\mu\widehat{\gamma}_1}(h_1) \cap E_{\nu\widehat{\gamma}_2}(h_2)\big) = M_{u_*}(\mathbb{T}) \qquad \text{almost surely.}$$

Step 3: conclude by the mass distribution principle. Since M_{u_*} is exact-dimensional with dimension

$$1 - q_1^2\widehat{\gamma}_1^{\,2} - q_2^2\widehat{\gamma}_2^{\,2},$$

any Borel set of full M_{u_*}-measure has Hausdorff dimension at least this value. This gives the claimed lower bound. $\square$

Proposition 6.24 (The C$\boldsymbol{\beta}$E lower-bound parameters lie in the one-point Seiberg window)**.** *Under the assumptions of Proposition 6.23, set*

$$S := q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} \in (0,1).$$

Then each of the scalar weighted inputs (a)–(c) *in Proposition 6.23 lies in the natural one-point insertion regime*

$$a < 1 + \frac{1}{2}c_X\lambda^2,$$

where c_X is the logarithmic covariance coefficient of the corresponding scalar field, λ is the chaos parameter, and $a = c_X\lambda u$ is the associated rooted weight. More precisely:

(i) for the scalar field $Z = u_1\chi + u_2\psi$ with chaos parameter $\lambda = 1$ and rooted parameter $u = 1$, one has

$$c_Z = 2S, \qquad a_Z = c_Z, \qquad a_Z < 1 + \frac{1}{2}c_Z;$$

(ii) for the scalar field χ with chaos parameter $\lambda = \gamma_1$ and rooted parameter $u = u_1 = q_1\gamma_1$, one has

$$c_\chi = \boldsymbol{\beta}^{-1}, \qquad a_\chi = \frac{q_1\gamma_1^2}{\boldsymbol{\beta}} = 2q_1\widehat{\gamma}_1^{\,2}, \qquad a_\chi < 1 + \widehat{\gamma}_1^{\,2} = 1 + \frac{1}{2}c_\chi\gamma_1^2;$$

(iii) for the scalar field ψ with chaos parameter $\lambda = \gamma_2$ and rooted parameter $u = u_2 = q_2\gamma_2$, one has

$$c_\psi = \boldsymbol{\beta}^{-1}, \qquad a_\psi = \frac{q_2\gamma_2^2}{\boldsymbol{\beta}} = 2q_2\widehat{\gamma}_2^{\,2}, \qquad a_\psi < 1 + \widehat{\gamma}_2^{\,2} = 1 + \frac{1}{2}c_\psi\gamma_2^2.$$

Proof. For (i), Proposition 6.23 gives $c_Z = 2S$ with $S < 1$, hence

$$a_Z = c_Z = 2S < 1 + S = 1 + \frac{1}{2}c_Z.$$

For (ii), using $u_1 = q_1\gamma_1$ and $c_\chi = \boldsymbol{\beta}^{-1}$,

$$a_\chi = c_\chi\gamma_1 u_1 = \frac{q_1\gamma_1^2}{\boldsymbol{\beta}} = 2q_1\widehat{\gamma}_1^{\,2}.$$

Since $q_1|\widehat{\gamma}_1| < 1$ and $|\widehat{\gamma}_1| < 1$,

$$a_\chi = 2|\widehat{\gamma}_1|(q_1|\widehat{\gamma}_1|) < 2|\widehat{\gamma}_1| < 1 + \widehat{\gamma}_1^{\,2},$$

where the last inequality is equivalent to $(1 - |\widehat{\gamma}_1|)^2 > 0$. The proof of (iii) is identical. $\square$

Remark 6.25 (Route B at the interval level)**.** Proposition 6.24 shows that the three scalar weighted inputs required in Proposition 6.23 lie in the natural one-point insertion window predicted by one-dimensional GMC with a boundary singularity. Proposition 6.20 reduces the corresponding circle weighted local masses to a family of microscopic interval GMC integrals with uniformly bounded covariance perturbations. The propositions below show that these microscopic integrals can be compared directly, in positive and negative moments, to the canonical limiting one-point insertion family on $[0,1]$. Combined with Proposition 6.28, this closes route B and removes the remaining conditional input from the exact lower bound.

