On Ruckle's conjecture for one-stage accumulation games: a coupling lift and a proved regime

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Abstract

We study the one-stage accumulation game introduced by Ruckle and further developed with Kikuta. A Hider allocates a nonnegative wealth budget h across n locations, a Searcher confiscates all wealth at r locations, and the Hider wins if the wealth remaining on the $s := n - r$ unsearched locations is at least 1. Ruckle conjectured that an optimal Hider strategy can always be chosen from a two-level family that places equal positive mass on some k locations and zero elsewhere.

We prove a quantitative partial result on this conjecture. Our first theorem establishes a sharp coupling lift from the without-replacement game to the corresponding i.i.d. extremal tail problem, with an explicit error equal to the collision probability. This yields the general upper bound $V(n, s, h) \leq p_s(h/n) + \big(1 - (n)_s/n^s\big)$, where $p_s(x)$ denotes the i.i.d. extremal tail value. Combining this lift with known identities for $p_s(x)$, we obtain a proved regime in which the value of the game is bracketed, up to an explicit finite-n error, by the better of two canonical two-level allocations: a concentrated allocation with $\lfloor h \rfloor$ unit masses and a spread allocation with $\lfloor sh \rfloor$ equal masses. In particular, the result is unconditional for $s = 3$, and it holds more generally whenever the identity $p_s(x) = \max\{1 - (1 - x)^s, (sx)^s\}$ is available at $x = h/n$.

Our proof combines a with/without-replacement coupling, the extremal theory of i.i.d. tail probabilities, and explicit finite-n hypergeometric corrections. In this way, the paper should be read as a transfer theorem from proved i.i.d. extremal-tail identities to quantitative statements about one-stage accumulation games, and hence as a proved quantitative regime rather than a complete resolution of the conjecture. A final section records an entropy-regularized selector on the conjectured two-level family.

Keywords. accumulation games; sampling without replacement; i.i.d. tail extremals; Erdős matching conjecture; entropic regularization.
MSC (2020). Primary 91A05; Secondary 05D05, 60C05, 60E15.

1 Introduction

1.1 The model and the conjecture

We study the one-stage accumulation game introduced by Ruckle and Kikuta [1, 13]. A Hider allocates a nonnegative wealth budget h across n labeled locations; a Searcher inspects r locations and confiscates all wealth there; the Hider succeeds if the total wealth remaining on

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the $s := n - r$ unsearched locations is at least 1. After the symmetry reduction of Alpern–Fokkink–Kikuta [14], the value can be written as

$$V(n, s, h) = \sup_{\substack{w \in \mathbb{R}^n_{\geq 0} \\ \sum_i w_i \leq h}} \mathbb{P}\Big(\sum_{i \in I} w_i \geq 1 \Big), \qquad I \sim \mathrm{Unif}\Big(\binom{[n]}{s} \Big).$$

Equivalently, $V(n, s, h)$ is the maximal fraction of s-subsets whose total allocated weight reaches the threshold.

The central open problem is the following two-level conjecture, stated explicitly in [14].

Conjecture 1.1 (Ruckle [14])**.** *For every (n, r, h) there exists an optimal Hider strategy of the following form: choose an integer $k \in \{1, \dots, n\}$, allocate equal weight h/k to k locations and 0 to the remaining $n - k$ locations, and then apply a uniform random permutation of location labels.*

Thus the conjecture predicts that canonical two-level allocations—equal positive weights on a support of size k and 0 elsewhere—always suffice for optimal play.

1.2 Known results and what this paper proves

The conjecture is known in several special cases, but remains open in general. Existing results establish it in a number of low-dimensional or structurally favorable regimes, while the general middle-density region remains unresolved. The main obstacle is that the objective is a discontinuous threshold count over subsets, rather than a smooth convex or concave functional of the weight vector. As a result, standard symmetrization and majorization arguments do not apply directly.

This paper proves a quantitative partial result on the conjecture. Our starting point is the i.i.d. extremal tail problem

$$p_s(x) := \sup \Big\{ \mathbb{P}(X_1 + \cdots + X_s \geq 1) : \; X_i \text{ i.i.d.}, \; X_i \geq 0, \; \mathbb{E}[X_i] \leq x \Big\}.$$

The conjectured explicit formula is

$$p_s(x) = M(x) := \max\{1 - (1 - x)^s, \; (sx)^s\}, \qquad 0 \leq x \leq 1/s. \tag{1}$$

This identity is known in several regimes, most importantly in the range $x \leq 3/(5s - 2)$ by Frankl–Kupavskii [16, Corollary 26]; see also [17], and on the full interval when $s = 3$ by Łuczak, Mieczkowska, and Šileikis [15].

Our first theorem gives a sharp comparison between the accumulation game and the i.i.d. problem.

Theorem 1.2 (With/without-replacement lift and an upper bound)**.** *For every $1 \leq s \leq n$ and $h \geq 0$,*

$$V(n, s, h) \leq p_s(h/n) + \Delta_{\mathrm{coll}}(n, s) \leq p_s(h/n) + \frac{\binom{s}{2}}{n}, \tag{2}$$

where

$$\Delta_{\mathrm{coll}}(n, s) := 1 - \frac{(n)_s}{n^s}.$$

Theorem 1.2 is unconditional. Its proof is elementary and sharp: the without-replacement winning probability equals the corresponding with-replacement i.i.d. tail probability conditioned on the absence of collisions, and differs from the unconditional i.i.d. tail probability by at most the collision probability.

