APPROACHING THE PRESCRIBED GAUSSIAN CURVATURE BY DISCRETE CONFORMALITY

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Abstract. We propose a discrete approach for approximating solutions to the prescribed Gaussian curvature problem in two-dimensional manifolds, based on the notion of discrete conformality. Our approach provides an efficient numerical method to compute the solution by minimizing a convex functional.

1. Introduction

The problem of the prescribed Gaussian curvature problem is classical and extensively studied in differential geometry. Suppose (M, g) is a 2-dimensional Riemannian manifold. Given a smooth function $\tilde{\kappa} : M \to \mathbb{R}$, the prescribed Gaussian curvature problem asks whether there exists a Riemannian metric $\tilde{g}$ on M such that the Gaussian curvature of $(M, \tilde{g})$ is exactly $\tilde{\kappa}$. By letting $\tilde{g} = e^{2\tilde{u}} g$ for some smooth function $\tilde{u} : M \to \mathbb{R}$, it suffices to solve the equation

$$\Delta_g \tilde{u} + \kappa = -e^{2\tilde{u}} \tilde{\kappa}, \tag{1}$$

where κ is the Gaussian curvature associated with the metric g, and Δ_g is the Laplace-Beltrami operator with respect to g. The prescribed Gaussian curvature problem has been extensively investigated, see [1], [4], [5], for examples.

This paper aims to give a discrete approach to the prescribed Gaussian curvature problem by approximating the solution u to the equation (1), for a class of 2-dimensional manifolds. A key ingredient is the theory of *discrete conformality*, which was developed in Luo [6] and Bobenko-Pinkall-Springborn [2] for *triangle meshes* based on *vetex scalings.*

Recall a well-known result for the existence of the solution to prescribed Gaussian curvature problem on surfaces with *genus* > 1 (Theorem 1.1 in [1], also see [4]).

Proposition 1.1. *Given a smooth function $\tilde{\kappa}(x) < 0$ on (M, g), there exists a unique function $\tilde{u}$ on M such that $e^{2\tilde{u}} g$ has the Gaussian curvature $\tilde{\kappa}$. Furthermore, such $\tilde{u}$ is smooth on M.*

This paper will describe a discrete approach to compute the conformal factor u in Proposition 1.1.

1.1. **Discrete conformality for polyhedral surfaces.** Let S be a closed topological surface and $\mathcal{T}$ be a triangulation of S. Let $F = F(\mathcal{T})$ denote the set of faces (2-simplexes) of $\mathcal{T}$, $E = E(\mathcal{T})$ denote the set of edges (1-simplexes) of $\mathcal{T}$, and $V = V(\mathcal{T})$ denote the set of vertices of $\mathcal{T}$. A *polyhedral metric* (PL metric) on $\mathcal{T}$ could be identified as an edge length function $l \in \mathbb{R}^E_{>0}$. We call the pair $(\mathcal{T}, l)$ a *polyhedral surface.* Given (T, l) and a triangle $\triangle ijk$ with the vertices $i, j, k \in V$, let θ^i_{jk} denote the inner angle at vertex i in the Euclidean triangle $\triangle ijk$.

Definition 1.2. *The discrete curvature is a function*

$$K : V \to (-\infty, 2\pi),$$

such that

$$K_i = 2\pi - \sum_{jk\in E:\triangle ijk\in F} \theta^i_{jk}.$$

Definition 1.3 (Discrete Conformality by Vertex Scaling, [6])**.** *Given a triangulation $\mathcal{T}$ on (S, V), two Euclidean polyhedral surfaces $(\mathcal{T}, l)$ and $(\mathcal{T}, l')$ are* discretely conformally equivalent*, if for some $u \in \mathbb{R}^{V(\mathcal{T})}$,*

$$l'_{ij} = e^{\frac{1}{2}(u_i+u_j)} l_{ij} \tag{2}$$

for any edge $ij \in E(\mathcal{T})$. We call such a u by discrete conformal factor*, and denote $l' = u * l$ if equation (2) holds.*