Definition 6.26 (Canonical interval insertion integral)**.** Fix a logarithmic covariance coefficient $c_X > 0$, a chaos parameter $\lambda \in \mathbb{R}$ with $c_X\lambda^2 < 2$, and a rooted weight $a \in \mathbb{R}$. Let $\widetilde{X}$ be any scalar log-correlated Gaussian field on $[0,1]$ with covariance

$$\mathrm{Cov}(\widetilde{X}(x), \widetilde{X}(y)) = c_X \log\frac{1}{|x-y|} + g(x,y),$$

where g is bounded and continuous, and let $\widetilde{M}_\lambda$ be the corresponding subcritical GMC measure. For $\kappa \in (0,1]$, define the one-point insertion integral

$$\mathcal{J}_\kappa^{(a,\lambda)} := \int_0^1 (x+\kappa)^{-a}\, \widetilde{M}_\lambda(dx).$$

Lemma 6.27 (Bounded perturbation transfer for insertion moments)**.** *Let $X^{(1)}$ and $X^{(2)}$ be scalar log-correlated Gaussian fields on $[0,1]$ with the same logarithmic covariance coefficient $c_X > 0$ and covariance kernels*

$$\operatorname{Cov}(X^{(i)}(x), X^{(i)}(y)) = c_X \log \frac{1}{|x-y|} + g_i(x,y), \qquad i = 1,2,$$

where g_1, g_2 are bounded and continuous on $[0,1]^2$. Fix $\lambda \in \mathbb{R}$ with $c_X\lambda^2 < 2$ and $a \geq 0$. Let $M_\lambda^{(1)}$ and $M_\lambda^{(2)}$ be the corresponding subcritical GMC measures, and define

$$\mathcal{J}_\kappa^{(i)} := \int_0^1 (x+\kappa)^{-a} M_\lambda^{(i)}(dx), \qquad \kappa \in (0,1],\ i = 1,2.$$

Then:

(i) for every $q \in (0,1)$ there exists $C_q < \infty$ such that

$$\sup_{\kappa\in(0,1]} \mathbb{E}[(\mathcal{J}_\kappa^{(1)})^q] \leq C_q \sup_{\kappa\in(0,1]} \mathbb{E}[(\mathcal{J}_\kappa^{(2)})^q],$$

and the same inequality holds with the roles of 1 and 2 reversed;

(ii) for every $p > 0$ there exists $C_p < \infty$ such that

$$\sup_{\kappa\in(0,1]} \mathbb{E}[(\mathcal{J}_\kappa^{(1)})^{-p}] \leq C_p \sup_{\kappa\in(0,1]} \mathbb{E}[(\mathcal{J}_\kappa^{(2)})^{-p}],$$

and again the same inequality holds with the roles of 1 and 2 reversed.

Proof. Choose standard regularizations $X_\delta^{(i)}$ such that

$$\operatorname{Cov}(X_\delta^{(i)}(x), X_\delta^{(i)}(y)) = c_X \log \frac{1}{|x-y|+\delta} + R_\delta^{(i)}(x,y),$$

with $R_\delta^{(i)}$ uniformly bounded on $[0,1]^2$, uniformly in $\delta \in (0,1]$. Set

$$\mathcal{J}_{\delta,\kappa}^{(i)} := \int_0^1 (x+\kappa)^{-a} \exp\Big(\lambda X_\delta^{(i)}(x) - \tfrac{1}{2}\lambda^2 \operatorname{Var}(X_\delta^{(i)}(x))\Big) dx.$$

Because the regularized covariance kernels differ by a uniformly bounded additive term, Kahane's convexity inequality applies with constants independent of δ and κ.

For the positive moments, fix $q \in (0,1)$ and $M > 0$, and let

$$F_{q,M}(x) := (x \wedge M)^q.$$

Then $F_{q,M}$ is bounded, continuous, and concave on $\mathbb{R}_+$, so for every δ, κ,

$$\mathbb{E}\big[F_{q,M}(\mathcal{J}_{\delta,\kappa}^{(1)})\big] \leq C_q\, \mathbb{E}\big[F_{q,M}(\mathcal{J}_{\delta,\kappa}^{(2)})\big].$$

For fixed $\kappa > 0$, the weight $x \mapsto (x+\kappa)^{-a}$ is bounded and continuous on $[0,1]$, hence subcritical chaos convergence implies

$$\mathcal{J}_{\delta,\kappa}^{(i)} \to \mathcal{J}_\kappa^{(i)} \qquad \text{in probability as } \delta \downarrow 0,$$

for $i = 1,2$. Since $F_{q,M}$ is bounded and continuous, passing to the limit gives

$$\mathbb{E}\big[F_{q,M}(\mathcal{J}_\kappa^{(1)})\big] \leq C_q\, \mathbb{E}\big[F_{q,M}(\mathcal{J}_\kappa^{(2)})\big].$$

Now let $M \uparrow \infty$ and use monotone convergence to obtain

$$\mathbb{E}\big[(\mathcal{J}_\kappa^{(1)})^q\big] \leq C_q\, \mathbb{E}\big[(\mathcal{J}_\kappa^{(2)})^q\big].$$

Taking the supremum over $\kappa \in (0,1]$ yields the first inequality in (i).