To obtain lower bounds, we introduce two explicit two-level allocations. The first is the concentrated allocation that places unit mass on $\lfloor h \rfloor$ locations and 0 elsewhere. The second is the spread allocation that distributes the budget evenly across $\lfloor sh \rfloor$ locations. Their exact win probabilities are

$$\begin{aligned} P_{\mathrm{C}}(n,s,h) &:= 1 - \frac{\binom{n-\lfloor h \rfloor}{s}}{\binom{n}{s}}, \\ P_{\mathrm{S}}(n,s,h) &:= \frac{\binom{\lfloor sh \rfloor}{s}}{\binom{n}{s}}, \\ W_{\mathrm{CS}}(n,s,h) &:= \max\{P_{\mathrm{C}}(n,s,h), P_{\mathrm{S}}(n,s,h)\}. \end{aligned}$$

These are canonical two-level allocations of the conjectured Ruckle form.

Define also

$$\Delta_{\mathrm{hyp}}(n,s) := \frac{s}{n} + \frac{\binom{s}{2}}{n-s+1}.$$

Our main quantitative partial result is the following sandwich theorem.

Theorem 1.3 (Sandwich bound under the i.i.d. identity)**.** *Assume $h \ge 1$ and $sh/n < 1$, and set $x := h/n \in (0, 1/s)$. If $p_s(x) = M(x)$ holds at this x, then*

$$M(x) - \Delta_{\mathrm{hyp}}(n,s) \;\le\; W_{\mathrm{CS}}(n,s,h) \;\le\; V(n,s,h) \;\le\; M(x) + \Delta_{\mathrm{coll}}(n,s). \tag{3}$$

Consequently,

$$0 \le V(n,s,h) - W_{\mathrm{CS}}(n,s,h) \le \Delta_{\mathrm{coll}}(n,s) + \Delta_{\mathrm{hyp}}(n,s). \tag{4}$$

In particular, the lower bound in (3) *is witnessed by one of the two explicit two-level allocations above.*

Thus, in every parameter region where the i.i.d. identity (1) is known, the value of the accumulation game is bracketed by explicit canonical two-level allocations with a fully explicit finite-n certificate. This is a genuine partial result on theorem 1.1: while we do not prove exact optimality of two-level allocations in full generality, we do prove that in the proved regime they determine the value up to a completely explicit error of order $O(s^2/n)$.

Two direct corollaries are especially useful. Every unconditional corollary in the paper is obtained by composing the unconditional lift theorem with a currently proved identity for the i.i.d. extremal quantity $p_s(x)$. Table 1 summarizes the role of these results in the paper.

Corollary 1.4 (Frankl–Kupavskii regime)**.** *Assume $h \ge 1$, $sh/n < 1$, and $x = h/n \le 3/(5s-2)$. Then* (3) *holds.*

Corollary 1.5 (Unconditional case $s = 3$)**.** *If $s = 3$, $h \ge 1$, and $3h/n < 1$, then* (3) *holds for all $x = h/n \in (0, 1/3)$.*

As a secondary result, we also study an entropy-regularized scale-selection problem on the two-level family. Restricting to support sizes $k \in \{1, \dots, n\}$, we define the exact win probability of the corresponding two-level allocation and maximize its expectation plus an entropy bonus

Table 1: The theorem package proved in this paper.

Result	Content
Lift theorem	Transfers the game value to the i.i.d. extremal tail problem with explicit collision error $\Delta_{\text{coll}}(n, s)$.
Sandwich theorem	Pins down $V(n, s, h)$ up to $\Delta_{\text{coll}}(n, s) + \Delta_{\text{hyp}}(n, s)$ once the identity $p_s(x) = M(x)$ is known at $x = h/n$.
FK corollary	Makes the sandwich theorem unconditional on the range $x \le 3/(5s - 2)$.
$s = 3$ corollary	Gives a fully unconditional theorem when $s = 3$.

over distributions on k. This produces a unique closed-form Gibbs selector and an exponential concentration bound onto near-optimal scales. In the present paper, this regularized problem serves as a mathematically explicit selection rule inside the conjectured family, rather than as a substitute for the main structural theorem.

A useful way to position these theorems relative to the existing accumulation-game literature is the following. The classical results identify parameter regimes in which exact two-level optimality can be established. Our result is complementary: it identifies parameter regimes in which the game value itself can be pinned down, up to an explicit finite-n error, by canonical two-level allocations. Thus the paper contributes a quantitative approximation principle of exactly the structural type suggested by Ruckle's conjecture.

1.3 What is proved exactly and what remains open

For clarity, we separate here the unconditional content of the paper from the part that depends on currently available information on the i.i.d. extremal problem.

(i) **Unconditional.** The with/without-replacement lift of theorem 1.2 is unconditional and holds for every n, s, h.

(ii) **Conditional transfer statement.** The sandwich theorem of theorem 1.3 is valid at every parameter point $x = h/n$ for which the identity $p_s(x) = M(x)$ is known.

(iii) **Unconditional corollaries.** The theorem is unconditional for all admissible x when $s = 3$, and unconditional on the regime $x \le 3/(5s - 2)$ for general s.

(iv) **What remains open.** We do not prove exact optimality of two-level allocations for every (n, s, h). Rather, we prove that on a currently provable range the value of the game is already controlled, up to explicit finite-n error, by canonical two-level allocations of the conjectured form.

