Reflected Brownian Motion and its Applications to Differential Geometry and Index Theory

DISSERTATION

Submitted in Partial Fulfillment of
the Requirements for
the Degree of

DOCTOR OF PHILOSOPHY

at the

NEW YORK UNIVERSITY
LEONARD N. STERN SCHOOL OF BUSINESS

by

Ziran Liu

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Ziran Liu

September 2023

To my family

ABSTRACT

Reflected Brownian Motion and its Applications to Differential Geometry and Index Theory

by

Ziran Liu

Advisors: Prof. Josh Reed, Ph.D. and Prof. Peter Lakner, Ph.D.

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This thesis is organized in two parts.

Part I studies reflected Brownian motion with drift in a wedge from the viewpoint of probability theory and stochastic-process limits. The main results establish existence and uniqueness for the submartingale problem with drift, the strong Markov property, several Feller properties, and quantitative information on the absorbed process and its hitting probability of the vertex. This part develops the probabilistic background that motivated

the present dissertation and provides a self-contained treatment of the wedge problem.

Part II turns to reflected Brownian motion on manifolds with boundary and to its applications to differential geometry and index theory. The guiding problem is to understand how reflected diffusion methods, mirror constructions, and boundary local-time asymptotics can be used in the study of local boundary value problems for Dirac-type operators and their associated index formulae. We formulate the local Dirac boundary problem, explain the heat-theoretic bridge from first-order boundary conditions to mixed second-order heat problems, review the relevant literature in heat-kernel index theory and stochastic analysis on manifolds with boundary, analyze in detail the method of Du and Hsu for the Gauss–Bonnet–Chern theorem with boundary, and record the current theorem-level progress of the present program. In particular, the second part isolates the critical boundary scaling slot, proves explicit supertrace cancellation results in simplified Clifford regimes, and identifies the residual localized coefficient that must ultimately be understood in order to obtain a probabilistic local index formula for Dirac-type operators with local boundary conditions.

Contents

List of Figures

Part I

Reflected Brownian Motion in a Wedge

Chapter 1

Background knowledge and motivations

1.1 Queueing theory

Queueing Theory is a field in probability theory. It has been widely studied, both in the theoretical side of itself and in its applications. For its nature, it always shows up in a universal collection of scenarios, such as economic models, tele-communication networks, industrial manufacture, computer systems, machine plants, etc.

In this section, we give an introduction to queueing theory. We will present the main definitions and results according to our purpose to further introduce heavy-traffic limit theorems and then reflected Brownian motions (RBM).

We begin by the basic objects in the study of Queueing Theory.

Let's take the classic customer-server model for convenience of explanation.

The queue-length (i.e., the number of customers in the queue) will be in the center of the concern in our study of queueing theory. Apparently, the queue-length is affected by two factors: the **arrival** of a customer into the queue and the **departure** of a customer from the queue.

The arrival of the sequence of customers is random, in queueing theory it is always characterized by characterizing the inter-arrival time or the number of customers arrived in a fixed amount of time. The idea is simple, one is to fix the number of arrival of customers and then we study the distribution of the inter-arrival time (the time needed for ONE new customer to arrive), the other is to fix the amount of time and see how many customer can arrive and characterize by giving its distribution.

To move on, we need the following **Kendall's notation**, which denotes the characteristics of a queue (or a queueing process).

Kendall's notation is a standard system used to describe and classify queueing models. This notation provides essential information about the primary characteristics of a queue.

In general, a queueing model in Kendall's notation is represented as:

$$A/S/c/K/m/Z$$

, in which the last three terms are optional, so we usually use it in a form of

$$A/S/c$$

.

Here's a breakdown of each component:

1. A **- Arrival Process:** Describes the statistical nature of the arrival process.

- M: Markovian or memoryless process (typically modeled as a Poisson process).
- D: Deterministic arrivals (constant time between successive arrivals).
- G: General distribution (no specific distribution is assumed).

2. S **- Service Time Distribution:** Describes the statistical distribution of the service times.

    - M: Memoryless service times (exponentially distributed).
    - D: Deterministic service times.
    - G: General distribution.

3. c - **Number of Servers:** A number representing how many servers/channels are in the system.

4. K - **System Capacity (optional):** Specifies the total number of entities (in service and in the queue) that the system can handle. If this component is omitted, it's assumed the system has infinite capacity.

5. m - **Population Size (optional):** Represents the source population size, or the total number of potential entities that could enter the system. If omitted, it's assumed the source population is infinite.

6. Z - **Queue Discipline (optional):** Describes the order or discipline in which entities in the queue are served.

    - $FIFO$ or $FCFS$: First In First Out (or First Come First Serve) — standard queue behavior.

- $LIFO$: Last In First Out.
- $SIRO$: Service in Random Order.
- PRI: Priority service.

In practice, the most common notation to come across is the $A/S/c$ format. The other parameters (like system capacity, population size, and queue discipline) are often omitted unless necessary for the context.

Some commonly seen examples include:

- $M/M/c$: A c servers queue where both the arrival process and service times are Markovian. This is one of the most basic and widely studied queueing models.
- $M/D/c$: A c servers queue where arrivals are Markovian, but service times are deterministic.
- $M/G/c$: A c servers queue where arrivals are Markovian, but service times follow a general distribution.

Now let us give some details of concepts and basic results of queueing theory for $M/M/1$ as a toy model .

1.2 M/M/1 queue

An $M/M/1$ queue is a model characterized by **single-server**, **Poisson arrival process**, and **exponential service times**.

Arrival Process: Let $N(t)$ be the number of arrivals in the interval $[0, t]$. The inter-arrival times are assumed to be exponentially distributed with parameter λ. Thus, $N(t)$ is a Poisson process with rate λ.

Service Times: Let S_i be the service time of the i^{th} customer. We assume S_i to be exponentially distributed with parameter μ, and independent of $N(t)$ and other S_j.

Number of Servers: There is a single server.

1.2.1 Key Parameters and Performance Measures

$$\lambda : \text{Average rate of customer arrivals.}$$

$$\mu : \text{Average rate at which a single server provides service.}$$

$$\rho = \frac{\lambda}{\mu} : \text{Server utilization.}$$

Performance measures of interest:

$$L : \text{Expected number of customers in the system.}$$

$$L_q : \text{Expected number of customers in the queue.}$$

$$W : \text{Expected waiting time in the system for a customer.}$$

$$W_q : \text{Expected waiting time in the queue for a customer.}$$

1.2.2 Equilibrium Equations

To derive the steady-state probabilities, consider the balance equations. Let p_n be the probability of having n customers in the system in the steady state.

Using the flow balance principle:

$$\lambda p_{n-1} = \mu p_n, \quad \text{for } n \geq 1 \tag{1.1}$$

From the above, we get:

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0, \quad \text{for } n \geq 1 \tag{1.2}$$

Where p_0 is the probability of having no customers in the system, and can be found using the normalization condition:

$$p_0 + p_1 + p_2 + \cdots = 1 \tag{1.3}$$

1.2.3 Furthermore on Performance Measures

Using the probabilities p_n:

$$L = \sum_{n=0}^{\infty} n p_n = \rho p_0 + 2\rho^2 p_0 + \cdots = \frac{\rho}{1-\rho} \tag{1.4}$$

From Little's formula:

$$L = \lambda W \implies W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda} \tag{1.5}$$

Similarly, for the queue:

$$L_q = L - \rho = \frac{\rho^2}{1 - \rho} \tag{1.6}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\rho}{\mu - \lambda} \tag{1.7}$$

1.3 Heavy-traffic conditions

In the study of queueing systems, understanding the system's behavior when it operates near its capacity is of great importance. Such scenarios, often termed as the 'heavy traffic' regime, provide insights into the potential bottlenecks, system resilience, and performance degradation as workloads approach system limits. As modern systems, from communication networks to manufacturing pipelines, increasingly push their operational boundaries, the heavy traffic analysis becomes a critical tool. This paper delves deep into the realm of heavy traffic in queueing processes, shedding light on the intricacies of system performance in near-capacity conditions, and offering analytical approximations that serve as vital instruments for system design, optimization, and management.

1.3.1 Heavy Traffic in Queueing Processes

Definition 1. *A queueing system is said to be in a* heavy traffic regime *when its workload is approaching the system's capacity. Formally, let λ be the arrival rate of customers to the system and μ be the service rate. The traffic intensity, denoted by ρ, is defined as:*

$$\rho = \frac{\lambda}{\mu}$$

.

A queueing system is in a heavy traffic condition when ρ approaches 1, i.e. $\rho \approx 1$, implying that the system's capacity is nearly saturated.

The **significance** of heavy-traffic: Studying queueing processes in heavy traffic is crucial because it allows us to understand the system's behavior under near-capacity conditions. Under heavy traffic, the variability in the queueing process and the associated waiting times often become pronounced, leading to potential service degradation.

The **Objective of Study** of heavy-traffic queueing processes: The primary aim of studying queueing systems in heavy traffic is to derive diffusion approximations. These approximations involve characterizing the behavior of the queue length or waiting time process as a stochastic differential equation, typically driven by Brownian motion or some related stochastic process.

The common frameworks are:

1. **Functional Central Limit Theorems (FCLT)**: Under heavy traffic conditions, appropriately centered and scaled queue lengths or waiting times can be shown to converge weakly to a specific stochastic process. This convergence is typically established

using functional central limit theorems.

2. **Diffusion Approximations**: Given the weak convergence from FCLT, the limiting stochastic process (usually a form of reflected Brownian motion) serves as a diffusion approximation for the queueing process under heavy traffic. Such approximations facilitate the computation of performance metrics like queue lengths, waiting times, and blocking probabilities in near-capacity regimes.

The importance of the study of the heavy-traffic regime is also in its **applications**: the insights derived from heavy-traffic studies enable better system design, capacity planning, and dynamic resource allocation. By understanding system behavior in its limits, we can design mechanisms to handle peak loads, ensuring system resilience and efficient utilization.