1.2. **Triangulations with mixed background constant curvatures.** Suppose $\mathcal{K} \in \mathbb{R}^{F(\mathcal{T})}_{<0}$ and $l \in \mathbb{R}^{E(\mathcal{T})}_{>0}$. Whenever well-defined, for any $\sigma \in F(\mathcal{T})$ we denote

(a) $g(\sigma, \mathcal{K}, l)$ as a Riemmanian metric on σ such that $(\sigma, g(\sigma, \mathcal{K}, l))$ is a geodesic triangle of the constant Gaussian curvature $\mathcal{K}(\sigma)$ such that the lengths of an edge e is

$$\frac{2}{-\mathcal{K}(\sigma)} \sinh^{-1}\left(\frac{-\mathcal{K}(\sigma)}{2} l(e)\right),$$

(b) $\theta^i_{jk} = \theta^i_{jk}(\mathcal{K}, l)$ as the value of inner angle at the vertex i in the geodesic triangle

$$(\triangle ijk, g(\triangle ijk, \mathcal{K}, l)),$$

and

(c) $K = K(\mathcal{K}, l) \in \mathbb{R}^V$ as a generalized discrete curvature such that

$$K_i = 2\pi - \sum_{jk \in E : \triangle ijk \in F} \theta^i_{jk}.$$

Remark 1.4. *Note that in item (c), the angle θ^i_{jk}'s at the same vertex i may not share the same metric as in the definition 1.2 of the* discrete curvature.

As observed in Bobenko-Pinkall-Springborn [2], if $\mathcal{K}(\sigma) = -1$, then $(\sigma, g(\sigma, \mathcal{K}, u * l))$ is discretely conformal to $(\sigma, g(\sigma, \mathcal{K}, l))$ as hyperbolic geodesic triangles with the discrete conformal factor u, in the notion of hyperbolic discrete conformality introduced in [2]. We can naturally extend their notion of hyperbolic discrete conformality to geodesic triangles of constant negative curvature $\mathcal{K}(\sigma)$. For fixed $\mathcal{K}$ and l, we denote $K(\mathcal{K}, u * l)$ by $K(u)$ for convenience.

Then by the variational principle developed by Luo [6] and Bobenko-Pinkall-Springborn [2]

$$\mathcal{F}(u) = \int \sum_{i \in V} K_i(u) du_i$$

is well-defined and locally strictly convex on the domain where $K(u)$ is well-defined.

Furthermore, Bobenko-Pinkall-Sprinborn [2] gave a simple and explicit formula extending $\mathcal{F}$ to a globally convex functional $\tilde{\mathcal{F}}$ on $\mathbb{R}^V$. So $K(u) = 0$ has at most one solution, and can be computed efficiently by minimizing a globally convex functional $\tilde{\mathcal{F}}$ on $\mathbb{R}^V$.

One can regard $\mathcal{K} \in \mathbb{R}^{F(\mathcal{T})}_{<0}$ as a discrete approximation of some smooth Gaussian curvature function $\tilde{\kappa} \in \mathbb{R}^S$. Thus, when $K(u) = 0$, one may regard $\cup_\sigma (\sigma, g(\sigma, \mathcal{K}, u*l))$ as an approximation of a Riemannian surface with the prescribed Gaussian curvature $\tilde{\kappa}$. Meanwhile, $u \in \mathbb{R}^{V(\mathcal{T})}$ may be regarded as a discrete approximation of the smooth conformal factor $u \in \mathbb{R}^S$ satisfying that $e^{2u} g$ has the Gaussian curvature $\tilde{\kappa}$.

1.3. **Geodesic triangulation, and discrete approximation of the prescribed curvature problem.** For triangulation $\mathcal{T}$, if any edge in $\mathcal{T}$ is a geodesic arc on S, we call such $\mathcal{T}$ a *geodesic triangulation.*

Suppose that $\mathcal{T}$ is a geodesic triangulation of (S, g) and V, E, F are defined the same way as in section 1. The edge length function $l \in \mathbb{R}^{E(\mathcal{T})}_{>0}$

is measured on (S,g), and the polyhedral surface $(\mathcal{T},l)$ is regarded as an approximation of (S,g).