The logical structure is therefore clean: the lift theorem is unconditional; the sandwich theorem is a transfer statement valid at every parameter point where the i.i.d. identity is known; and the FK-range and $s = 3$ statements are unconditional corollaries obtained by plugging in currently proved identities. What remains open is the full exact structural conjecture beyond these transfer-based regimes.

1.4 Relation to the existing accumulation-game and i.i.d. literatures

Two comparisons are especially important for interpreting the present theorem. First, relative to Alpern–Fokkink–Kikuta [14], our result is complementary rather than overlapping. AFK prove exact structural optimality of two-level allocations in several special regimes and derive general upper bounds for the game value. By contrast, our main theorem is a quantitative transfer principle: on every parameter point at which the i.i.d. identity is known, it converts that information into an explicit finite-n sandwich theorem for the original accumulation game. Second, relative to the i.i.d. extremal-tail literature itself, our contribution is not a new proof of the identity for $p_s(x)$; instead, it is the observation that a sharp with/without-replacement coupling makes those i.i.d. identities directly usable for one-stage accumulation games. In this sense, the paper should be read as occupying the interface between exact structural results for accumulation games and extremal identities for i.i.d. sums.

1.5 Methodological novelty

The technical novelty of the paper lies in a three-step transfer argument.

(i) We prove a sharp with/without-replacement coupling that isolates the exact collision error.

(ii) We import the best currently available i.i.d. extremal-tail identities into the accumulation game.

(iii) We complement the upper bound by two explicit two-level lower bounds and quantify the finite-n hypergeometric correction explicitly.

The resulting theorem is both asymptotic and nonasymptotic: it yields a clean leading-order formula and an explicit finite-n certificate.

1.6 Proof strategy and notation

The proof has a deliberately modular structure. After the symmetry reduction, the game value becomes a threshold probability under sampling without replacement. We then compare this object to an i.i.d. tail probability under sampling with replacement. The comparison is exact up to the collision event and yields the upper bound in theorem 1.2. The remaining work is extremal: known identities for the i.i.d. quantity $p_s(x)$ give the upper half of the sandwich, while two explicit two-level allocations provide the lower half. The finite-n discrepancy is therefore completely controlled by a collision term and a hypergeometric correction.

This separation among three quantities—the exact game value $V(n, s, h)$, the i.i.d. extremal value $p_s(x)$, and the exact values of canonical two-level allocations—is the main organizational principle of the paper. It is what makes the transfer theorem transparent, reusable, and immediately improvable whenever stronger information on $p_s(x)$ becomes available.

1.7 Related work

Accumulation games and adjacent threshold-search models. Accumulation games were introduced by Ruckle [1] and developed by Kikuta and Ruckle in noisy-search and continuous-location variants [2, 13]. Alpern, Fokkink, and Kikuta [14] gave the modern one-stage formulation of Ruckle's conjecture, reformulated the objective in terms of heavy subsets, and proved a number of special cases, including $s \in \{2, n-2\}$ together with several additional low-dimensional and small-h regimes. Closely related threshold-hiding models arise in cyclic caching and allied resource-hiding problems [3, 5, 4]. Our paper remains in the one-stage accumulation-game framework of [14], but contributes a new proved regime and a quantitative approximation theorem for canonical two-level allocations.

Search and inspection games with multiple hidden resources. More broadly, the model belongs to a family of search and inspection games with limited search resources and heterogeneous hidden objects. Inspection games in arms control already emphasized the role of scarce inspection capacity [6]. Weighted and multiple-object search games later analyzed how heterogeneous values, capacities, and multiple search teams change equilibrium hiding patterns [7, 8, 9]. Recent hide-and-seek models with capacities and imperfect detection continue this line in a more explicitly OR-facing direction [10]. These papers differ from ours in objective—expected detection, expected search time, or search cost rather than threshold survival—but they provide a natural neighboring literature in which resource concentration versus dispersion is also central.

Extremal tail probabilities and hypergraph methods. The i.i.d. extremal-tail problem underlying our upper bound fits into a classical line going back to Hoeffding's inequalities for bounded sums [11]. Extremal-distribution questions for tail probabilities were subsequently studied by Meester [12] and, in the nonnegative i.i.d. setting relevant here, by Łuczak, Mieczkowska, and Šileikis [15]. Frankl and Kupavskii [16, 17] strengthened this line substantially by connecting the problem to extremal hypergraph techniques and the Erdős matching conjecture. Our contribution is a transfer theorem: once a regime for the i.i.d. identity is known, it yields a corresponding quantitative statement for the without-replacement accumulation game.

Interpretation of the present result. From the perspective of theorem 1.1, our theorem should be read as a quantitative partial result rather than a surrogate conjecture. It does not assert exact optimality of two-level allocations for every parameter triple (n, s, h); instead, it proves that on a proved range of h/n, canonical two-level allocations already determine the value up to an explicit finite-n correction. This formulation is natural when exact structural classification appears substantially harder than sharp value approximation.

Regularized selection within the conjectured family. The entropy-based scale selector developed later in the paper is secondary to the main theorem. Its role is to provide a smooth and explicit selection principle among two-level allocations, particularly near the transition region where several support sizes may perform nearly equally well. We do not claim that this regularized problem resolves theorem 1.1; rather, it refines the use of the conjectured family once the family itself has been identified as near-optimal.