1.4 Diffusion approximations

We have already known the heavy traffic conditions, now we give two examples as to show the "limit theorems" under this kind of heavy traffic conditions.

1.4.1 $M/M/1$ Queue

Theorem 1.1. *Consider an $M/M/1$ queueing system with arrival rate λ and service rate μ. Let $Q(t)$ be the number of customers in the system at time t, and define the traffic intensity $\rho = \frac{\lambda}{\mu}$. If we denote $B(t)$ as a standard Brownian motion, then as the traffic intensity $\rho \to 1$, the scaled and centered queue length process converges in distribution to a reflected Brownian motion:*

$$\sqrt{\frac{1-\rho}{\rho^2}}(Q(\rho^{-2}t) - \rho t) \Rightarrow B^+(t) \tag{1.8}$$

where $\Rightarrow$ indicates convergence in distribution, and $B^+(t)$ is the standard Brownian motion $B(t)$ reflected to remain non-negative.

Proof Idea: The result can be derived by considering the Lindley recursion for the waiting times and applying a functional central limit theorem after proper scaling and centering. The functional central limit theorem for the $M/M/1$ queue, leading to a reflected Brownian motion approximation in heavy traffic, can be found in:

Reference: In [29], Feller, W. (1971). *An introduction to probability theory and its applications (Vol. 2)*. John Wiley & Sons.

1.4.2 $M/G/1$ Queue

Theorem 1.2. *Consider an $M/G/1$ queueing system operating under a First-Come-First-Serve (FCFS) discipline with arrival rate λ. Let S represent the generic service time with $E[S] = 1/\mu$ and variance $Var(S) = \sigma^2$. Define the traffic intensity as $\rho = \lambda E[S]$. Let $W(t)$ denote the virtual waiting time at time t. If we assume that the second moment $E[S^2] < \infty$, then as $\rho \to 1$, the scaled and centered virtual waiting time process converges in distribution to a centered reflected Brownian motion:*

$$\sqrt{\frac{1-\rho}{\lambda\sigma^2}}(W(\rho^{-1}t) - (1-\rho)\rho^{-1}t) \Rightarrow W^*(t) \tag{1.9}$$

where $W^*(t)$ is a reflected Brownian motion with zero mean and variance σ^2, reflected to stay non-negative.

Proof Idea: This result involves analyzing the $M/G/1$ waiting time recursion and then applying a functional central limit theorem, with appropriate scaling and centering, to the sequence of waiting times. The limiting process is the reflected Brownian motion due to the nature of the queue, ensuring that waiting times remain non-negative.

The functional central limit theorem and heavy traffic results for the $M/G/1$ queue can be found in:

Reference: In [47], Kingman, J. F. C. (1961). The single server queue in heavy traffic. *Proceedings of the Cambridge Philosophical Society, 57*(4), 902-904.

Additionally, Whitt's book provides a deeper dive into these approximations and more:

Reference: In [75], Whitt, W. (2002). *Stochastic-process limits.* Springer.

1.5 Reflected Brownian motion (RBM) as a main model of the limiting-process in the diffusion approximation

1.5.1 Functional Central Limit Theorems (FCLT) and the Rise of RBM in Queueing Systems

The Functional Central Limit Theorem, often referred to in the context of stochastic processes as the "invariance principle", establishes the convergence of rescaled versions of partial sums of stochastic processes to a limiting continuous Gaussian process.

Theorem 1.3 (Donsker's Invariance Principle). *Let $\{X_i\}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{V}ar[X_i] = \sigma^2 < \infty$. Define the* *partial sum process* $S_n(t) = \sum_{i=1}^{[nt]} X_i$*. Then, as* $n \to \infty$*, the rescaled partial sum process*

$$W_n(t) = \frac{1}{\sqrt{n}} S_n(t)$$

converges in distribution in the Skorohod space $D[0,1]$ *(of right-continuous functions with left limits on* $[0,1]$*) to a standard Brownian motion* $B(t)$*.*

Proof Idea:

1. *Finite Dimensional Distributions*: Begin by proving that for any finite collection of times $0 \le t_1 < t_2 < \cdots < t_k \le 1$, the joint distribution of $(W_n(t_1), W_n(t_2), \ldots, W_n(t_k))$ converges to the joint distribution of $(B(t_1), B(t_2), \ldots, B(t_k))$. This uses the classical Central Limit Theorem.

2. *Tightness*: To show convergence in the Skorohod space, it's necessary to prove that the family of stochastic processes $\{W_n(t)\}$ is tight. This means that for any $\epsilon > 0$, there exists a compact set $K \subseteq D[0,1]$ such that $\mathbb{P}(W_n \in K) > 1 - \epsilon$ for all n.

3. *Kolmogorov's Continuity Criterion*: Use Kolmogorov's criterion to establish the existence of a version of the Brownian motion that's continuous almost surely, which ensures convergence in $D[0,1]$.

4. *Merging the Results*: Combining the results from the above steps gives the desired convergence in distribution in the Skorohod space $D[0,1]$.

Reference:

Donsker, M. D. (1952). "Justification and Extension of Kolmogorov's Limit Theorems." *Proceedings of the National Academy of Sciences of the United States* *of America*, 38(3), 263-267.

Within the realm of queueing systems, this becomes particularly pertinent as we investigate the limiting behavior of queueing processes under suitable normalization, especially as the system is driven towards its capacity, commonly referred to as the "heavy traffic" regime.

To begin with, consider a queueing process $\{Q(t), t \geq 0\}$, where $Q(t)$ represents the queue length at time t. Under the usual conditions of stationarity and ergodicity, one can study the normalized centered and scaled process, $Q_n(t)$, defined as:

$$Q_n(t) = \frac{Q(nt) - n\mu t}{\sqrt{n}} \tag{1.10}$$

Here, μ represents the mean rate of arrivals (or service, depending on the context). The FCLT posits that, for a broad class of queueing models, $Q_n(t)$ converges weakly to a Gaussian process as $n \to \infty$.

However, the direct application of the FCLT might yield a Brownian motion which can take negative values, contradicting the physical reality of queues. This leads us to the realm of RBM. The Reflected Brownian Motion (RBM), defined as a Brownian motion reflected to remain non-negative, serves as the limiting process in these cases. Mathematically, if $B(t)$ is a standard Brownian motion, the RBM, $R(t)$, can be described as:

$$R(t) = B(t) + |\inf_{s \leq t} B(s)| \tag{1.11}$$

This reflection mechanism at the origin ensures that the process remains non-negative, thus preserving the intrinsic nature of queueing systems.

Furthermore, in systems where overloads are frequent, leading to long queues, the behavior of the queue length process, when suitably normalized, mimics the RBM. This is especially pronounced in systems operating under heavy traffic conditions, where the operational capacity closely matches the demand. The reflection at zero for the RBM effectively captures the system's inherent resilience in preventing queue lengths from becoming negative.

In conclusion, while the FCLT provides the foundational pathway to understand the limiting behaviors of queueing systems, it's the nuanced characteristics of RBM that align perfectly with the physical realities of queues, making it the primary candidate as the limiting process in diffusion approximations.

1.5.2 Reflected Brownian Motion: An Intrinsic Model for Queueing Systems

Within the domain of queueing theory, the Reflected Brownian Motion (RBM) and its variants have emerged as a frequently used model for the nuanced and multifaceted behavior of queuing systems, especially when venturing into the realms of their limiting behavior. This section meticulously examines the salient features of RBM that account for its primacy and resonance with real-world queueing dynamics.

1.5.2.1 Non-Negativity and Physical Constraints

The very nature of a queue precludes the possibility of negative lengths. Any model aspiring to be a representative of queueing dynamics needs to fundamentally respect this axiom. It's not merely about mathematical consistency, but more pertinently about physical realism. Queues, in their essence, can either be empty or possess a certain positive length. The concept of a queue having a negative length is an anathema to both theoretical constructs and real-world observations.

While the Brownian motion does capture the inherent stochasticity, its trajectory isn't constrained to remain non-negative, making it an inappropriate representation for queue lengths. Here, RBM gains precedence. By introducing a reflection mechanism at the boundary (typically at zero), RBM ensures adherence to the non-negativity principle.

Formally, let $B(t)$ denote a standard Brownian motion. The Reflected Brownian Motion, $R(t)$, can be defined as:

$$R(t) = B(t) + \sup_{0 \leq s \leq t} (-\min_{0 \leq u \leq s} B(u)) \tag{1.12}$$

This mathematical formulation guarantees that the value of the process $R(t)$ remains non-negative at all times.

1.5.2.2 Stochastic Variability: Capturing Randomness in Arrivals and Services

Another quintessential aspect of queueing systems is the inherent randomness in the arrivals and service processes. While determinism might provide some average insights, it fails to capture the intricacies and volatilities of practical scenarios. Hence, there's an intrinsic need for a stochastic model.

Regular Brownian motion, with its inherent randomness, certainly comes close but falls short due to its aforementioned unboundedness. RBM, inheriting the stochastic properties of Brownian motion while being disciplined by its reflection mechanism, emerges as an optimal fit. The stochastic properties of RBM effectively encapsulate the randomness in arrivals and services, providing a holistic representation of the system.

1.5.2.3 Reflection Mechanism: A Mirror to Real-world Dynamics

Beyond mere mathematical convenience, the reflection property of RBM carries profound real-world implications. Consider the scenario of a server system nearing its capacity. The buildup of the queue might lead to adaptive measures: increased service rates, activation of auxiliary servers, or even client redirection. Such mechanisms, designed to mitigate or prevent system overload, parallel the reflection mechanism intrinsic to RBM.

In a mathematical sense, when the process approaches a boundary (often zero in the context of queue lengths), the reflection ensures it remains within the permissible domain. In real-world analogies, this reflection can be likened to systemic responses invoked to manage and maintain service quality and efficiency.