Suppose $\tilde{\kappa}(x)<0$ is a smooth function on S, by Theorem 1.1, there exists a smooth function $\tilde{u}$ such that $e^{2\tilde{u}}g$ has the Gaussian curvature $\tilde{\kappa}$.

We will find a discrete approximation of $\tilde{u}$ using the information of $(\mathcal{T},l)$ and $\tilde{\kappa}$. For all $\sigma\in F(\mathcal{T})$, we let $\mathcal{K}(\sigma)=\tilde{\kappa}(x)$ for some $x\in\sigma$. Then by minimizing a convex functional on $\mathbb{R}^V$, we expect to find a unique discrete conformal factor $u\in\mathbb{R}^V$ such that $K(\mathcal{K},u*l)=0$ (i.e. $K(u)=0$). Such a discrete conformal factor u should be an approximation of $\tilde{u}$ in the sense of item (b) in Theorem 1.6.

We need some regularity of the triangulation to prove the convergence. Recall from section 1, for $i,j,k\in V$, $jk\in E$ and $ijk\in F$, we let θ^i_{jk} be the value of the inner angle at vertex i in the triangle $\triangle ijk$.

Definition 1.5. *A polyhedral surface $(\mathcal{T},l)$ is called* ϵ-acute *if*

$$\theta^i_{jk}\leq\frac{\pi}{2}-\epsilon$$

for all inner angle.

In this paper, if $x\in\mathbb{R}^A$ is a vector for some finite set A, we use $|x|$ to denote the infinity norm of x, i.e., $|x|=|x|_\infty=\max_{i\in A}|x_i|$. Now we state our main theorem.

Theorem 1.6. *Suppose (S,g) is a connected closed orientable smooth Riemannian surface with genus >1, $\tilde{\kappa}(x)<0$ is a smooth function on S, and $\tilde{u}$ is the smooth function on S such that $e^{2\tilde{u}}g$ has Gaussian curvature $\tilde{\kappa}(x)$. Assume $\mathcal{T}$ is a geodesic triangulation of (S,g), and $l\in\mathbb{R}^{E(T)}_{>0}$ denotes the edge length in (S,g). For any triangle σ in $F(\mathcal{T})$, let $\mathcal{K}(\sigma)=\tilde{\kappa}(x)$ for some $x\in\sigma$. Then for any $\epsilon>0$, there exist constants $\delta=\delta(S,g,\tilde{\kappa},\epsilon)>0$ and $C=C(M,g,\tilde{\kappa},\epsilon)$ such that if $(\mathcal{T},l)$ is ϵ-acute and $|l|<\delta$, then*

(a) there exists a unique discrete conformal factor $u\in\mathbb{R}^{V(\mathcal{T})}$, such that $K(\mathcal{K},u*l)=0$, and

(b) $\left|u-\tilde{u}|_{V(\mathcal{T})}\right|\leq C|l|$.

The main idea of the proof of Theorem 1.4 follows [7]. We describe a key discrete elliptic estimate in Section 2 and the formula of $\partial K/\partial u$ in Section 3. The proof of Theorem 1.4 is given in Section 4.

2. Discrete calculus on graphs

Assume $G=(V,E)$ is an undirected connected simple graph. Let ij denote an edge in E with endpoints i and j, and $i\sim j$ represent that the vertices i and j are connected by an edge $ij\in E$. Let $\mathbb{R}^E$ and $\mathbb{R}^E_A$ be vector spaces of dimension $|E|$ such that

(a) a vector $x\in\mathbb{R}^E$ is represented symmetrically, i.e., $x_{ij}=x_{ji}$, and

(b) a vector $x\in\mathbb{R}^E_A$ is represented anti-symmetrically, i.e., $x_{ij}=-x_{ji}$.