1.8 Notation at a glance

We use the following symbols throughout. The exact accumulation-game value is

$$V(n, s, h),$$

the corresponding i.i.d. extremal-tail value is

$$p_s(x), \qquad x = h/n,$$

and the two canonical two-level benchmark values are

$$P_{\mathrm{C}}(n, s, h), \qquad P_{\mathrm{S}}(n, s, h), \qquad W_{\mathrm{CS}}(n, s, h) = \max\{P_{\mathrm{C}}(n, s, h), P_{\mathrm{S}}(n, s, h)\}.$$

The leading-order comparison is governed by

$$M(x) = \max\{1 - (1 - x)^s, (sx)^s\},$$

and the two explicit finite-n correction terms are

$$\Delta_{\mathrm{coll}}(n, s) = 1 - \frac{(n)_s}{n^s}, \qquad \Delta_{\mathrm{hyp}}(n, s) = \frac{s}{n} + \frac{\binom{s}{2}}{n - s + 1}.$$

With this notation, the main quantitative statement of the paper is the sandwich

$$M(x) - \Delta_{\mathrm{hyp}}(n, s) \le W_{\mathrm{CS}}(n, s, h) \le V(n, s, h) \le M(x) + \Delta_{\mathrm{coll}}(n, s).$$

Organization

Section 2 formalizes the game, proves a symmetry reduction, and records a WLOG truncation to $w \in [0, 1]^n$. Section 3 proves the with/without-replacement lift. Section 4 recalls the i.i.d. extremal-tail problem and the currently proved ranges for (1). Section 5 proves the upper and sandwich bounds, together with the explicit two-level lower bounds. Section 6 develops the entropic scale-selection problem on the two-level family.

2 Model, symmetry reduction, and standing assumptions

Fix integers $n \ge 1$ and $r \in \{0, 1, \ldots, n\}$ and set

$$s := n - r \in \{0, 1, \ldots, n\}.$$

Let $[n] := \{1, 2, \ldots, n\}$.

Definition 2.1 (One-stage accumulation game)**.** Fix $h \ge 0$. A (pure) Hider action is a vector $w = (w_1, \ldots, w_n)$ with $w_i \ge 0$ and $\sum_{i=1}^n w_i \le h$. A (pure) Searcher action is a subset $J \subset [n]$ with $|J| = r$; the Searcher confiscates wealth in J and the Hider wins if

$$\sum_{i \notin J} w_i \ge 1.$$

Let $V(n, s, h)$ denote the value of the resulting zero-sum game, i.e. the Hider's optimal winning probability.

Remark 2.2 (Trivial boundary cases)**.** If $s = 0$ (equivalently $r = n$), the Hider always loses since the remaining wealth is 0. We therefore focus on $1 \le s \le n$.

2.1 A symmetry reduction

The game is invariant under simultaneous permutations of location labels. The following reduction is standard and appears (with further structure) in [14]; we include a short proof for completeness.

Proposition 2.3 (Reduction to a uniform s-subset)**.** *For $1 \le s \le n$, the value satisfies*

$$V(n,s,h) = \sup_{w \in \mathbb{R}^n_{\ge 0}:\ \sum_i w_i \le h} \mathbb{P}\Big(\sum_{i \in I} w_i \ge 1\Big), \qquad I \sim \mathrm{Unif}\Big(\binom{[n]}{s}\Big). \tag{5}$$

Equivalently,

$$V(n,s,h) = \sup_{w: \sum w_i \le h} \frac{1}{\binom{n}{s}} \#\Big\{I \in \binom{[n]}{s} : \sum_{i \in I} w_i \ge 1\Big\}.$$

Proof. It is equivalent for the Searcher to choose the confiscated r-subset J or the unsearched s-subset $I := [n] \setminus J$; the Hider wins iff $\sum_{i \in I} w_i \ge 1$.

Let $\mathcal{S} := \binom{[n]}{s}$ be the Searcher's pure strategy space in the I-formulation. The symmetric group $\mathfrak{S}_n$ acts transitively on $\mathcal{S}$ by permuting labels. For any mixed Searcher strategy τ on $\mathcal{S}$, let $\bar{\tau}$ be its group average:

$$\bar{\tau} := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi\tau,$$

where $(\pi\tau)(I) := \tau(\pi^{-1} I)$. Then $\bar{\tau}$ is $\mathfrak{S}_n$-invariant, hence uniform on $\mathcal{S}$.

For any fixed Hider mixed strategy σ over feasible allocations, linearity and the payoff invariance under joint permutation imply

$$u(\sigma, \bar{\tau}) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} u(\sigma, \pi\tau) \le \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \sup_{\sigma'} u(\sigma', \pi\tau) = \sup_{\sigma'} u(\sigma', \tau),$$

so replacing τ by its uniform symmetrization does not increase the Hider's best-response payoff. Hence the Searcher has an optimal uniform strategy on $\mathcal{S}$. Against a uniform I, the Hider optimally chooses a deterministic allocation w (randomization can only convex-combine payoffs), yielding (5). $\square$

2.2 Standing assumptions

Remark 2.4 (Standing assumption)**.** We restrict to the nontrivial regime

$$\frac{sh}{n} < 1, \qquad h \ge 1. \tag{SA}$$

If $sh/n \ge 1$, the uniform allocation $w_i = h/n$ makes all s-sets winning; if $h < 1$, no allocation can reach the threshold.