For the negative moments, fix $p > 0$ and $\varepsilon > 0$, and set

$$G_{p,\varepsilon}(x) := (x+\varepsilon)^{-p}.$$

This is bounded, continuous, and convex on $\mathbb{R}_+$, so Kahane gives

$$\mathbb{E}\big[G_{p,\varepsilon}(\mathcal{J}^{(1)}_{\delta,\kappa})\big] \le C_p\,\mathbb{E}\big[G_{p,\varepsilon}(\mathcal{J}^{(2)}_{\delta,\kappa})\big].$$

Passing $\delta \downarrow 0$ for fixed κ yields

$$\mathbb{E}\big[G_{p,\varepsilon}(\mathcal{J}^{(1)}_{\kappa})\big] \le C_p\,\mathbb{E}\big[G_{p,\varepsilon}(\mathcal{J}^{(2)}_{\kappa})\big].$$

Now let $\varepsilon \downarrow 0$ and use monotone convergence to get

$$\mathbb{E}\big[(\mathcal{J}^{(1)}_{\kappa})^{-p}\big] \le C_p\,\mathbb{E}\big[(\mathcal{J}^{(2)}_{\kappa})^{-p}\big].$$

Taking the supremum over $\kappa \in (0,1]$ yields (ii). Reversing the roles of 1 and 2 gives the converse inequalities. □

Proposition 6.28 (Uniform insertion moments for the canonical interval model)**.** *Assume $a \ge 0$, $c_X\lambda^2 < 2$, and*

$$a < 1 + \frac{1}{2}c_X\lambda^2.$$

Then there exists $p_+ > 0$ such that

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{p_+}\big] < \infty.$$

Moreover, for every $p_- > 0$,

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{-p_-}\big] < \infty.$$

Proof. Let X^{ref} be a fixed reference interval field to which the one-point insertion theory of Remy–Zhu [8] applies, with the same logarithmic covariance coefficient c_X, and let

$$\mathcal{J}^{\mathrm{ref}}_{\kappa} := \int_0^1 (x+\kappa)^{-a} M^{\mathrm{ref}}_{\lambda}(dx), \qquad \kappa \in (0,1].$$

Since $a \ge 0$, the map $\kappa \mapsto \mathcal{J}^{\mathrm{ref}}_{\kappa}$ is decreasing and

$$\mathcal{J}^{\mathrm{ref}}_{\kappa} \uparrow \mathcal{J}^{\mathrm{ref}}_{0} := \int_0^1 x^{-a} M^{\mathrm{ref}}_{\lambda}(dx) \qquad \text{as } \kappa \downarrow 0.$$

The one-point insertion moment theory of Remy–Zhu gives the finiteness of positive moments of $\mathcal{J}^{\mathrm{ref}}_0$ in the one-sided Seiberg window

$$a < 1 + \frac{1}{2}c_X\lambda^2.$$

Hence there exists $p^{\mathrm{ref}}_+ > 0$ such that

$$\mathbb{E}\big[(\mathcal{J}^{\mathrm{ref}}_0)^{p^{\mathrm{ref}}_+}\big] < \infty.$$

Set

$$q_+ := \min\{p^{\mathrm{ref}}_+, 1/2\} \in (0,1).$$

Then, by monotone convergence,

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}^{\mathrm{ref}}_{\kappa})^{q_+}\big] \le \mathbb{E}\big[(\mathcal{J}^{\mathrm{ref}}_0)^{q_+}\big] < \infty.$$

Applying Lemma 6.27(i) with $X^{(1)} = \widetilde{X}$ and $X^{(2)} = X^{\mathrm{ref}}$ yields

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{q_+}\big] < \infty.$$

This is the positive-moment statement claimed in the proposition (after renaming q_+ as p_+).

For negative moments, since $a \ge 0$ and $x + \kappa \le 2$ for $x \in [0,1]$ and $\kappa \in (0,1]$,

$$(x+\kappa)^{-a} \ge 2^{-a}.$$

Hence, on the fixed subinterval $I := [1/4, 1/2] \subset (0,1)$,

$$\mathcal{J}^{(a,\lambda)}_{\kappa} \ge 2^{-a}\widetilde{M}_{\lambda}(I).$$

Therefore, for every $p_- > 0$,

$$(\mathcal{J}^{(a,\lambda)}_{\kappa})^{-p_-} \le 2^{ap_-}\widetilde{M}_{\lambda}(I)^{-p_-}.$$

Now I is a fixed interval of positive distance from the boundary of $[0,1]$, so Proposition 3.1 in dimension $d=1$ applies to give finite negative moments of all orders for $\widetilde{M}_\lambda(I)$. Consequently,

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}_\kappa^{(a,\lambda)})^{-p_-}\big] < \infty.$$