A vector $x\in\mathbb{R}^E_A$ is called a *flow* on G. An *edge weight* η on G is a vector in $\mathbb{R}^E$. Given an edge weight η, the *gradient* $\nabla f=\nabla_\eta f$ of a vertex function $f\in\mathbb{R}^V$ is a flow in $\mathbb{R}^E_A$ defined as

$$(\nabla f)_{ij}=\eta_{ij}(f_j-f_i)$$

where $f_i=f(i)$. Given a flow $x\in\mathbb{R}^E_A$, its *divergence* $div(x)$ is a vector in $\mathbb{R}^V$ such that

$$div(x)_i=\sum_{j\sim i}x_{ij}.$$

Given an edge weight η, the associated *Laplacian* $\Delta=\Delta_\eta:\mathbb{R}^V\to\mathbb{R}^V$ is defined as $\Delta f=div(\nabla_\eta f)$, i.e.,

$$(\Delta f)_i=\sum_{j\sim i}\eta_{ij}(f_j-f_i).$$

A Laplacian Δ_η is a linear transformation on $\mathbb{R}^V$, and can be identified as a $|V|\times|V|$ symmetric negative semi-definite matrix.

Now we introduce the notion of C-isoperimetry for a graph $G=(V,E)$ associated with a positive vector $l\in\mathbb{R}^E_{>0}$. Given any $V_0\subset V$, denote

$$\partial V_0=\{ij\in E: i\in V_0, j\notin V_0\},$$

and then define the *l-perimeter* of V_0 and the *l-area* of V_0 as

$$|\partial V_0|_l=\sum_{ij\in\partial V_0}l_{ij}\quad\text{and}\quad|V_0|_l=\sum_{i,j\in V_0, ij\in E}l_{ij}^2$$

respectively.

For a constant $C>0$, such a pair (G,l) is called *C-isoperimetric* if for any $V_0\subset V$

$$\min\{|V_0|_l,|V|_l-|V_0|_l\}\le C\cdot|\partial V_0|_l^2.$$

We will see, from part (b) of Lemma 4.2, that a uniform C-isoperimetric condition is satisfied by regular geodesic triangulations approximating a closed smooth surface. The following discrete elliptic estimate is a key ingredient of the proofs of our main theorems.

Lemma 2.1 (Wu-Zhu [7])**.** *Assume (G,l) is C_1-isoperimetric, and $x \in \mathbb{R}^E_A, \eta \in \mathbb{R}^E_{>0}, C_2 > 0, C_3 > 0$ are such that*

(i) $|x_{ij}| \leq C_2 l_{ij}^2$ for any $ij \in E$, and

(ii) $\eta_{ij} \geq C_3$ for any $ij \in E$.

Then

$$|\Delta_\eta^{-1} \circ div(x))| \leq \frac{4C_2\sqrt{C_1+1}}{C_3}|l| \cdot |V|_l^{1/2}.$$

Furthermore, if $y \in \mathbb{R}^V$ and $C_4 > 0$ and $D \in \mathbb{R}^{V\times V}$ is a non-zero diagonal matrix such that

$$|y_i| \leq C_4 D_{ii}|l| \cdot |V|_l^{1/2}$$

for any $i \in V$, then

$$|(D-\Delta_\eta)^{-1}(div(x)+y)| \leq \left(C_4 + \frac{8C_2\sqrt{C_1+1}}{C_3}\right)|l| \cdot |V|_l^{1/2}.$$

3. Differential of the discrete curvatures and angles

The following proposition can be easily derived from Proposition 6.1.7 in [2] or Lemma 3.4 in [7].