In essence, the RBM's alignment with the core principles of queueing systems isn't coincidental. It emerges as an outcome of an intricate interplay of mathematical formulations and their resonances with real-world dynamics, making RBM an unparalleled model in the domain of queueing theory.

1.5.3 Insights and Implications from RBM-based Models

The salient attributes of Reflected Brownian Motion (RBM) in elucidating the nuanced intricacies of queueing systems cannot be overstated. While its descriptive power serves as a foundational scaffold, its analytical prowess opens the avenue for myriad insights and actionable stratagems. This section delves deep into these implications, intertwining them with mathematical formalisms.

1.5.3.1 System Dynamics and Behavioral Analysis

The RBM model's fidelity to realistic queueing dynamics offers an unparalleled lens into the intricate mechanics of such systems, especially in the vicinity of their operational thresholds.

1. **Queue Variability and Stochastic Analysis**: The RBM, represented as

$$R(t) = B(t) + \sup_{0 \leq s \leq t} (- \min_{0 \leq u \leq s} B(u)),$$

captures the stochastic variability of the queue lengths. The stochastic properties such as variance and autocorrelation can provide insights into traffic predictability. Peaks in variance, for instance, can signal potential system bottlenecks. See [80] for reference.

2. **Threshold Behavior and System Overflows**: From the reflection property, one can derive thresholds, say T, beyond which the system experiences overflows. Mathematically, the reflection point offers a pivotal insight:

$$P(R(t) \geq T),$$

where P denotes the probability. A higher value could be an indicator of frequent overflows, necessitating infrastructural enhancements or adaptive protocols. For more details, see [46].

1.5.3.2 Resource Management and Optimization

With the granular insights furnished by RBM, system architects and administrators are equipped with a potent toolkit for resource orchestration.

1. **Dynamic Allocation and Stochastic Control**: Based on the trajectory of the RBM, one can devise a stochastic control strategy $u(t)$ that modulates resource allocation in real-time. The objective could be to minimize the expected queue length, formulated as:

$$\mathbb{E}[\int_0^T R(t)u(t)dt],$$

where $\mathbb{E}$ denotes expectation and T is a suitable time horizon. For more details, see [39].

2. **Optimal Utilization and Performance Analysis**: Through the rigorous analysis of the RBM, optimal strategies $u^*(t)$ can be discerned, ensuring resources are optimally utilized. This minimizes operational costs while maximizing throughput, a balance given by:

$$\min_{u(t)} \mathbb{E}[\int_0^T c(u(t))dt + h(R(T))],$$

where c represents cost and h encapsulates a suitable performance metric. Further details can be found in [13]

3. **Metrics Derivation**: Metrics such as mean waiting time W, throughput λ, and service rate μ can be derived from the RBM model. By analyzing the steady-state properties, one can express:

$$W = \frac{L}{\lambda},$$

where L represents the mean number of customers in the system, yielding insights into system efficiency and areas of enhancement. See [66] for further details.

To encapsulate, the probing into Reflected Brownian Motion as a modeling paradigm for queueing systems isn't only a pure academic exercise. It has been spread over academic research (both theoretical and applied), engineering, industrial and a lot more, each mathematical twist and turn shedding light on profound and actionable insights with a vast spectrum of real-world ramifications.

Chapter 2

Reflected Brownian motion and its formulations

2.1 1-dim RBM on the positive half of real line

Reflected Brownian Motion in one dimension is a cornerstone in stochastic process analysis, given its wide applicability and profound theoretical importance.

2.1.1 An explicit pathwise formula of 1-Dimensional RBM

A particularly good character of 1-dim RBM is that it possesses an explicit path wise formula due to the simplicity of the structure of its state-space.

RBM in one dimension is denoted by $Z(t)$ and is defined as:

$$Z(t) = B(t) + \sup_{0\leq s\leq t}\left(-\min_{0\leq u\leq s} B(u)\right), \tag{2.1}$$

where $B(t)$ represents a standard Brownian motion.

2.1.2 1-dim Skorokhod Problem

The Skorokhod problem provides a formulation for the reflection mechanism in RBM. Given a continuous function $x(t)$, the problem seeks a pair of functions $(z(t), l(t))$ such that:

$$z(t) = x(t) + l(t) \geq 0, \tag{2.2}$$

$$l(t) \text{ is non-decreasing,} \tag{2.3}$$

$$l'(t) = 0 \text{ wherever } y(t) > 0. \tag{2.4}$$

2.1.3 Stationary Distribution

For a 1-dimensional RBM Z_t, under conditions when the driving Brownian motion has a drift μ toward the origin and variance σ^2, (i.e. negative drift for RBM on the positive half real line) the stationary distribution is:

$$\mathbb{P}(Z_t \leq x) = 1 - \exp\left\{\frac{2\mu x}{\sigma^2}\right\}, \text{ for } \mu < 0. \tag{2.5}$$

See Harrison [36] page 15 for details. It should be pointed out that in Harrison [36], the 1-dim has been investigated very extensively, but in another name called "regulated Brownian motion".

2.1.4 Tanaka's Formula

2.1.4.1 Local Time

For a one-dimensional Brownian motion B_t, the local time L_t^a at level a up to time t is defined as follows:

$$L_t^a = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|B_s - a| < \varepsilon\}} \, ds.$$

Intuitively, this measures the "amount of time" the Brownian motion has spent around the level a up to time t, accumulating whenever the process is near a.

2.1.4.2 The theorem of Tanaka's Formula

Let B_t be a one-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for all $t \geq 0$:

$$|B_t| = B_0 + \int_0^t \operatorname{sgn}(B_s) \, dB_s + \frac{1}{2} L_t^0.$$

Here, sgn denotes the sign function, defined by:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Proof

The proof utilizes Itô's formula for semimartingales. We aim to apply Itô's formula to the function $f(x) = |x|$. The challenge is that f is not differentiable at 0, but a careful treatment of the terms yields the result.

1. For $x \neq 0$, $f'(x) = \text{sgn}(x)$. For $x = 0$, we'll define $f'(0) = 0$, keeping in mind the derivative does not actually exist at this point.

2. Applying Itô's formula to f and considering the properties of the quadratic variation of the Brownian motion:

$$df(B_t) = f'(B_t)\, dB_t + \frac{1}{2} f''(B_t)\, d[B]_t.$$

   Given that $f''(B_t)$ does not exist, the second term poses an issue. However, due to the behavior of the Brownian motion, this term is only problematic where $B_t = 0$.

3. Observing the behavior of B_t around 0, the local time L_t^0 accounts for the "missing" term in the Itô's formula application, giving the Tanaka's formula in its entirety.

2.2 RBM in an Orthant

RBM in an orthant is the roughly speaking the second complicated case than RBM on the positive half of real line. However, being in an orthant ($dim \geq 2$) is still essentially make the problem go to another essentially more complicated level. We summarize the seminal results among many exciting ones here, to show the monuments of the research on this topic, as well as some possible futuristic research problems and concerns

2.2.1 RBM in an Orthant: Fundamental Results

1. **Existence and Uniqueness:** There exists a unique solution to the Skorokhod problem in the orthant, corresponding to RBM.

   Reference: Harrison, Reiman (1981) [37].

2. **Stationary Distributions:** Under specific conditions, RBM in an orthant admits a stationary distribution. The existence of such a distribution provides a foundation for many applications and further theoretical advancements.

   Reference: Williams (1987) [78].

3. **Decomposition Principle:** Certain network structures allow RBM in an orthant to be decomposed into simpler, lower-dimensional RBMs.

   Reference: Dai (1995) [18].

2.2.2 Applications and Extensions

1. **Queueing Networks:** RBM in orthants can be utilized to model and analyze queueing networks, especially in heavy traffic scenarios.

   Reference: Harrison (1987) [39].

2. **Stochastic Control & Optimization:** RBMs can be controlled to optimize performance measures. This area merges control theory, optimization, and stochastic processes.

   Reference: Sznitman (1991) [72].

3. **Higher-Dimensional Analysis:** As systems become more intricate, there's been a surge in interest in analyzing RBM behavior in higher-dimensional orthants, especially

concerning corner reflections.

Reference: Dupuis and Williams (1994) [27].

2.2.3 Trends in the Research and Challenges (as of 2023)

1. **Scaling and Computational Aspects:** With the rise of large-scale systems and networks, there's a growing emphasis on the computational aspects of RBMs. Numerical methods, simulations, and approximations are areas of active research.

2. **Interplay with Modern Domains:** There's a push towards integrating the insights from RBM research with modern domains like machine learning, data centers, and cloud computing.

3. **Multi-class Networks:** Systems with multiple classes of jobs/customers/agents present a richer structure and complexity. This has become a focal point of recent investigations.

2.2.4 Open Problems and Challenges (as of 2023)

1. **Generalized Reflections:** While the behavior of RBM at boundaries is well-understood, how RBM behaves at the corners of higher-dimensional orthants, especially with non-standard reflection laws, remains an area to delve into.

2. **Transient Behavior:** Much of the existing work has focused on stationary behavior. Understanding the transient behavior of RBMs, especially in complex networks, remains challenging.

3. **Integration with Modern Statistical Methods:** With the avalanche of data in many applications, integrating RBM models with statistical methods to estimate model parameters and validate theoretical results is a growing need.

4. **Non-Markovian Models:** Extending the analysis of RBM to scenarios where the underlying driving processes are not Markovian is an ambitious and challenging frontier.

2.3 Skorokhod problem

The Skorokhod problem, which seeks regulated functions constrained by given barriers, has found significant applicability in studying reflected Brownian motion (RBM) [67]. By providing a robust framework for understanding reflection mechanisms, it allows for a deeper exploration of RBM's behavior at boundaries [36]. The multidimensional extensions further elucidate the complexities of RBM in higher-dimensional spaces [27]. This relationship between the Skorokhod problem and RBM serves as a bedrock in applications from queueing theory to advanced computational systems [78].