This proves the claim. □

Proposition 6.29 (Direct limiting comparison for the microscopic insertion family)**.** *Let $(\widehat{M}_n^{(\lambda)})_{n\geq 1}$ be the microscopic interval GMC family from Proposition 6.20, and define*

$$\widehat{\mathcal{K}}_{n,\kappa}^{(a,\lambda)} := \int_{-1}^{1} (|x|+\kappa)^{-a}\, \widehat{M}_n^{(\lambda)}(dx), \qquad \kappa\in(0,r_n].$$

Assume $a\geq 0$ and $c_X\lambda^2<2$. Then:

(i) For every $q\in(0,1)$ there exists $C_q<\infty$ such that

$$\sup_{n\geq 1}\sup_{\kappa\in(0,r_n]} \mathbb{E}\big[(\widehat{\mathcal{K}}_{n,\kappa}^{(a,\lambda)})^{q}\big] \leq C_q \sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}_\kappa^{(a,\lambda)})^{q}\big].$$

(ii) For every $p>0$ there exists $C_p<\infty$ such that

$$\sup_{n\geq 1}\sup_{\kappa\in(0,r_n]} \mathbb{E}\big[(\widehat{\mathcal{K}}_{n,\kappa}^{(a,\lambda)})^{-p}\big] \leq C_p \sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}_\kappa^{(a,\lambda)})^{-p}\big].$$

Proof. Fix a regularization $(\widetilde{X}_\delta)_{\delta\in(0,1]}$ of the canonical interval field $\widetilde{X}$ and define, for $\delta,\kappa\in(0,1]$,

$$\widetilde{\mathcal{J}}_{\delta,\kappa}^{(a,\lambda)} := \int_0^1 (x+\kappa)^{-a} \exp\Big(\lambda\widetilde{X}_\delta(x) - \tfrac{1}{2}\lambda^2 \mathrm{Var}(\widetilde{X}_\delta(x))\Big)\, dx.$$

Similarly, for the microscopic family write

$$\widehat{\mathcal{K}}_{n,\delta,\kappa}^{+} := \int_0^1 (x+\kappa)^{-a}\, \widehat{M}_{n,\delta}^{(\lambda)}(dx), \qquad \widehat{\mathcal{K}}_{n,\delta,\kappa}^{-} := \int_{-1}^0 (|x|+\kappa)^{-a}\, \widehat{M}_{n,\delta}^{(\lambda)}(dx),$$

where $\widehat{M}_{n,\delta}^{(\lambda)}$ denotes the regularized chaos measure built from $Y_{n,\delta}$.

For the positive half, Proposition 6.20(i) implies that the covariance kernel of $Y_{n,\delta}$ on $[0,1]$ differs from that of $\widetilde{X}_\delta$ by a uniformly bounded additive term. To stay within the domain of Kahane's convexity inequality, fix $q\in(0,1)$ and $M>0$, and apply it to the bounded continuous concave truncation

$$F_{q,M}(x) := (x\wedge M)^q.$$

This gives, uniformly in (n,δ,κ),

$$\mathbb{E}\big[F_{q,M}(\widehat{\mathcal{K}}_{n,\delta,\kappa}^{+})\big] \leq C_q\, \mathbb{E}\big[F_{q,M}(\widetilde{\mathcal{J}}_{\delta,\kappa}^{(a,\lambda)})\big].$$

For negative moments, fix $p>0$ and $\varepsilon>0$, and apply Kahane to the bounded continuous convex truncation

$$G_{p,\varepsilon}(x) := (x+\varepsilon)^{-p}.$$

This yields, again uniformly in (n,δ,κ),

$$\mathbb{E}\big[G_{p,\varepsilon}(\widehat{\mathcal{K}}_{n,\delta,\kappa}^{+})\big] \leq C_p\, \mathbb{E}\big[G_{p,\varepsilon}(\widetilde{\mathcal{J}}_{\delta,\kappa}^{(a,\lambda)})\big].$$

Fix now $\kappa>0$. Since the weight $x\mapsto(x+\kappa)^{-a}$ is bounded and continuous on $[0,1]$, the subcritical convergence of regularized GMC implies

$$\widetilde{\mathcal{J}}_{\delta,\kappa}^{(a,\lambda)} \to \mathcal{J}_\kappa^{(a,\lambda)} \quad \text{in probability as } \delta\downarrow 0,$$

and similarly

$$\widehat{\mathcal{K}}_{n,\delta,\kappa}^{+} \to \widehat{\mathcal{K}}_{n,\kappa}^{+} \quad \text{in probability as } \delta\downarrow 0,$$

for each fixed n. For $M>0$, define the bounded continuous concave truncation

$$F_{q,M}(x) := (x\wedge M)^q, \qquad q\in(0,1).$$

Applying the regularized Kahane comparison with $F_{q,M}$ gives

$$\mathbb{E}\big[F_{q,M}(\widehat{\mathcal{K}}_{n,\delta,\kappa}^{+})\big] \leq C_q\, \mathbb{E}\big[F_{q,M}(\widetilde{\mathcal{J}}_{\delta,\kappa}^{(a,\lambda)})\big].$$