Proposition 3.1. *Given $(\mathcal{T},l)$, $\mathcal{K} \in \mathbb{R}^{F(\mathcal{T})}_{<0}$, and $u \in \mathbb{R}^{V(\mathcal{T})}$ such that $K(u)$ is well-defined, denote*

$$\hat{\theta}^i_{jk}(u) = \frac{1}{2}(\pi + \theta^i_{jk}(u) - \theta^j_{ik}(u) - \theta^k_{ij}(u))$$

and

$$\lambda_{ij,k} = \frac{\mathcal{K}(\triangle ijk)^2 \cdot (u*l)^2_{ij}}{\mathcal{K}(\triangle ijk)^2 \cdot (u*l)^2_{ij} + 4}$$

for all triangle $\triangle ijk \in F(\mathcal{T})$. Then

$$\frac{\partial K}{\partial u}(u) = D(u) - \Delta_{\eta(u)}$$

where

$$\eta_{ij}(u) = \frac{1}{2}\cot\hat{\theta}^k_{ij}(u)(1-\lambda_{ij,k}) + \frac{1}{2}\cot\hat{\theta}^{k'}_{ij}(u)(1-\lambda_{ij,k'}),$$

and $D = D(u)$ is a diagonal matrix such that

$$D_{ii}(u) = \sum_{\triangle ijk \in F}(\cot\hat{\theta}^k_{ij}(u)\lambda_{ij,k} + \cot\hat{\theta}^{k'}_{ij}(u)\lambda_{ij,k'}).$$

4. Proof of the Main Theorems

In this section we first introduce two geometric lemmas, and then prove Theorem 1.6 in Subsections 4.2.

4.1. Two geometric lemmas.

The following lemma was first proved in [3], and was also proved in [7].

Lemma 4.1. *Suppose (S,g) is a closed Riemannian surface, and $u \in C^\infty(S)$. Then there exists $C = C(S,g,u) > 0$ such that for any $x, y \in S$,*

$$|d_{e^{2u}g}(x,y) - e^{\frac{1}{2}(u(x)+u(y))} d_g(x,y)| \le C d_g(x,y)^3,$$

where $d_{e^{2u}g}(x,y)$ and $d_g(x,y)$ are the distance metrics from x to y under $e^{2u}g$ and g respectively.

The following Lemma 4.2 is similar to Lemma 4.4 in [7] and can be proved almost in the same way.

Lemma 4.2. *Suppose (S,g) is a closed Riemannian surface, and $\mathcal{T}$ is a geodesic triangulation of (S,g) with V, E the set of vertices and edges. Let $l \in \mathbb{R}^{E(\mathcal{T})}$ be the geodesic lengths of the edges and assume that $(\mathcal{T}, l)$ is ϵ-acute.*

(a) Given $\tilde{u} \in C^\infty(S)$, there exists a constant $\delta = \delta(S,g,\tilde{u},\epsilon) > 0$ such that if $|l| < \delta$ then there exists a geodesic triangulation $\mathcal{T}'$ in $(S, e^{2\tilde{u}}g)$ such that $V(\mathcal{T}') = V(\mathcal{T})$, and $\mathcal{T}'$ is homotopic to $\mathcal{T}$ relative to $V(\mathcal{T})$. Furthermore, $(S, V, \bar{l})$ is $\frac{1}{2}\epsilon$-acute where $\bar{l} \in \mathbb{R}^{E(\mathcal{T}')}$ denotes the geodesic lengths of the edges of $\mathcal{T}'$ in $(S, e^{2\tilde{u}}g)$.

(b) There exist constants $\delta = \delta(S,g,\epsilon)$ and $C = C(S,g,\epsilon)$ such that if $|l| < \delta$, then $(\mathcal{T}, l)$ is C-isoperimetric.

4.2. Proof of Theorem 1.6.

For a constant $\epsilon > 0$, we assume that $(\mathcal{T}, l)$ is ϵ-acute and $|l| \le \delta$ where $\delta = \delta(S,g,\tilde{\kappa},\epsilon) < 1$ is a sufficiently small constant to be determined. By Lemma 4.2, if δ is sufficiently small there exists a geodesic triangulation $\mathcal{T}'$ of $(S, e^{2\tilde{u}}g)$ homotopic to $\mathcal{T}$ relative to $V(\mathcal{T}) = V(\mathcal{T}')$. Let $\bar{l} \in \mathbb{R}^{E(\mathcal{T})} \cong R^{E(\mathcal{T}')}$ denote the geodesic lengths of edges of $\mathcal{T}'$ in $(S, e^{2\tilde{u}}g)$.