2.3.1 Skorokhod's Original Problem and Solution

In the **Paper** [68], by Skorokhod, A. V. (1956), "Limit theorems for stochastic processes". He first introduced a space of regulated functions, and posed the problem of finding a regulated version of a function that remains greater than a fixed barrier. And then he Provided conditions under which solutions exist. This is the first time the Skorokhod problem showed up.

2.3.2 Reflected Brownian Motion in an Orthant

In the **Paper** [38], by Harrison, J. M., and Reiman, M. I. (1981), "Reflected Brownian motion on an orthant", they introduced and rigorously formalized RBM in multidimensional spaces, and demonstrated existence and uniqueness of RBM in a non-negative orthant, and then showed that this RBM is a strong Markov process.

2.3.3 Multidimensional RBM and Stationary Distributions

In the **Paper** [40], by Harrison, J. M., and Williams, R. J. (1987). Multidimensional reflected Brownian motions having exponential stationary distributions." They studied RBM in multiple dimensions, derived conditions for RBM in an orthant to have exponential stationary distributions, and examined the construction and properties of these distributions.

2.3.4 The Skorokhod Problem in the Quadrant

In the **Paper** [26], by, Dupuis, P., and Ishii, H. (1991), "On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications." They investigated the Skorokhod problem in the two-dimensional quadrant, and explored Lipschitz continuity of the solution to the Skorokhod problem and conditions for its existence, and showed some applications of these results to stochastic differential equations with RBM.

2.3.5 Skorokhod Problem in smooth and non-smooth bounded Domains

In the **Paper** [52] by Paper: Lions, P.-L., and Sznitman, A.-S. (1984). "Stochastic Differential Equations with Reflecting Boundary Conditions." They developed a systematic framework for analyzing stochastic differential equations with reflecting boundary conditions, particularly in bounded Lipschitz domains, presented a new approach to solving the Skorokhod problem, which provided clearer insights into the structure and properties of solutions, and showed the strong connections between their framework and the behavior of RBM, establishing the foundation for further studies on reflected diffusions in various geometric domains, by using penalization methods.

2.3.6 Skorokhod Problem in General Domains

In the **Paper** [65] by Saisho, Y. (1987), "Stochastic differential equations for multi-dimensional domain with reflecting boundary." They extended Skorokhod problem to more general domains, and investigated the existence, uniqueness, and properties of solutions in these contexts, and then Demonstrated connections between RBM and the Skorokhod problem in these broader settings.

2.4 Submartingale problem

The first place where the submartingale problem shows up is the paper by Stroock and Varadhan [69]. In that paper, they formulated the submartingale problem for "diffusion processes with boundary conditions in a smooth domain"as an analogue to the martingale problem for "diffusion processes" [71] and [70].

The idea behind the submartingale problem is to characterize a diffusion associated to a second-order elliptic operator in some domain with boundary conditions, by a unique measure on the space of continuous functions (i.e. the space of sample paths of diffusion processes), which satisfies an initial condition and makes an integral functional of the underlying diffusion process a submartingale.

To this end, a diffusion process getting reflected in a constrained domain is characterized in an intrinsic way. By intrinsic, we mean that we don't need a boundary process which "pushes" the diffusion process back into the domain when it gets to the boundary, noticing that the boundary process has a different state space other than the state space of the diffusion process its own. Although, in most of the cases, such a boundary process can be proved to exist in a hindsight way.

However, the form of submartingale problem in our concern is from the paper by Varadhan and Williams [74], a submartingale problem in a more singular domain, the wedge-shaped area.

Definition 2 (Submartingale Problem). *A family of probability measures* $\{\mathbb{P}^z, z \in S\}$ *on* $(C_S, \mathcal{M})$ *is said to solve the submartingale problem if for each* $z \in S$*, the following three conditions hold,*

1. $\mathbb{P}^z(Z(0) = z) = 1$;

2. *For each* $f \in C_b^2(S)$*, the process*

$$\left\{ f(Z(t)) - \frac{1}{2}\int_0^t \Delta f(Z(s))ds, t \geq 0 \right\}$$

is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z)$ *whenever* f *is constant in a neighbourhood of the origin and satisfies* $D_i f \geq 0$ *on* ∂S_i *for* $i = 1, 2$*;*

3. $\mathbb{E}^z_\mu \left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}} dt \right] = 0.$

2.5 A brief guide to the literature from orthants to wedges and related domains

At this point it is useful to separate several related but genuinely different streams of work. First, there is the classical orthant theory of Harrison, Reiman, Williams and others, where reflected Brownian motion is often constructed through the Skorokhod map and where one has strong Markov, semimartingale, and stationary-distribution results under suitable reflection data [38, 40, 78]. Second, there is the broader Skorokhod-problem literature in smooth and non-smooth domains, including the work of Lions–Sznitman, Saisho, Dupuis–Ishii, and later developments on reflected SDEs and extended Skorokhod maps [52, 65, 26, 60]. Third, there is the singular-domain literature, especially wedges and cusps, where the process may fail to be a semimartingale and the pushing term may have infinite variation [74, 77, 23, 20].

The wedge problem studied in Part I sits exactly at the intersection of these streams. It is close enough to queueing-motivated reflected Brownian motion that the heavy-traffic and Skorokhod-map intuition remains essential, but it is singular enough that the classical semimartingale/sample-path tools are not universally available. This is why the submartingale problem becomes central here: it provides an intrinsic formulation that does not presuppose bounded-variation pushing terms, while still being robust enough to support existence, uniqueness, and Markov/Feller analysis.

A second important point is that the addition of a non-zero drift is not merely cosmetic. In queueing limits, the drift records first-order imbalance and is therefore part of the limiting model itself. In the orthant case one can often incorporate such a drift directly into the Skorokhod-map construction. In the wedge setting, however, once the process leaves the semimartingale regime this is no longer automatic. This is precisely where the present work enters the literature: it extends the wedge submartingale theory from the driftless case of [74] to the constant-drift case, while preserving the sharp phase transition in the parameter

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

Chapter 3

Introducing the underlying research problem

To set up the research problem, we study 2-dimensional Brownian motion with constant drift $\mu \in \mathbb{R}^2$ constrained to a wedge S in $\mathbb{R}^2$. This process may also be referred to as reflected Brownian motion (RBM) with drift in a wedge, and we denote the process itself by Z. For concreteness, we define the wedge in polar coordinates by $\{r \geq 0, 0 \leq \theta \leq \xi\}$ for some $0 < \xi < 2\pi$. Loosely speaking, the behavior of Z may be characterized as follows. In the interior of S, Z behaves as a 2-dimensional Brownian motion. On the other hand, the behavior of Z on the boundary of S is characterized by two reflection angles θ_1 and θ_2, depending upon whether the lower boundary ∂S_1 or upper boundary ∂S_2 has been reached. Both $-\pi/2 < \theta_1, \theta_2, < \pi/2$ and the angles are measured from their inward-facing normals, with positive angles corresponding to reflection toward the vertex of the wedge and negative angles away. See Figure below for an illustration.

One way to define RBM in a wedge is using a sample-path approach [17, 19, 38, 73] where

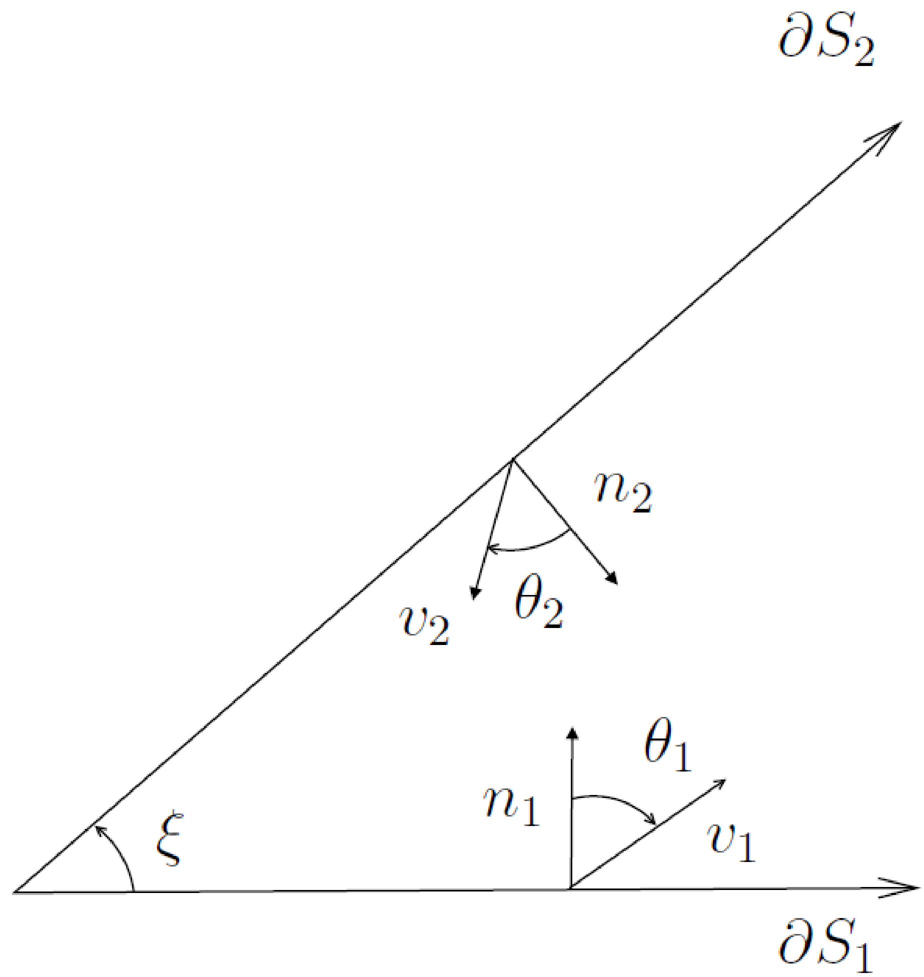


Figure 3.1: RBM in a Wedge

Z is explicitly characterized as the sum of a 2-dimensional Brownian motion on an arbitrary probability space [45, 51, 63] and a constraining or pushing process which satisfies the specifications related to the directions of reflection given above. This sample-path approach works with or without drift for some but not all parameter regimes of $(\xi, \theta_1, \theta_2)$. It tends not to work in regimes where Z is known not to be a semi-martingale [77] and the pushing process has infinite variation. Recent progress in this direction has however been made [43, 60].