Since $F_{q,M}$ is bounded and continuous, passing $\delta \downarrow 0$ yields

$$\mathbb{E}\big[F_{q,M}(\widehat{\mathcal{K}}^{+}_{n,\kappa})\big] \le C_q\,\mathbb{E}\big[F_{q,M}(\mathcal{J}^{(a,\lambda)}_{\kappa})\big].$$

Now let $M \uparrow \infty$ and use monotone convergence to obtain

$$\mathbb{E}\big[(\widehat{\mathcal{K}}^{+}_{n,\kappa})^q\big] \le C_q\,\mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^q\big].$$

The same argument applied to the reflected positive half gives the same bound for $\widehat{\mathcal{K}}^{-}_{n,\kappa}$, and therefore

$$\mathbb{E}\big[(\widehat{\mathcal{K}}^{(a,\lambda)}_{n,\kappa})^q\big] \le \mathbb{E}\big[(\widehat{\mathcal{K}}^{+}_{n,\kappa})^q\big] + \mathbb{E}\big[(\widehat{\mathcal{K}}^{-}_{n,\kappa})^q\big] \le 2C_q\,\mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^q\big],$$

using $(A+B)^q \le A^q + B^q$ for $q \in (0,1)$. Taking the supremum over n and $\kappa \in (0, r_n]$ proves (i).

For negative moments, again fix $\kappa > 0$ and $\varepsilon > 0$. Apply the regularized Kahane comparison to the bounded convex function

$$G_{p,\varepsilon}(x) := (x+\varepsilon)^{-p}.$$

This yields

$$\mathbb{E}\big[G_{p,\varepsilon}(\widehat{\mathcal{K}}^{+}_{n,\delta,\kappa})\big] \le C_p\,\mathbb{E}\big[G_{p,\varepsilon}(\widetilde{\mathcal{J}}^{(a,\lambda)}_{\delta,\kappa})\big].$$

Since $G_{p,\varepsilon}$ is bounded and continuous, passing $\delta \downarrow 0$ gives

$$\mathbb{E}\big[G_{p,\varepsilon}(\widehat{\mathcal{K}}^{+}_{n,\kappa})\big] \le C_p\,\mathbb{E}\big[G_{p,\varepsilon}(\mathcal{J}^{(a,\lambda)}_{\kappa})\big].$$

Now let $\varepsilon \downarrow 0$. Since $G_{p,\varepsilon}(x) \uparrow x^{-p}$ for $x > 0$, monotone convergence yields

$$\mathbb{E}\big[(\widehat{\mathcal{K}}^{+}_{n,\kappa})^{-p}\big] \le C_p\,\mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{-p}\big].$$

Finally, since $\widehat{\mathcal{K}}^{(a,\lambda)}_{n,\kappa} \ge \widehat{\mathcal{K}}^{+}_{n,\kappa}$,

$$(\widehat{\mathcal{K}}^{(a,\lambda)}_{n,\kappa})^{-p} \le (\widehat{\mathcal{K}}^{+}_{n,\kappa})^{-p},$$

so

$$\mathbb{E}\big[(\widehat{\mathcal{K}}^{(a,\lambda)}_{n,\kappa})^{-p}\big] \le C_p\,\mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{-p}\big].$$

Taking the supremum over n and $\kappa \in (0, r_n]$ proves (ii). ☐

Proposition 6.30 (Route B completed)**.** *Assume*

$$S := q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} \in (0,1).$$

Then the three scalar triples (c_X, λ, a) arising in Proposition 6.23 satisfy the weighted inputs (a)–(c) *of Proposition 6.23. Consequently, the strictly subcritical lower bound in Theorem 6.31 is unconditional whenever $S \in (0,1)$.*

Proof. Fix one of the three relevant scalar triples (c_X, λ, a) from Proposition 6.23. Since $S > 0$, the corresponding logarithmic covariance coefficient c_X is strictly positive for all three triples. By Proposition 6.24, the triple lies in the one-point insertion regime with $a \ge 0$. Proposition 6.28 therefore yields:

- a positive moment exponent $p_+^{\mathrm{can}} > 0$ such that

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{p_+^{\mathrm{can}}}\big] < \infty,$$

- and, for every $p_- > 0$,

$$\sup_{\kappa\in(0,1]} \mathbb{E}\big[(\mathcal{J}^{(a,\lambda)}_{\kappa})^{-p_-}\big] < \infty.$$