For simplicity, we will frequently use the notion $A = O(B)$ to denote that if $\delta = \delta(S,g,\tilde{\kappa},\epsilon)$ is sufficiently small, then $|A| \le C \cdot |B|$ for some constant $C = C(S,g,\tilde{\kappa},\epsilon)$. For example, we have that

(a) $l_{ij} = O(l_{jk})$ for any $\triangle ijk \in F(\mathcal{T})$,

(b) $(\tilde{u} * l)_{ij} = O(l_{ij})$,

(c) $\bar{l}_{ij} = O(l_{ij})$, and

(d) $l_{ij} = O(\frac{2}{-\mathcal{K}(\triangle ijk)} \sinh^{-1}(\frac{-\mathcal{K}(\triangle ijk)}{2} l_{ij}))$.

The remaining of the proof is divided into two steps.

(1) Firstly we show that
$$(\tilde{u} * l)_{ij} - \bar{l}_{ij} = O(l_{ij}^3)$$
and
$$K(\tilde{u}) = div(x) + y$$
for some $x \in \mathbb{R}^E_A$ and $y \in \mathbb{R}^V$ such that $x_{ij} = O(l_{ij}^2)$ and $y_i = O(l_{ij}^3)$.

(2) Secondly, we construct a smooth path $\underline{u}(t) : [0,1] \to \mathbb{R}^V$ with $u(0) = \tilde{u}|_V$ such that the following identity holds
$$K(\underline{u}(t)) = (1-t)K(\tilde{u}).$$
Furthermore, we show that $|\underline{u}'(t)| = O(|l|)$, and then $K(\underline{u}(1)) = 0$ and $\underline{u}(1) - \tilde{u} = O(|l|)$.

Therefore the vertex function $\underline{u}(1)$ is the wanted discrete conformal factor for $(\mathcal{T}, l)$ in Theorem 1.6.

4.2.1. *Part 1.* By Lemma 4.2, $(S, V, \bar{l})$ is $\frac{1}{2}\epsilon$-acute if δ is sufficiently small. For simplicity we denote $\tilde{u}|_{V(\mathcal{T})}$ by $\tilde{u}$. By Lemma 4.1, we have that
$$\bar{l}_{ij} - (\tilde{u} * l)_{ij} = O(l_{ij}^3).$$
Notice that
$$\tilde{\kappa}(x) - \mathcal{K}(\triangle ijk) = O(l_{ij})$$
for all $\triangle ijk \in F(\mathcal{T})$ and $x \in \triangle ijk$. Denote $\tilde{\theta}^i_{jk}$ as the inner angle of the geodesic triangle $(\triangle ijk, e^{2\tilde{u}}g)$ in $\mathcal{T}'$. By standard estimates on geodesic triangles with bounded Gaussian curvature, we have that
$$\alpha^i_{jk} := \tilde{\theta}^i_{jk} - \theta^i_{jk}(\tilde{u}) = O(l_{ij}^2), \tag{3}$$
and
$$\alpha^i_{jk} + \alpha^j_{ik} + \alpha^k_{ij} = O(l_{ij}^3). \tag{4}$$

So $(\mathcal{T}, \tilde{u}*l)$ is $\frac{1}{3}\epsilon$-acute if δ is sufficiently small. Let $x \in \mathbb{R}^E_A$ and $y \in \mathbb{R}^V$ be such that