2.3 A WLOG truncation

Lemma 2.5 (Truncation to $[0,1]$)**.** *Let $w \in \mathbb{R}^n_{\ge 0}$ be feasible, and define $w'_i := \min\{w_i, 1\}$. Then w' is feasible and for every s-subset I,*

$$\sum_{i \in I} w_i \ge 1 \iff \sum_{i \in I} w'_i \ge 1.$$

Consequently, in (5) we may restrict the supremum to allocations $w \in [0,1]^n$.

Proof. Feasibility is clear since $w_i' \le w_i$ implies $\sum_i w_i' \le \sum_i w_i \le h$. Fix any $I \in \binom{[n]}{s}$. If $w_i \le 1$ for all $i \in I$, then $w'(I) = w(I)$. Otherwise there exists $i \in I$ with $w_i > 1$, in which case $w_i' = 1$ and hence $w'(I) \ge 1$; also $w(I) \ge w_i > 1$. Thus the events $\{w(I) \ge 1\}$ and $\{w'(I) \ge 1\}$ coincide for every I. □

3 A with/without-replacement lift

Definition 3.1 (Without and with replacement subset sums)**.** Fix $w \in [0,1]^n$.

- **(Without replacement)** Let $I \sim \mathrm{Unif}\big(\binom{[n]}{s}\big)$ and set
$$S_{\mathrm{wor}}(w) := \sum_{i \in I} w_i.$$
- **(With replacement)** Let $J_1, \dots, J_s$ be i.i.d. uniform on $[n]$ and set
$$S_{\mathrm{wr}}(w) := \sum_{t=1}^{s} w_{J_t}.$$

Let D be the event that $J_1, \dots, J_s$ are all distinct.

Lemma 3.2 (Conditional equality and a sharp error bound)**.** *For all* $w \in [0,1]^n$*,*
$$\mathbb{P}\big(S_{\mathrm{wor}}(w) \ge 1\big) = \mathbb{P}\big(S_{\mathrm{wr}}(w) \ge 1 \mid D\big).$$

Moreover,
$$\big|\mathbb{P}(S_{\mathrm{wor}}(w) \ge 1) - \mathbb{P}(S_{\mathrm{wr}}(w) \ge 1)\big| \le \mathbb{P}(D^c) = 1 - \frac{(n)_s}{n^s} \le \frac{\binom{s}{2}}{n}. \tag{6}$$

Proof. On D, the ordered tuple $(J_1, \dots, J_s)$ is uniform over all s-tuples of distinct elements; forgetting the order yields a uniform random s-subset I. Since $S_{\mathrm{wr}}(w)$ depends only on the set of sampled indices, conditioning on D matches the distribution of $S_{\mathrm{wor}}(w)$. This yields the conditional equality.

For the inequality, write by total probability
$$\mathbb{P}(S_{\mathrm{wr}} \ge 1) = \mathbb{P}(S_{\mathrm{wr}} \ge 1 \mid D)\mathbb{P}(D) + \mathbb{P}(S_{\mathrm{wr}} \ge 1 \mid D^c)\mathbb{P}(D^c).$$

Substitute $\mathbb{P}(S_{\mathrm{wr}} \ge 1 \mid D) = \mathbb{P}(S_{\mathrm{wor}} \ge 1)$ and rearrange:
$$\mathbb{P}(S_{\mathrm{wor}} \ge 1) - \mathbb{P}(S_{\mathrm{wr}} \ge 1) = \mathbb{P}(D^c)\Big(\mathbb{P}(S_{\mathrm{wor}} \ge 1) - \mathbb{P}(S_{\mathrm{wr}} \ge 1 \mid D^c)\Big),$$

so the absolute value is at most $\mathbb{P}(D^c)$. Finally,
$$\mathbb{P}(D) = \frac{(n)_s}{n^s} = \prod_{j=0}^{s-1}\left(1 - \frac{j}{n}\right) \ge 1 - \sum_{j=0}^{s-1} \frac{j}{n} = 1 - \frac{\binom{s}{2}}{n},$$

using the inequality $\prod_j (1 - a_j) \ge 1 - \sum_j a_j$ for $a_j \in [0,1]$. □

4 The i.i.d. tail problem and an explicit EMC-based range

Definition 4.1 (i.i.d. extremal tail probability)**.** For $s \geq 1$ and $x \geq 0$, define

$$p_s(x) := \sup \Big\{ \mathbb{P}(X_1 + \cdots + X_s \geq 1) : \ X_i \text{ i.i.d.}, \ X_i \geq 0, \ \mathbb{E}[X_i] \leq x \Big\}.$$

By scaling, the threshold is normalized to 1. Note $p_s(x) = 1$ for $x \geq 1/s$.

Theorem 4.2 (Łuczak–Mieczkowska–Šileikis; Frankl–Kupavskii)**.** *For every $s \geq 1$ and $0 \leq x \leq 1/s$, the conjectured formula*

$$p_s(x) = \max\{1 - (1 - x)^s, \ (sx)^s\} \tag{7}$$

holds on the following proven ranges:

(i) It holds for $s = 3$ and all $x \in [0, 1/3]$ [15].

(ii) For general s, it holds for $x \leq 1/(2s - 1)$ [15].

(iii) For general s, it holds for $x \leq 3/(5s - 2)$ [16, Corollary 26]; see also [17].