A more probabilistic approach to defining RBM in a wedge was given by Varadhan and Williams [74]. In this case, Z is defined as the solution to a submartingale problem. This approach yields existence and uniqueness results for all parameter regimes but at several points the proofs of [74] rely heavily on the assumption that Z behaves as a standard Brownian motion inside of S. This is not an issue for parameter regimes where the sample-path approach described above may be applied because it is amenable to Brownian motions with drift, and the recent paper [44] demonstrates equivalence between the sample-path and the submartingale approach in such settings. On the other hand, in parameter regimes where the sample-path approach cannot be applied, extending the results of [74] in the direction of allowing Z to behave as a Brownian motion with drift in the interior of S remains an open problem. In this paper, we resolve this open problem for the case of a constant drift.

Remark 3.1. We conjecture that our results could be further generalized for the case when the drift is a bounded function of the current state, but this generalization is beyond the scope of the present paper.

As introduced in Chapter 1, the primary motivation comes from queueing theory where semi-martingale RBM with drift has long been known to serve as the weak limit of both the properly scaled queue length [12, 35, 37, 62] and workload [11, 14, 57, 79] processes of different queueing systems in heavy-traffic. In such queueing settings, the drift term arises as the result of an imbalance between the input and output processes to the system. The limiting RBM in these cases is often defined using the sample-path approach via the conventional Skorokhod map [38, 67, 75]. More recently, using the extended Skorokhod map [60], RBM with drift which is not a semi-martingale has been proven [61] to be the weak limit of the properly scaled unfinished work process of the generalized processor sharing model in heavy traffic. In this example, the sample-path approach is still employed to define the limiting process with the help of the extended Skorokhod map [60]. We conjecture however that there exist other applied queueing settings where the limiting heavy-traffic process is an RBM with drift which is not a semi-martingale and cannot be rigorously defined via the sample-path approach. One of these settings is the coupled processor model [15, 28]. In such situations, before proving any limit theorems, it is necessary to first establish the existence of RBM with drift through other means such as the submartingale problem.

Remark 3.2. We also mention that there exists a related stream of literature studying the behavior of reflected Brownian in smooth domains with cusps. The paper [23] appears to be the first to study reflected Brownian motion confined to a cusp in the plane. In this case, RBM is defined as the solution to a corresponding submartingale problem. Existence and uniqueness results are then proven by conformally mapping RBM in the upper half plane to the cusp and applying a time change. In the follow-up paper [20], it is shown that depending on the geometry of the problem in [23], RBM in a cusp in the plane may or may not turn out to be a semi-martingale. The results in [20] are similar to those in [77] where a wedge instead of a cusp is considered. The authors in [30] use a Dirichlet form approach to construct a diffusion process contained in a d-dimensional Lipschitz domain with cusps. Moreover, conditions are provided under which the constraining process is of bounded variation. The paper [16] considers RBM in a cusp in the plane as the solution to a stochastic differential equation with reflection (SDER). It is proven that in this case there exists a unique weak solution to the corresponding SDER.

To study RBM, this thesis focuses on the submartingale problem with drift in a wedge, and its associated absorption problem and hitting probabilities. We establish the existence and uniqueness of the submartingale problem with drift, and Markov and Feller properties. The formulation of the research problem is stated as follows.

3.1 Historical background and novelty of Part I

The driftless wedge problem of Varadhan and Williams [74] already exhibits the main singular feature of the geometry: depending on the reflection data, the process may or may not be a semimartingale, and the behavior at the vertex is subtle. Later work clarified the semimartingale boundary of the model and connected the wedge problem with more general reflected-diffusion and Skorokhod-map constructions [77, 60, 44]. The novelty of Part I is that it treats the *constant-drift* case at the level of the submartingale problem, including regimes in which the usual pathwise reflected-SDE machinery is unavailable or at least not presently adequate.

This is not just a cosmetic extension of the driftless theory. In heavy traffic, the drift records first-order imbalance and is therefore part of the limiting model. From that point of view one needs a wedge theory that allows drift without assuming from the beginning that the process is already given by a semimartingale Skorokhod decomposition. The thesis provides exactly this extension.

3.2 Summary of the main results of Part I

The first principal result is the phase transition for the submartingale problem with drift. With

$$\alpha = \frac{\theta_1 + \theta_2}{\xi},$$

Part I proves that:

- if $\alpha < 2$, then for every constant drift $\mu \in \mathbb{R}^2$ there exists a unique solution to the submartingale problem with drift;
- if $\alpha \geq 2$, then no such solution exists.

In the existence regime one also obtains a decomposition

$$Z = X + Y,$$

where X is a Brownian motion with drift μ and Y solves the extended Skorokhod problem in the wedge. In the non-semimartingale range $1 \leq \alpha < 2$, the process Z is shown not to be a semimartingale, while Y has precise variation and Dirichlet-process properties.

The second principal block of results concerns the absorbed process at the vertex. The thesis proves existence and uniqueness of the absorbed process for every $\alpha \in \mathbb{R}$ and derives a sharp description of the hitting behavior of the vertex. In particular, the hitting probability is controlled by the geometry of the reflection directions and the drift vector, and in the range $\alpha \geq 1$ the event of hitting the vertex is no longer a trivial zero–one event for all drifts.

The third block of results concerns probabilistic regularity: the solution of the submartingale problem with drift has the strong Markov property and the relevant Feller properties in the existence regime. These statements are essential if the wedge process is to be used as a genuine limit object in queueing and stochastic-network applications.

3.3 Roadmap of Part I

The logical structure of Part I is as follows. Chapter 1 introduces the queueing-theoretic and diffusion-approximation background. Chapter 2 reviews reflected Brownian motion, the Skorokhod problem, and the submartingale problem in the broader reflected-diffusion literature. Chapter 3 formulates the wedge problem with drift precisely and states the main problems. Chapter 4 treats the absorbed process and the hitting behavior of the vertex. Chapter 5 then develops the existence, uniqueness, Markov, Feller, variation, and Dirichlet-process results for the full submartingale problem with drift.

We first set up some notation to make our further statement clear.

Let $C_S = C(\mathbb{R}_+, S)$ and, for each $t \geq 0$, let $Z(t) : C_S \to S$ denote the coordinate map $Z(t)(\omega) = \omega(t)$ for $\omega \in C_S$. Also, let $Z = \{Z(t), t \geq 0\}$ denote the coordinate mapping process on C_S. Let $\mathcal{M}_t = \sigma(Z(s), 0 \leq s \leq t)$ be the underlying natural filtration with terminal σ-algebra $\mathcal{M} = \sigma(Z(s), s \geq 0)$. For each $n \geq 1$ and domain $\Omega \subseteq \mathbb{R}^2$, we denote by $C_b^n(\Omega)$ the set of n times bounded continuously differentiable functions on Ω. We assume that the wedge S is positioned so that one side of it is the positive horizontal half line, and the angle of the wedge is ξ. We define ∂S_1 and ∂S_2 as the two sides of the wedge so that neither includes the vertex, i.e., $\partial S_1 = \{(x, 0) : \ x > 0\}$ and $\partial S_2 = \{r(\cos\xi, \sin\xi) : \ r > 0\}$. Next (see Figure 3.1), we denote by v_1 and v_2 the reflection directions on the boundaries ∂S_1 and ∂S_2, respectively. For convenience, we assume that each v_i is normalized such that $v_i \cdot n_i = 1$, where n_i is the inward facing normal vector on ∂S_i for $i = 1, 2$. Finally, for $i = 1, 2$, we set the directional derivative operator $D_i = v_i \cdot \nabla$, with ∇ being the gradient operator, the dot is the inner product, and denote by Δ the Laplacian operator.

3.4 The Absorbed Process Problem with Drift

Let

$$\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}$$

be the stopping time with respect to $\{\mathcal{M}_t, t \geq 0\}$ representing the first time that Z reaches the vertex of the wedge. Results in this subsection concern the RBM in a wedge up until τ_0. Results of this type were provided in [74] for the driftless case and we extend many of them to the case of a constant drift μ. We begin with the following definition.

Definition 3 (The Absorbed Process Problem). *A family of probability measures $\{\mathbb{P}_\mu^{z,0}, z \in S\}$ on $(C_S, \mathcal{M})$ is said to solve the absorbed process problem with drift $\mu \in \mathbb{R}^2$ if for each $z \in S$, the following three conditions hold,*

1. $\mathbb{P}_\mu^{z,0}(Z(0) = z) = 1$;

2. *The process*

$$\left\{ f(Z(t \wedge \tau_0)) - \int_0^{t \wedge \tau_0} \mu \cdot \nabla f(Z(s)) ds - \frac{1}{2} \int_0^{t \wedge \tau_0} \Delta f(Z(s)) ds, t \geq 0 \right\}$$

   is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^{z,0})$, for each $f \in C_b^2(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$;

3. $\mathbb{P}_\mu^{z,0}(Z(t) = 0, \forall t \geq \tau_0) = 1$.

Problem 3.3. For the above absorbed process problem with drift,

1. if there is a solution to this problem, is it unique?

2. once such a solution to this problem exists, what is the hitting probability to the origin (the vertex of the wedge)?