(5)
$$x_{ij} = \frac{\alpha^i_{jk} - \alpha^j_{ik}}{3} + \frac{\alpha^i_{jk'} - \alpha^j_{ik'}}{3} \quad \text{and} \quad y_i = \frac{1}{3} \sum_{jk:\triangle ijk \in F(\mathcal{T})} (\alpha^i_{jk} + \alpha^j_{ik} + \alpha^k_{ij})$$

where $\triangle ijk$ and $\triangle ijk'$ are adjacent triangles. Then we have the following identity

$$div(x)_i + y_i = K_i(\tilde{u}).$$

By (3), (4), (5) and the fact that any vertex $i \in V(\mathcal{T}')$ has at most $2\pi/(\epsilon/2)$ neighbors, we have

$$x_{ij} = O(l^2_{ij}), \tag{6}$$

and

$$y_i = O(l^3_{ij}). \tag{7}$$

4.2.2. *Part 2.* Let

$$\tilde{\Omega} = \{u \in \mathbb{R}^V : u*l \text{ satisfies the triangle inequalities and } (S, V, u*l) \text{ is } \frac{\epsilon}{4}\text{-acute}\}$$

and

$$\Omega = \{u \in \tilde{\Omega} : |u - \tilde{u}| \leq 1,\ (S, V, u*l) \text{ is } \frac{\epsilon}{5}\text{-acute}\}.$$

Since $(\mathcal{T}, \tilde{u}*l)$ is $\frac{1}{3}\epsilon$-acute, $\tilde{u}$ is in the interior of Ω. Now consider the following ODE on $int(\tilde{\Omega})$,

(8)
$$\begin{cases} u'(t) = (\Delta_{\eta(u)} - D(u))^{-1} K(\tilde{u}) = (\Delta_{\eta(u)} - D(u))^{-1}(div(x) + y) \\ u(0) = \tilde{u} \end{cases},$$

where $D(u)$ and $\eta(u)$ are defined as in Proposition 3.1. For any triangle $\triangle ijk$ and $u \in \tilde{\Omega}$, we have

$$D_{ii}(u) \geq \epsilon' l^2_{ij}, \quad \text{and} \quad \eta_{ij}(u) \geq \epsilon' \tag{9}$$

for some constant $\epsilon' = \epsilon'(S, g, \tilde{\kappa}, \epsilon) > 0$.

The right-hand side of equation (8) is a smooth function of u, so the ODE (8) has a unique solution $u(t)$ satisfying

$$\frac{dK(\underline{u}(t))}{dt} = \frac{\partial K}{\partial u}(u)\underline{u}'(t) = (D(u) - \Delta_{\eta(u)})(\Delta_{\eta(u)} - D(u))^{-1} K(\tilde{u}) = -K(\tilde{u}).$$

Therefore

$$K(\underline{u}(t)) = (1 - t)K(\tilde{u}).$$

Assume the maximum existence interval of $u(t)$ in Ω is $[0, T_0)$ where $T_0 \in (0, \infty]$. By Lemma 4.2, when δ is sufficiently small, $(\mathcal{T}, l)$ is C-isoperimetric for some constant $C = C(S, g, \epsilon)$. Then for any $u \in \Omega$, $(S, V, u * l)$ with triangulation $\mathcal{T}$ is $(e^{4(|\tilde{u}|+1)}C)$-isoperimetric. It is not difficult to see

$$|V|_l = O(1)$$

and

$$1 = O(|V|_l).$$

Then by Lemma 2.1 and equation (6),(7),(9), for any $t \in [0, T_0)$

$$|u'(t)| = O(|l| \cdot |V|_l^{1/2}) = O(|l|). \tag{10}$$

If $T_0 \le 1$,

$$|\underline{u}(T_0) - \tilde{u}| = O(|l|) \quad \text{and} \quad \theta^i_{jk}(\underline{u}(T_0)) - \theta^i_{jk}(\tilde{u}) = O(|l|), \tag{11}$$

and then $\underline{u}(T_0) \in \text{int}(\Omega)$ if δ is sufficiently small. But this contradicts with the maximality of T_0. So $T_0 > 1$ and $K(u(1)) = 0$ and $|\underline{u}(1) - \tilde{u}| = O(|l|)$.

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