5 Main bounds for the accumulation game

5.1 An upper bound via i.i.d. tails

Proof of Theorem 1.2. Set $x := h/n$. By theorems 2.3 and 2.5, it suffices to take the supremum in (5) over $w \in [0, 1]^n$. Fix such w. By theorem 3.2,

$$\mathbb{P}(S_{\text{wor}}(w) \geq 1) \leq \mathbb{P}(S_{\text{wr}}(w) \geq 1) + \mathbb{P}(D^c).$$

The random variable $S_{\text{wr}}(w)$ is a sum of s i.i.d. draws from the empirical distribution on $\{w_1, \ldots, w_n\}$, whose mean is $\frac{1}{n} \sum_i w_i \leq x$. Hence $\mathbb{P}(S_{\text{wr}}(w) \geq 1) \leq p_s(x)$ by definition of p_s. Taking the supremum over feasible w yields (2). □

Remark 5.1 (A simple comparison to Markov bounds)**.** Since $1 - (1 - x)^s \leq sx$ for $x \in [0, 1]$ and $(sx)^s \leq sx$ for $x \leq 1/s$, we have $M(x) \leq sx$ on the entire nontrivial regime $x \in (0, 1/s)$. Thus, whenever the explicit formula $p_s(x) = M(x)$ is available, theorem 1.2 refines the Markov-type upper bound $V(n, s, h) \leq sh/n$ (or $\lfloor sh \rfloor / n$ as in [14, Theorem 4]) by replacing sx with the strictly smaller $M(x)$, up to the collision term $\Delta_{\text{coll}}(n, s)$.

5.2 Two canonical two-level lower bounds

Definition 5.2 (Two canonical two-level strategies)**.** Let $k_1 := \lfloor h \rfloor$ and $k_2 := \lfloor sh \rfloor$.

- **Unit weights**: place weight 1 on k_1 locations and 0 elsewhere.
- $1/s$**-plateau**: place weight h/k_2 on k_2 locations and 0 elsewhere.

Lemma 5.3 (Exact win probabilities of the canonical strategies)**.** *Define*

$$A_{n,s}(k) := \frac{\binom{n-k}{s}}{\binom{n}{s}}, \qquad B_{n,s}(k) := \frac{\binom{k}{s}}{\binom{n}{s}},$$

with the convention $B_{n,s}(k) = 0$ for $k < s$. Then the unit-weights strategy wins with probability $1 - A_{n,s}(k_1)$. The $1/s$-plateau strategy wins with probability at least $B_{n,s}(k_2)$. Consequently,

$$V(n, s, h) \geq \max\left\{1 - A_{n,s}(k_1),\ B_{n,s}(k_2)\right\}. \tag{8}$$

Proof. For unit weights, the Hider loses iff the unsearched s-set avoids all k_1 unit piles, which has probability $A_{n,s}(k_1)$.

For the plateau strategy, $h/k_2 \geq 1/s$ by definition of $k_2 = \lfloor sh \rfloor$. Hence any unsearched s-set entirely contained in the support has total weight at least 1, so the win probability is at least $\mathbb{P}(I \subseteq K) = B_{n,s}(k_2)$. □

5.3 Hypergeometric-to-binomial approximation with explicit error

Lemma 5.4 (A product representation)**.** *For integers $n \geq s \geq 1$ and $k \in \{0, 1, \ldots, n\}$,*

$$A_{n,s}(k) = \prod_{j=0}^{s-1} \left(1 - \frac{k}{n-j}\right), \qquad B_{n,s}(k) = \prod_{j=0}^{s-1} \frac{k-j}{n-j}.$$

Proof. Directly from falling-factorial definitions. □

Lemma 5.5 (A telescoping bound for products)**.** *If $0 \leq u_j \leq v_j \leq 1$ for $j = 0, 1, \ldots, s-1$, then*

$$0 \leq \prod_{j=0}^{s-1} v_j - \prod_{j=0}^{s-1} u_j \leq \sum_{m=0}^{s-1} (v_m - u_m).$$

Proof. Write

$$\prod_{j=0}^{s-1} v_j - \prod_{j=0}^{s-1} u_j = \sum_{m=0}^{s-1} \Big(\prod_{j<m} v_j\Big)(v_m - u_m)\Big(\prod_{j>m} u_j\Big),$$

which is obtained by expanding the difference one factor at a time (a standard telescoping identity). Since all factors lie in $[0, 1]$, each term is between 0 and $(v_m - u_m)$, giving the bound. □

Lemma 5.6 (Explicit approximation bounds)**.** *Let $n \geq s \geq 1$.*

(a) If $0 \leq k \leq n - s$, then

$$0 \leq (1 - k/n)^s - A_{n,s}(k) \leq \frac{k}{n} \cdot \frac{\binom{s}{2}}{n - s + 1}.$$

(b) If $s \leq k \leq n$, then

$$0 \leq (k/n)^s - B_{n,s}(k) \leq \frac{n-k}{n} \cdot \frac{\binom{s}{2}}{n - s + 1}.$$

Proof. (a) By theorem 5.4, write $A_{n,s}(k) = \prod_{j=0}^{s-1} u_j$ with $u_j := 1 - \frac{k}{n-j}$ and let $v_j := 1 - \frac{k}{n}$. Then $0 \le u_j \le v_j \le 1$ because $n - j \le n$. Hence by theorem 5.5,

$$(1 - k/n)^s - A_{n,s}(k) \le \sum_{m=0}^{s-1} \left(\frac{k}{n-m} - \frac{k}{n} \right) = \sum_{m=0}^{s-1} \frac{km}{n(n-m)}.$$

Since $n - m \ge n - s + 1$ and $\sum_{m=0}^{s-1} m = \binom{s}{2}$, we obtain the stated upper bound.