3.5 The Submartingale Problem with Drift

Definition 4 (Submartingale Problem with Drift)**.** *A family of probability measures* $\{\mathbb{P}^z_\mu, z \in S\}$ *on* $(C_S, \mathcal{M})$ *is said to solve the submartingale problem with drift* $\mu \in \mathbb{R}^2$ *if for each* $z \in S$*, the following three conditions hold,*

1. $\mathbb{P}^z_\mu(Z(0) = z) = 1;$

2. *For each* $f \in C^2_b(S)$*, the process*

$$\left\{ f(Z(t)) - \int_0^t \mu \cdot \nabla f(Z(s))ds - \frac{1}{2}\int_0^t \Delta f(Z(s))ds, t \geq 0 \right\}$$

is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$ *whenever* f *is constant in a neighbourhood of the origin and satisfies* $D_i f \geq 0$ *on* ∂S_i *for* $i = 1, 2;$

3. $\mathbb{E}^z_\mu\left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}}dt\right] = 0.$

Problem 3.4. For the above submartingale problem with drift,

1. if there is a solution to this problem, is it unique?

2. once such a solution to this problem exists, does it have Markov property and Feller property and other probabilistic properties?

Chapter 4

Absorption problem and hitting probability

4.1 The problem and the main results

Recall from Chapter 3 that

$$\tau_0 = \inf\{t \geq 0 : Z(t) = 0\}$$

is the stopping time with respect to $\{\mathcal{M}_t, t \geq 0\}$ representing the first time that Z reaches the vertex of the wedge.

Results in this section concern the RBM in a wedge up until τ_0. Results of this type were provided in [74] for the driftless case, and we extend many of them to the case of a constant drift μ. We begin with the following definition. For convenience, we display the figure of the state space here again.

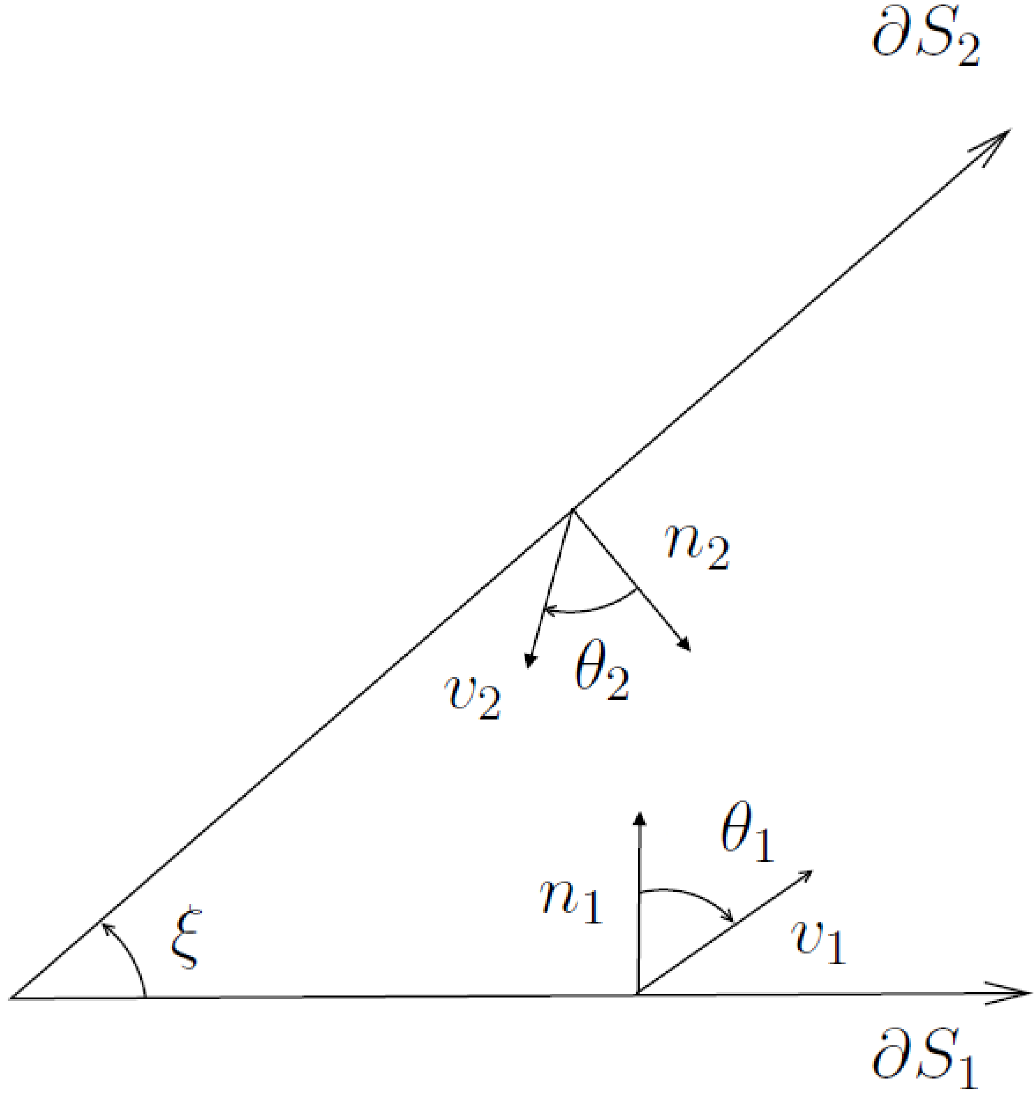


Figure 4.1: RBM in a wedge

Definition 5 (The Absorbed Process Problem). *A family of probability measures* $\{\mathbb{P}_\mu^{z,0}, z \in S\}$ *on* $(C_S, \mathcal{M})$ *is said to solve the absorbed process problem with drift* $\mu \in \mathbb{R}^2$ *if for each* $z \in S$, *the following three conditions hold:*

1. $\mathbb{P}_\mu^{z,0}(Z(0) = z) = 1$;

2. The process

$$\left\{ f(Z(t\wedge\tau_0)) - \int_0^{t\wedge\tau_0} \mu\cdot\nabla f(Z(s))\,ds - \frac{1}{2}\int_0^{t\wedge\tau_0} \Delta f(Z(s))\,ds,\ t\geq 0 \right\}$$

is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}_\mu^{z,0})$, for each $f \in C_b^2(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$;

3. $\mathbb{P}_\mu^{z,0}(Z(t) = 0, \forall t \geq \tau_0) = 1.$

Theorem 4.1. *For each $\mu \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, there exists a unique solution to the absorbed process problem.*

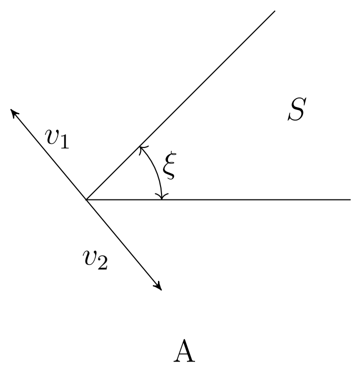
The above theorem is particularly interesting if $\alpha \geq 2$, since Theorem 5.1 does not cover that case. The existence of a solution to the absorbed process problem easily follows from the existence of a solution to the submartingale problem whenever $\alpha < 2$. However, the uniqueness part of Theorem 4.1 does not follow in an obvious way from Theorem 5.1, even in the $\alpha < 2$ case. Our proof of Theorem 4.1 applies to all $\alpha \in \mathbb{R}$.

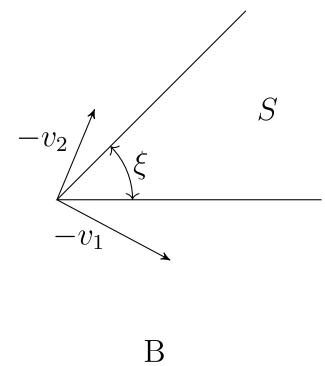
Next, we state a series of results on the hitting probability of the vertex for the absorbed process in the case of a constant drift.

Theorem 4.2. *If $\alpha \leq 0$, then for each $\mu \in \mathbb{R}^2$ and $z \in S$, $\mathbb{P}_\mu^{z,0}(\tau_0 = \infty) = 1$.*

The hitting probability of the vertex is more varied in the case of $\alpha \geq 1$, and before proceeding we must make some observations on the geometry of the wedge. For $n \geq 1$ and a set of vectors $\{a_1, \dots, a_n\} \subset \mathbb{R}^2$, let $\mathrm{co}(a_1, \dots, a_n)$ denote the closed convex cone generated by $\{a_1, \dots, a_n\}$. We illustrate two relevant cases for α below.

In the above diagrams, case A corresponds to $\alpha = 1$, which occurs if and only if $\mathrm{co}(v_1, v_2)$





is a line. Case B corresponds to $\alpha > 1$, which occurs if and only if $\text{co}(-v_1, -v_2)$ contains S. In both cases, that is, whenever $\alpha \geq 1$, we have $\text{co}(v_1, v_2) \cap S = \{0\}$. Note also that $\alpha \geq 1$ implies $\xi < \pi$.

Theorem 4.3. *If $\alpha \geq 1$, then*

$$\mathbb{P}_{\mu}^{z,0}(\tau_0 < \infty) > 0 \text{ for each } \mu \in \mathbb{R}^2 \text{ and } z \in S. \tag{4.1}$$

Moreover, if in addition to the $\alpha \geq 1$ condition we also have that

$$co(v_1, v_2, \mu) \cap S = \{0\}, \tag{4.2}$$

then for each $z \in S$,

$$\mathbb{P}_{\mu}^{z,0}(\tau_0 < \infty) = 1. \tag{4.3}$$

We note that in the case $\alpha > 1$, condition (4.2) can be cast in an algebraic form. Let R be the 2×2 matrix such that its i-th column vector is v_i, for $i = 1, 2$. If $\alpha > 1$, then condition (4.2) is equivalent to the requirement that the vector $R^{-1}\mu$ has at least one non-negative component.