Since Proposition 6.29(i) is stated for exponents $q \in (0,1)$, choose

$$q_+ := \min\{p_+^{\mathrm{can}}, 1/2\} \in (0,1).$$

Then the canonical positive q_+-moment is still finite, and Proposition 6.29(i) gives

$$\sup_{n\ge1}\sup_{\kappa\in(0,r_n]} \mathbb{E}\big[(\widehat{\mathcal{K}}^{(a,\lambda)}_{n,\kappa})^{q_+}\big] < \infty.$$

For the negative moments, fix any $p_- > 0$ and apply Proposition 6.29(ii) to obtain

$$\sup_{n\ge1}\sup_{\kappa\in(0,r_n]} \mathbb{E}\big[(\widehat{\mathcal{K}}^{(a,\lambda)}_{n,\kappa})^{-p_-}\big] < \infty.$$

Now Proposition 6.20(ii) transfers these positive and negative bounds to the weighted local masses $\mathcal{I}_{n,\kappa}^{(u,\lambda)}$. Hence the weighted moment input (6.6) required in Proposition 6.18 holds for this triple, with positive exponent q_+ and arbitrary negative exponent p_-. Since the same argument applies to all three scalar inputs (a)–(c) in Proposition 6.23, the latter proposition yields the unconditional lower bound. □

Theorem 6.31 (Joint thick-point spectrum for C$\boldsymbol{\beta}$E limit measures)**.** *Fix $\boldsymbol{\beta} > 0$ and let $\gamma_1, \gamma_2 \in \mathbb{R}$ satisfy $|\widehat{\gamma}_i| < 1$, where $\widehat{\gamma}_i := \gamma_i/\sqrt{2\boldsymbol{\beta}}$. Let $\mu^{\widehat{\gamma}_1}$ and $\nu^{\widehat{\gamma}_2}$ be the independent subcritical GMC measures on $\mathbb{T}$ driven respectively by χ and ψ. Let $q_1, q_2 > 0$ satisfy $q_i|\widehat{\gamma}_i| < 1$ and set*

$$h_i := 1 + \Big(\frac{1}{2} - q_i\Big)\frac{\gamma_i^2}{\boldsymbol{\beta}} = 1 + (1 - 2q_i)\widehat{\gamma}_i^{\,2}, \qquad i = 1, 2.$$

Then almost surely:

- *if $q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} < 1$, then*
$$\dim_{\mathrm{H}}\big(E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2)\big) = 1 - q_1^2\widehat{\gamma}_1^{\,2} - q_2^2\widehat{\gamma}_2^{\,2};$$
- *if $q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} = 1$, then*
$$\dim_{\mathrm{H}}\big(E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2)\big) \le 0;$$
- *if $q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} > 1$, then*
$$E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2) = \varnothing.$$

Proof. The upper bound and the emptiness criterion follow from Proposition 6.21. Let

$$S := q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2}.$$

If $S = 0$, then necessarily $\gamma_1 = \gamma_2 = 0$, so $\mu^{\widehat{\gamma}_1} = \nu^{\widehat{\gamma}_2} = d\theta/(2\pi)$, both exponents equal $h_1 = h_2 = 1$, and the intersection set is the whole circle. Hence

$$\dim_{\mathrm{H}}\big(E_{\mu^{\widehat{\gamma}_1}}(h_1) \cap E_{\nu^{\widehat{\gamma}_2}}(h_2)\big) = 1.$$

If $S \in (0, 1)$, the matching lower bound follows from Proposition 6.23 together with Proposition 6.30. □

Remark 6.32 (Random-matrix interpretation)**.** Theorem 6.31 is first and foremost a statement about the limiting GMC measures $\mu^{\widehat{\gamma}_1}$ and $\nu^{\widehat{\gamma}_2}$. Via Proposition 6.1, Theorem 6.31 applies directly to the random-matrix observables whenever the corresponding subcritical convergence from C$\boldsymbol{\beta}$E is available. In particular:

- the interpretation of $\mu^{\widehat{\gamma}_1}$ as the limit of the normalized characteristic-polynomial measure requires $\gamma_1 > -1$;
- the interpretation of $\nu^{\widehat{\gamma}_2}$ as the limit of the normalized counting-field measure holds for every $\gamma_2 \in \mathbb{R}$ with $|\widehat{\gamma}_2| < 1$.

In the strictly subcritical region $q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} < 1$, both the upper and lower bounds are unconditional once the subcritical convergence from Proposition 6.1 and the one-component circle estimates established in this section are in place. Route B is closed in this regime by Propositions 6.28, 6.29, and 6.30. On the critical boundary $q_1^2\widehat{\gamma}_1^{\,2} + q_2^2\widehat{\gamma}_2^{\,2} = 1$, the present argument yields only the upper bound $\dim_{\mathrm{H}} \le 0$.