(b) Similarly, write $B_{n,s}(k) = \prod_{j=0}^{s-1} u_j$ with $u_j := \frac{k-j}{n-j}$ and let $v_j := \frac{k}{n}$. One checks $0 \le u_j \le v_j \le 1$ because $\frac{k-j}{n-j} \le \frac{k}{n}$ is equivalent to $j(n-k) \ge 0$. Apply theorem 5.5 to obtain

$$(k/n)^s - B_{n,s}(k) \le \sum_{m=0}^{s-1} \left(\frac{k}{n} - \frac{k-m}{n-m} \right) = \sum_{m=0}^{s-1} \frac{m(n-k)}{n(n-m)} \le \frac{n-k}{n} \cdot \frac{\binom{s}{2}}{n-s+1}.$$

□

Lemma 5.7 (A Lipschitz bound for power maps)**.** *For any integer $s \ge 1$ and $u, v \in [0, 1]$,*

$$|u^s - v^s| \le s|u - v|, \qquad |(1-u)^s - (1-v)^s| \le s|u - v|.$$

Proof. Mean value theorem (since $|\frac{d}{dt} t^s| \le s$ and $|\frac{d}{dt}(1-t)^s| \le s$ on $[0, 1]$). □

5.4 An explicit sandwich bound in the Frankl–Kupavskii range

Proof of Theorem 1.3. We recall the notation

$$\Delta_{\text{coll}}(n, s) := 1 - \frac{(n)_s}{n^s}, \qquad \Delta_{\text{hyp}}(n, s) := \frac{s}{n} + \frac{\binom{s}{2}}{n-s+1}, \qquad M(x) := \max\{1-(1-x)^s,\ (sx)^s\}.$$

Set $x := h/n$ and assume $p_s(x) = M(x)$.

Upper bound. By theorem 1.2, $V \le p_s(x) + \Delta_{\text{coll}}(n, s)$. By assumption $p_s(x) = M(x)$, hence

$$V(n, s, h) \le M(x) + \Delta_{\text{coll}}(n, s).$$

Lower bound via the canonical two-level allocations. By theorem 5.3,

$$W_{\text{CS}}(n, s, h) = \max\{1 - A_{n,s}(k_1),\ B_{n,s}(k_2)\}, \qquad k_1 = \lfloor h \rfloor,\ k_2 = \lfloor sh \rfloor.$$

We bound each canonical benchmark against the corresponding component of $M(x)$.

First, using theorem 5.6(a) and $|k_1/n - x| \le 1/n$ together with theorem 5.7,

$$\left|\big(1 - A_{n,s}(k_1)\big) - \big(1 - (1-x)^s\big)\right| \le \frac{s}{n} + \frac{\binom{s}{2}}{n-s+1} = \Delta_{\text{hyp}}(n, s).$$

Second, using theorem 5.6(b) and $|k_2/n - sx| \le 1/n$,

$$\left|B_{n,s}(k_2) - (sx)^s\right| \le \frac{s}{n} + \frac{\binom{s}{2}}{n-s+1} = \Delta_{\text{hyp}}(n, s).$$

Taking the maximum preserves this error bound, so

$$W_{\text{CS}}(n, s, h) \ge M(x) - \Delta_{\text{hyp}}(n, s).$$

Since one always has $W_{\text{CS}}(n, s, h) \le V(n, s, h)$, this proves (3); subtracting $W_{\text{CS}}(n, s, h)$ from the middle inequality yields (4). □

Corollary 5.8 (Unconditional sandwich for $s = 3$)**.** *Let $n \geq 3$, $h \geq 1$, and $3h/n < 1$, and set $x := h/n \in (0, 1/3)$. Then*

$$M(x) - \Delta_{\text{hyp}}(n, 3) \leq W_{\text{CS}}(n, 3, h) \leq V(n, 3, h) \leq M(x) + \Delta_{\text{coll}}(n, 3),$$
$$M(x) = \max\{1 - (1 - x)^3,\ (3x)^3\}.$$

In particular, one of the two canonical two-level allocations of theorem 5.2 achieves $V(n, 3, h) \geq M(x) - \Delta_{\text{coll}}(n, 3) - \Delta_{\text{hyp}}(n, 3)$.

Proof. By theorem 4.2(i), the identity $p_3(x) = M(x)$ holds for all $x \in [0, 1/3]$ [15]. Apply theorem 1.3. □

Corollary 5.9 (Asymptotic value for fixed s in the FK range)**.** *Fix $s \geq 1$ and $x \in (0, 1/s)$ with $x \leq 3/(5s - 2)$. If $h = h_n$ and $n \to \infty$ satisfy $h_n/n \to x$, then*

$$V(n, s, h_n) = M(x) + O\left(\frac{s^2}{n}\right),$$

with an explicit finite-n certificate given by equations (3) and (4). In particular, $V(n, s, h_n) \to M(x)$ as $n \to \infty$.