Remark 4.4. Theorem 4.3 leaves open the possibility that $\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) = 1$ whenever $\alpha \geq 1$. This, however, is not the case, as the following counterexample shows. Let $\alpha \in \mathbb{R}$ be arbitrary and let the drift $\mu \in \mathbb{R}^2$ be given by $\mu = \|\mu\|(\cos\eta, \sin\eta) \neq 0$, where $\eta \in (0, \xi)$. Then, it is not hard to show using the proposition below that $\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) < 1$ for each $z \in S \setminus \{0\}$.

Proposition 4.5. *Let S be the 2-d wedge defined above, let S^0 be the interior of S, let B be a 2-d standard Brownian motion with zero drift started at the origin under a probability measure P, and let $\mu \in \mathbb{R}^2$ be given by $\mu = \|\mu\|(\cos\eta, \sin\eta) \neq 0$, where $\eta \in (0, \xi)$. Then, if $0 < \xi < \pi$, for each $z \in S^0$,*

$$P(z + B_t + \mu t \in S^0, t \geq 0) > 0.$$

Using the proposition above and Theorem 4.3, we may now deduce that if $\alpha \geq 1$, $\eta \in (0, \xi)$, and $\mu = \|\mu\|(\cos\eta, \sin\eta) \neq 0$, then

$$\mathbb{P}_\mu^{z,0}(\tau_0 < \infty) \in (0, 1) \quad \text{for each} \quad z \in S \setminus \{0\}.$$

This implies that when $\alpha \geq 1$, hitting the vertex is no longer a 0-1 event for certain values of μ, which contrasts with the driftless result of [74].

4.2 Proof of Theorems 4.2, 4.3, and Proposition 4.5

We need to recall Proposition 5.18. According to this proposition, for every $w \in C(\mathbb{R}_+, \mathbb{R}^2)$ with $w(0) \in S$ there exists a triple (ϕ, η, T_0) such that items 1–4 and (i), (ii) hold. Since B is the coordinate mapping process on $C(\mathbb{R}_+, \mathbb{R}^2)$, we may replace w in item 1 with B, and write

$$\phi(t) = B(t) + R\eta(t), \quad t \in [0, T_0).$$

We also know that B is a two-dimensional Brownian motion with drift μ started at z under $\hat{\mathbb{P}}^z_\mu$ for every $z \in S$, hence we can write

$$\phi(t) = z + W(t) + \mu t + R\eta(t), \quad t \in [0, T_0), \tag{4.4}$$

where W is a standard two-dimensional Brownian motion started at zero under $\hat{\mathbb{P}}^z_\mu$.

The measure $\mathbb{P}^{z,0}_\mu$ was defined as the measure induced by Γ on $\mathcal{M}$ under $\hat{\mathbb{P}}^z_\mu$. From this and from $\tau_0 \circ \Gamma = T_0$ it follows that

$$\mathbb{P}^{z,0}_\mu(\tau_0 < \infty) = \hat{\mathbb{P}}^z_\mu(T_0 < \infty). \tag{4.5}$$

We shall use (4.5) repeatedly in the proofs below.

Proof of Theorem 4.2. By Theorem 2.2 in [74], we have $\mathbb{P}^{z,0}_0(\tau_0 = \infty) = 1$, thus by (4.5) also $\hat{\mathbb{P}}^z_0(T_0 = \infty) = 1$. For every $n \in \mathbb{N}_+$, the measures $\hat{\mathbb{P}}^z_0$ and $\hat{\mathbb{P}}^z_\mu$ are mutually absolutely continuous on $\mathcal{W}_n$, so $\hat{\mathbb{P}}^z_0(T_0 < n) = 0$ implies $\hat{\mathbb{P}}^z_\mu(T_0 < n) = 0$. Then $\hat{\mathbb{P}}^z_\mu(T_0 = \infty) = 1$ follows, and this together with (4.5) gives the required result. □

Proof of Theorem 4.3. First, we are going to show (4.1). By the $\alpha \geq 1$ condition there exists a vector $b \in \mathbb{R}^2$ such that $b \cdot z < 0$ for all $z \in S$, $z \neq 0$, and $b \cdot v_i \geq 0$ for $i = 1, 2$. Indeed, if $\alpha \geq 1$, then $\mathrm{co}(-v_1, -v_2)$ is either a line containing S within one side, or it is a wedge with an angle less than π containing S. In either case, the existence of such a vector follows. Then, by identity (4.4), for each $z \in S$,

$$0 \geq b \cdot \phi(t) = b \cdot z + b \cdot W_t + b \cdot v_1 \eta_1(t) + b \cdot v_2 \eta_2(t) + b \cdot \mu t \geq b \cdot z + b \cdot W_t + b \cdot \mu t,$$

for $t < T_0$, $\hat{\mathbb{P}}^z_\mu$-a.s., and so

$$\hat{\mathbb{P}}^z_\mu(0 \geq b \cdot z + b \cdot W_t + b \cdot \mu t,\ t < T_0) = 1.$$

Therefore,

$$\begin{aligned} & \hat{\mathbb{P}}^z_\mu(0 \geq b \cdot z + b \cdot W_t + b \cdot \mu t,\ t < \infty) \\ \geq\ & \hat{\mathbb{P}}^z_\mu(\{0 \geq b \cdot z + b \cdot W_t + b \cdot \mu t, t < T_0\} \cap \{T_0 = \infty\}) \\ =\ & \hat{\mathbb{P}}^z_\mu(T_0 = \infty). \end{aligned}$$

This implies

$$\hat{\mathbb{P}}^z_\mu(-b \cdot z \geq b \cdot W_t + b \cdot \mu t, t < \infty) \geq \hat{\mathbb{P}}^z_\mu(T_0 = \infty). \tag{4.6}$$

However,

$$\hat{\mathbb{P}}^z_\mu(-b \cdot z \geq b \cdot W_t + b \cdot \mu t,\ t < \infty) < 1,$$

and together with (4.5) this proves the result.

Suppose now that (4.2) also holds, in addition to $\alpha \geq 1$. Then $\mathrm{co}(v_1, v_2, \mu)$ is either a wedge with an angle less than π, or a half-space, or a line. Then the same is true for $\mathrm{co}(-v_1, -v_2, -\mu)$, and if it is a wedge or a half-space then it contains S, and if it is a line then it contains S in one side. In all cases, we can select b so that in addition to $b \cdot z < 0$ for all $z \in S$, $z \neq 0$, and $b \cdot v_i \geq 0$ for $i = 1, 2$, we also have $b \cdot \mu \geq 0$. In that case,

$$\hat{\mathbb{P}}^z_\mu(-b \cdot z \geq b \cdot W_t + b \cdot \mu t,\ t < \infty) = 0,$$

so (4.6) and (4.5) imply (4.3). □

Proof of Proposition 4.5. Suppose first that $0 < \xi \leq \pi/2$. Let $z \in S^0$ and $\mu \in \mathbb{R}^2$ be given by $\mu = \|\mu\|(\cos \eta, \sin \eta) \neq 0$, where $\eta \in (0, \xi)$. Next, set

$$X_t \quad = \quad z + B_t + \mu t, \quad t \geq 0, \tag{4.7}$$

where B_t is a standard 2-d Brownian motion.

Now translate the origin of the coordinate axes to z and then rotate the axis in a counterclockwise direction by the angle η. By the translational and rotational invariance of Brownian motion, in these new coordinates the process X may be written as

$$\hat{X}_t = \hat{B}_t + \|\mu\|\hat{e}_1 t, \quad t \geq 0,$$

where $\hat{B}$ is a standard Brownian motion and $\hat{e}_1 = (1, 0)$. Next, in the new coordinate system denote by $\mathcal{L}_1$ the line corresponding to ∂S_1, and by $\mathcal{L}_2$ the line corresponding to ∂S_2. Then, in the new coordinate system, the interior of S may be expressed as

$$\mathcal{S}^0 \quad = \quad \{\hat{z} \in \mathbb{R}^2 : \mathcal{L}_1(\hat{z}_1) < \hat{z}_2 < \mathcal{L}_2(\hat{z}_1)\},$$

where $\mathcal{L}_i(\hat{z}_1)$ is a coordinate uniquely determined by the relation $(\hat{z}_1, \mathcal{L}_i(\hat{z}_1)) \in \mathcal{L}_i$, for $i = 1, 2$.

Hence, to complete the proof for the case $0 < \xi \leq \pi/2$, it suffices to show that

$$P(\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1),\ t \geq 0) \quad > \quad 0.$$

First note that since $0 < \eta < \xi \leq \pi/2$, we may write

$$\mathcal{L}_1(\hat{z}_1) \quad = \quad -a - b\hat{z}_1 \quad \text{and} \quad \mathcal{L}_2(\hat{z}_1) \ = \ c + d\hat{z}_1,$$

for $a, b, c, d > 0$. Hence,

$$\begin{aligned} & \{\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1),\ t \geq 0\} \\ = \quad & \{-a - b\|\mu\|t - b\hat{B}_t^1 < \hat{B}_t^2 < c + d\|\mu\|t + d\hat{B}_t^1,\ t \geq 0\}. \end{aligned} \tag{4.8}$$

From (4.8), it now follows after some algebra that

$$\begin{aligned} & \{\hat{B}_t^1 > \max(-a/2b, -c/2d) - (\|\mu\|/2)t,\ t \geq 0\} \\ & \cap\{-(a/2) - (b/2)\|\mu\|t < \hat{B}_t^2 < (c/2) + (d/2)\|\mu\|t,\ t \geq 0\} \\ \subseteq \quad & \{\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1),\ t \geq 0\}. \end{aligned}$$

Next, by the independence of $\hat{B}^1$ and $\hat{B}^2$, we have

$$\begin{aligned} & P(\mathcal{L}_1(\hat{X}_t^1) < \hat{X}_t^2 < \mathcal{L}_2(\hat{X}_t^1),\ t \geq 0) \\ \geq \quad & P(\hat{B}_t^1 > \max(-a/2b, -c/2d) - (\|\mu\|/2)t,\ t \geq 0) \\ & \times P(-(a/2) - (b/2)\|\mu\|t < \hat{B}_t^2 < (c/2) + (d/2)\|\mu\|t,\ t \geq 0). \end{aligned}$$

However, since $a, b, c, d, \|\mu\| > 0$, it follows by (4.3) of Doob [24] that the two probabilities on the right-hand side above are greater than zero. This completes the proof for the case $0 < \xi \leq \pi/2$.