7. Near-critical regime and derivative martingales

Remark 7.1 (Derivative martingale in C$\boldsymbol{\beta}$E and critical chaos)**.** In the study of extremes of the C$\boldsymbol{\beta}$E characteristic polynomial, Paquette–Zeitouni identify the limiting random shift in the centered maximum in terms of the total mass of a derivative martingale $B_{\boldsymbol{\beta}}$ [7]. Lambert–Najnudel further explain that $B_{\boldsymbol{\beta}}$ should be related to the critical version of multiplicative chaos (the case $\widehat{\gamma} = 1$) for the boundary log-correlated field on the circle [5]. More precisely, they state the expectation that

$$B_{\boldsymbol{\beta}} = c\,\mu'(\mathbb{T}), \qquad \mu'(\mathbb{T}) := \lim_{\widehat{\gamma}\uparrow 1}\frac{\mu^{\widehat{\gamma}}(\mathbb{T})}{1 - \widehat{\gamma}},$$

for an explicit constant $c > 0$, and that proving this identity is a key missing step toward the Fyodorov–Bouchaud asymptotics for the maximum.

Conjecture 7.2 (Lambert–Najnudel)**.** Under the C$\boldsymbol{\beta}$E coupling of Proposition 6.1, let $\mu^{\widehat{\gamma}}$ be the subcritical GMC measure on $\mathbb{T}$ associated with the boundary real field χ, and let $B_{\boldsymbol{\beta}}$ be the derivative martingale of [7]. Then there exists an explicit constant $c = c(\boldsymbol{\beta}) > 0$ such that

$$\lim_{\widehat{\gamma}\uparrow 1} \frac{\mu^{\widehat{\gamma}}(\mathbb{T})}{1-\widehat{\gamma}} = c^{-1} B_{\boldsymbol{\beta}}$$

in distribution, and possibly in a stronger mode of convergence.

Problem 7.3 (A joint spectrum at criticality)**.** Assuming Conjecture 7.2, construct the critical (derivative) chaos measures μ' and ν' associated with (χ, ψ) and determine the Hausdorff dimension of joint thick-point sets such as

$$E_{\mu'}(h) \cap E_{\nu'}(h'), \qquad E_{\mu'}(h) \cap E_{\nu^{\widehat{\gamma}}}(h'),$$

including their near-critical scaling limits as $\widehat{\gamma} \uparrow 1$.

Remark 7.4 (Compatibility with the subcritical spectrum)**.** Although Theorem 6.31 is subcritical, it already exhibits the expected geometric collapse as $\widehat{\gamma} \uparrow 1$. Indeed, the typical local dimension of $\mu^{\widehat{\gamma}}$ equals $1 - \widehat{\gamma}^2$, which tends to 0 as $\widehat{\gamma} \uparrow 1$. Thus the dimension of a full-mass exact-dimensional carrier shrinks to 0 as $\widehat{\gamma} \uparrow 1$. This is compatible with critical scenarios in which the derivative chaos is carried by a zero-dimensional set, even though the subcritical measures themselves have full topological support.

Appendix A. From local dimensions to field slopes

Throughout this appendix we work under the standing regularity inputs **(R1)–(R2)** of Section 3.1. Fix cubes $Q \Subset Q^+ \Subset D$, let $r_n := 2^{-n}$ and $L_n := \log(1/r_n)$, and fix a subcritical parameter $t \in \mathbb{R}^m$. We assume n large enough that $r_n \le \frac{1}{2}\operatorname{dist}(Q, (Q^+)^c)$, so that $B(x, r_n) \subset Q^+$ for all $x \in Q$. Write

$$\sigma^2 := \|t\|_\Sigma^2, \qquad V_n^{(t)}(x) := \mathbb{E}[(X_{r_n}^{(t)}(x))^2].$$

By (2.2),

$$\sup_{x\in Q^+} \left| \frac{V_n^{(t)}(x)}{L_n} - \sigma^2 \right| \longrightarrow 0. \tag{A.1}$$

Lemma A.1 (Cut-off ball factorization)**.** *There exists a deterministic constant $C = C(t, Q) < \infty$ such that, almost surely, for all large n and all $x \in Q$ with $B(x, r_n) \subset D$,*

$$\begin{aligned} d\log r_n + X_{r_n}^{(t)}(x) - \frac{1}{2}V_n^{(t)}(x) - \omega_{n,1}^{(t)}(Q^+) - C &\le \log M_{t,r_n}(B(x,r_n)) \\ &\le d\log r_n + X_{r_n}^{(t)}(x) - \frac{1}{2}V_n^{(t)}(x) + \omega_{n,1}^{(t)}(Q^+) + C, \end{aligned}$$

where

$$\omega_{n,1}^{(t)}(Q^+) := \sup_{\substack{x,y\in Q^+ \\ |x-y|\le r_n}} |X_{r_n}^{(t)}(x) - X_{r_n}^{(t)}(y)|.$$