Proof. Immediate from theorem 1.3 since $\Delta_{\text{coll}}(n, s) = O(s^2/n)$ and $\Delta_{\text{hyp}}(n, s) = O(s^2/n)$ for fixed s. □

6 Entropic scale selection on the two-level (Ruckle) family

6.1 Two-level family and its exact win probability

Definition 6.1 (Two-level allocation at scale k)**.** For $k \in \{1, \ldots, n\}$, fix a set $K \subset [n]$ with $|K| = k$ and define the truncated level

$$a_k := \min\{h/k, 1\} \in [0, 1].$$

Define the two-level vector

$$w_i^{(k)} = \begin{cases} a_k, & i \in K, \\ 0, & i \notin K. \end{cases}$$

Define the corresponding win probability

$$\Phi_{n,s,h}(k) := \mathbb{P}\big(S_{\text{wor}}(w^{(k)}) \geq 1\big).$$

Lemma 6.2 (Hypergeometric tail representation)**.** *Let $J := |I \cap K|$ for $I \sim \mathrm{Unif}\big(\binom{[n]}{s}\big)$. Then $J \sim \mathrm{Hypergeom}(n, k, s)$ and*

$$\Phi_{n,s,h}(k) = \mathbb{P}\left(J \geq m(k)\right) = \sum_{j=m(k)}^{s} \frac{\binom{k}{j}\binom{n-k}{s-j}}{\binom{n}{s}}, \qquad m(k) := \left\lceil \frac{k}{h} \right\rceil.$$

Proof. Let $I \sim \mathrm{Unif}\big(\binom{[n]}{s}\big)$ and set $J := |I \cap K|$. Then $J \sim \mathrm{Hypergeom}(n, k, s)$ and

$$S_{\text{wor}}(w^{(k)}) = \sum_{i \in I} w_i^{(k)} = J a_k, \qquad a_k = \min\{h/k, 1\}.$$

The win condition is $J a_k \geq 1$. If $h \leq k$, then $a_k = h/k$ and this is equivalent to $J \geq k/h$; if $h > k$, then $a_k = 1$ and the condition is $J \geq 1$. In both cases the threshold equals $m(k) = \lceil k/h \rceil$, which proves the formula. □

6.2 An entropic regularization over scales

Definition 6.3 (Entropic scale-selection problem)**.** Fix $\lambda > 0$. Over the simplex $\Delta_n := \{\pi \in \mathbb{R}^n_{\geq 0} : \sum_{k=1}^n \pi_k = 1\}$, consider

$$\max_{\pi \in \Delta_n} \left\{ \sum_{k=1}^{n} \pi_k \, \Phi_{n,s,h}(k) + \lambda H(\pi) \right\}, \qquad H(\pi) := -\sum_{k=1}^{n} \pi_k \log \pi_k.$$

Proposition 6.4 (Unique Gibbs optimizer)**.** *For every $\lambda > 0$, the entropic problem theorem 6.3 has a unique maximizer π^λ, given by*

$$\pi_k^\lambda = \frac{\exp\left(\Phi_{n,s,h}(k)/\lambda\right)}{\sum_{j=1}^{n} \exp\left(\Phi_{n,s,h}(j)/\lambda\right)}.$$

Proof. The objective is strictly concave in π because H is strictly concave on Δ_n. KKT conditions yield $\Phi(k) - \lambda(\log \pi_k + 1) + \alpha = 0$, hence the Gibbs form. □

Lemma 6.5 (Concentration onto near-optimal scales)**.** *Let $\Phi_{\max} := \max_{1 \leq k \leq n} \Phi_{n,s,h}(k)$ and fix $\delta > 0$. Define*

$$G_\delta := \{k \in \{1, \ldots, n\} : \; \Phi_{n,s,h}(k) \geq \Phi_{\max} - \delta\}.$$

Then for the Gibbs optimizer π^λ,

$$\pi^\lambda(G_\delta^c) \leq (n-1)e^{-\delta/\lambda}.$$

Proof. If $k \notin G_\delta$ then $\Phi(k) \leq \Phi_{\max} - \delta$ and $\exp(\Phi(k)/\lambda) \leq e^{-\delta/\lambda} \exp(\Phi_{\max}/\lambda)$. Comparing the total Gibbs mass outside G_δ to the mass of one maximizer gives the claim. □

Remark 6.6 (Canonical scales are near-optimal on the FK range)**.** Assume (SA) and $x = h/n \leq 3/(5s-2)$. Theorem 1.3 implies that one of the two canonical scales $k_1 = \lfloor h \rfloor$ or $k_2 = \lfloor sh \rfloor$ achieves a win probability within $\Delta_{\text{coll}}(n,s) + \Delta_{\text{hyp}}(n,s)$ of $M(x)$. In particular, restricted to the two-level family, the best scale is witnessed (up to this explicit error) by one of these two canonical constructions.

7 Concluding remarks

The paper establishes a transfer principle from i.i.d. extremal-tail identities to one-stage accumulation games. Once an identity for $p_s(x)$ is known at $x = h/n$, the accumulation-game value inherits an explicit upper bound through the coupling lift, and canonical two-level allocations supply a matching lower bound up to a finite-n correction. The resulting sandwich theorem gives a proved quantitative regime and, in that regime, a genuine partial result on Ruckle's conjecture.

The result is deliberately quantitative rather than exact. We do not classify all optimal supports or all optimal weight vectors, and we do not resolve theorem 1.1 in full. What we do prove is that on a mathematically explicit region the conjectured two-level family is already value-determining up to a completely explicit error. In that sense, the paper isolates a precise interface between extremal probability and accumulation games.

A natural next step is therefore immediate: enlarge the currently known range of the i.i.d. identity $p_s(x) = M(x)$, or prove it in full. Any such improvement would automatically enlarge the regime covered by our theorems. A second direction is to upgrade value control by canonical two-level allocations into exact structural optimality on larger parameter regions.

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