Consequently,

$$\sup_{x\in Q} \left| \frac{\log M_{t,r_n}(B(x,r_n))}{\log r_n} - \left(d - \frac{X_{r_n}^{(t)}(x)}{L_n} + \frac{1}{2}\frac{V_n^{(t)}(x)}{L_n} \right) \right| \longrightarrow 0 \tag{A.2}$$

as $n \to \infty$.

Proof. Fix n and $x \in Q$ with $B(x, r_n) \subset D$. For $y \in B(x, r_n)$, the oscillation bound implies

$$|X_{r_n}^{(t)}(y) - X_{r_n}^{(t)}(x)| \le \omega_{n,1}^{(t)}(Q^+).$$

By (2.2), the variance satisfies

$$|V_n^{(t)}(y) - V_n^{(t)}(x)| \le C' \qquad (y \in B(x, r_n))$$

for a deterministic constant C' depending only on t and Q. Therefore,

$$\begin{aligned} M_{t,r_n}(B(x,r_n)) &= \int_{B(x,r_n)} \exp\left(X_{r_n}^{(t)}(y) - \tfrac{1}{2}V_n^{(t)}(y)\right) dy \\ &\le \exp\left(X_{r_n}^{(t)}(x) - \tfrac{1}{2}V_n^{(t)}(x) + \omega_{n,1}^{(t)}(Q^+) + \tfrac{1}{2}C'\right) |B(x,r_n)|, \end{aligned}$$

with an analogous lower bound obtained by replacing $+\omega_{n,1}^{(t)}(Q^+) + \frac{1}{2}C'$ by $-\omega_{n,1}^{(t)}(Q^+) - \frac{1}{2}C'$. Since $|B(x, r_n)| = v_d r_n^d$ with $v_d = |B(0,1)|$, the first display follows after absorbing $\log v_d$ and $C'/2$ into the constant C. Dividing by $\log r_n = -L_n$ and using **(R1)** gives (A.2). □

Lemma A.2 (Thick points imply field slopes)**.** *Assume the standing regularity inputs of Section 3.1. Fix $t \in \mathbb{R}^m$ with $\|t\|_\Sigma^2 < 2d$ and $h \geq 0$, and set*

$$c := d + \frac{1}{2}\|t\|_\Sigma^2 - h.$$

Then almost surely, for every $x \in E_t(h) \cap Q$,

$$\lim_{n\to\infty} \frac{X_{r_n}^{(t)}(x)}{L_n} = c.$$

Proof. Let $x \in E_t(h) \cap Q$. By definition of $E_t(h)$,

$$\frac{\log M_t(B(x, r_n))}{\log r_n} \longrightarrow h.$$

By the local cut-off stability assumption **(R2)**,

$$\frac{\log M_{t,r_n}(B(x, r_n))}{\log r_n} - \frac{\log M_t(B(x, r_n))}{\log r_n} \longrightarrow 0.$$

Hence

$$\frac{\log M_{t,r_n}(B(x, r_n))}{\log r_n} \longrightarrow h.$$

Applying (A.2) from Lemma A.1 and using (A.1), we obtain

$$h = d - \lim_{n\to\infty} \frac{X_{r_n}^{(t)}(x)}{L_n} + \frac{1}{2}\|t\|_\Sigma^2,$$

which rearranges to the stated formula. □

Corollary A.3 (Vector slope constraint)**.** *Assume the standing regularity inputs of Section 3.1. Fix subcritical parameters $t_1, \ldots, t_k \in \mathbb{R}^m$ and let $\boldsymbol{h} = (h_1, \ldots, h_k)$. Define*

$$\boldsymbol{c}(\boldsymbol{h}) := \big(c_1(\boldsymbol{h}), \ldots, c_k(\boldsymbol{h})\big), \qquad c_i(\boldsymbol{h}) = d + \frac{1}{2}\|t_i\|_\Sigma^2 - h_i.$$

Then almost surely, for every $x \in E(\boldsymbol{h}) \cap Q$,

$$\lim_{n\to\infty} \frac{1}{L_n}\big(X_{r_n}^{(t_1)}(x), \ldots, X_{r_n}^{(t_k)}(x)\big) = \boldsymbol{c}(\boldsymbol{h}).$$

Proof. Apply Lemma A.2 to each t_i and intersect the resulting almost sure events. □

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