Now suppose that $\pi/2 < \xi < \pi$ and $z \in S^0$ and $0 < \eta < \xi$. In this case, we show that there exists a wedge $\bar{S} \subset S$ such that $z \in \bar{S}^0$ and $P(X_t \in \bar{S}^0, t \geq 0) > 0$, which is sufficient to complete the proof. First, suppose that $0 < \eta < \pi/2$. In this case, the wedge $\bar{S}$ can be defined in the usual way by setting $\xi = \pi/2$ and placing the vertex of $\bar{S}$ at a point $\bar{z} \in S^0$ such that $\bar{z} < z$. The results above then yield the desired conclusion. Next, suppose $\pi/2 \leq \eta < \xi$. Then set

$$\bar{S} = v + \{r \geq 0,\ \pi - (\xi + \eta)/2 \leq \theta \leq (\xi + \eta)/2\},$$

where the vertex $v = (v_1, v_2) \in S^0$ is such that $v_1 = z_1$ and $v_2 < z_2$. The results above again yield the desired result. $\square$

Chapter 5

Submartingale problem with drift, its Markov and Feller properties

5.1 Submartingale problem with drift, its Markov and Feller properties

Before stating our main results, we still recall the main notations and definitions from Chapter 3. Let $C_S = C(\mathbb{R}_+, S)$ and, for each $t \geq 0$, let $Z(t) : C_S \to S$ denote the coordinate map $Z(t)(\omega) = \omega(t)$ for $\omega \in C_S$. Also, let $Z = \{Z(t), t \geq 0\}$ denote the coordinate mapping process on C_S. Let $\mathcal{M}_t = \sigma(Z(s), 0 \leq s \leq t)$ be the underlying natural filtration with terminal σ-algebra $\mathcal{M} = \sigma(Z(s), s \geq 0)$. For each $n \geq 1$ and domain $\Omega \subseteq \mathbb{R}^2$, we denote by $C_b^n(\Omega)$ the set of n times bounded continuously differentiable functions on Ω. We assume that the wedge S is positioned so that one side of it is the positive horizontal half line, and the angle of the wedge is ξ. We define ∂S_1 and ∂S_2 as the two sides of the wedge so that neither includes the vertex, i.e., $\partial S_1 = \{(x,0):\ x>0\}$ and $\partial S_2 = \{r(\cos\xi, \sin\xi):\ r>0\}$. Next (see Figure 3.1), we denote by v_1 and v_2 the reflection directions on the boundaries ∂S_1 and ∂S_2, respectively. For convenience, we assume that each v_i is normalized such that $v_i \cdot n_i = 1$, where n_i is the inward facing normal vector on ∂S_i for $i = 1, 2$. Finally, for $i = 1, 2$, we set the directional derivative operator $D_i = v_i \cdot \nabla$, with ∇ being the gradient operator, the dot is the inner product, and denote by Δ the Laplacian operator.

Again, for our convenience, we attach the figure of the state-space here again.

5.2 The problem and the main results

Definition 6 (Submartingale Problem with Drift). *A family of probability measures $\{\mathbb{P}^z_\mu, z \in S\}$ on $(C_S, \mathcal{M})$ is said to solve the submartingale problem with drift $\mu \in \mathbb{R}^2$ if for each $z \in S$, the following three conditions hold,*

1. $\mathbb{P}^z_\mu(Z(0) = z) = 1$;
2. *For each $f \in C^2_b(S)$, the process*

$$\left\{ f(Z(t)) - \int_0^t \mu \cdot \nabla f(Z(s))ds - \frac{1}{2}\int_0^t \Delta f(Z(s))ds, t \geq 0 \right\}$$

   is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$ whenever f is constant in a neighbourhood of the origin and satisfies $D_i f \geq 0$ on ∂S_i for $i = 1, 2$;

3. $\mathbb{E}^z_\mu \left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}} dt \right] = 0.$

The above definition bears a relationship to the extended Skorokhod problem (ESP) developed in [60]. We shall recall the definition of the ESP below. Let $d(\cdot)$ be a set-valued

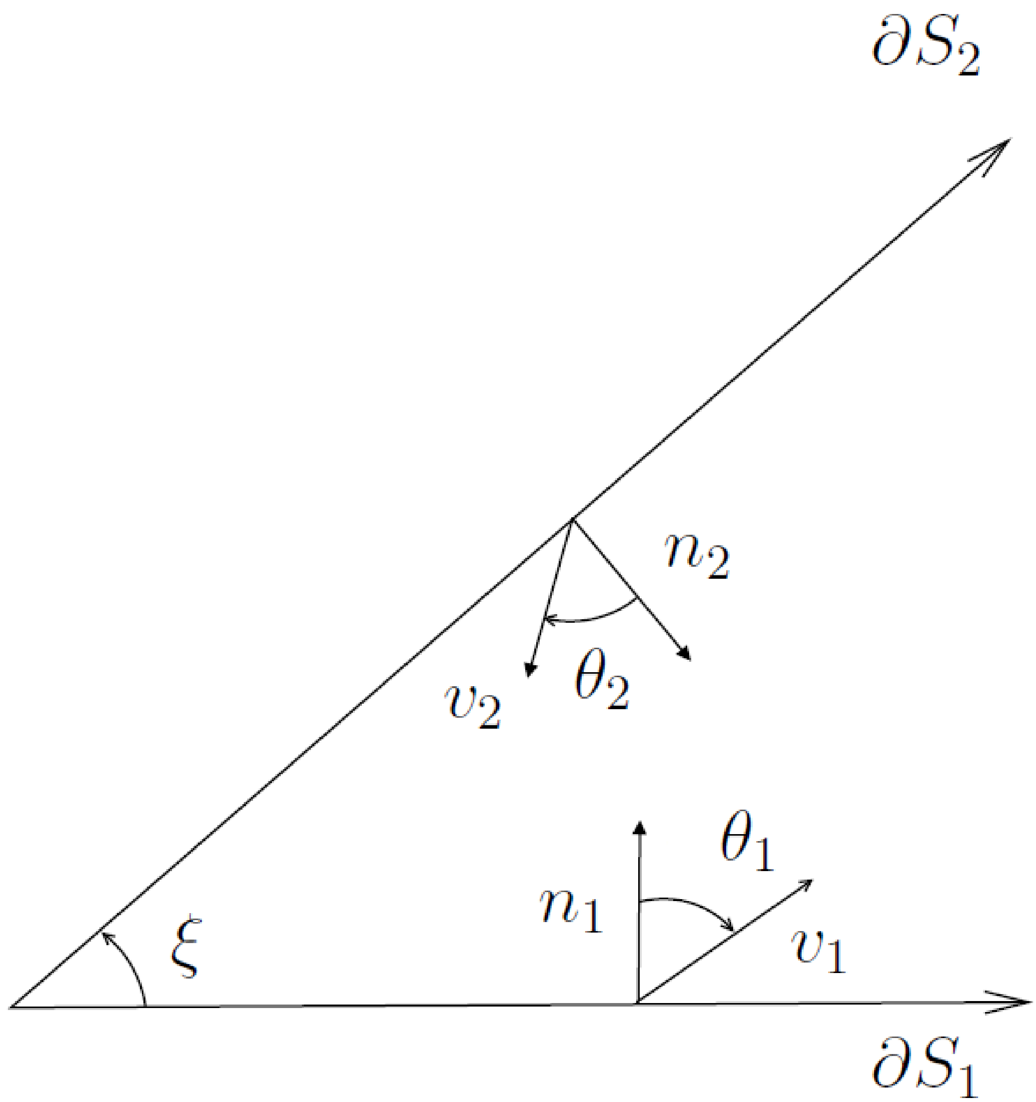


Figure 5.1: RBM in a Wedge

map from ∂S, the boundary of S, to the class of subsets of $\mathbb{R}^2$ satisfying the following two conditions:

(d1) for any $x \in \partial S$, the image $d(x)$ is a non-empty closed convex cone in $\mathbb{R}^2$ with the vertex being the origin;

(d2) the graph $\{(x, d(x));\ x \in \partial S\}$ is closed.

For convenience, we extend the definition of $d(\cdot)$ to S by setting $d(x) = \{0\}$ for all $x \in S^\circ$, where S° is the interior of S. For a set $A \subset \mathbb{R}^2$, let $\text{co}(A)$ be the closed convex cone generated by A.

Definition 7 (Extended Skorokhod Problem (ESP))**.** *A pair of processes $(\phi, \eta) \in C_S \times C(\mathbb{R}_+, \mathbb{R}^2)$ is said to solve the ESP $(S, d(\cdot))$ for $\psi \in C(\mathbb{R}_+, \mathbb{R}^2)$ such that $\psi(0) \in S$ if $\phi(0) = \psi(0)$, and if for all $t \in \mathbb{R}_+$, the following properties hold,*

1. *$\phi(t) = \psi(t) + \eta(t)$;*
2. *$\phi(t) \in S$;*
3. *For every $s \in [0, t]$;*
$$\eta(t) - \eta(s) \in co\Big[\cup_{u \in (s,t]} d(\phi(u))\Big].$$

Item 2 in the above definition is redundant since we already required that $\phi \in C_S$, but we kept that item as it appears in the original definition in [60].

Just like in [74], let
$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$
The quantity α plays a prominent role.

Theorem 5.1. *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ there exists a unique solution $\{\mathbb{P}^z_\mu, z \in S\}$ to the submartingale problem with drift. In addition, the following statements hold:*

1. *There exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion with drift μ started at z under $\mathbb{P}^z_\mu$;*

2. Setting $Y = Z - X$, the pair (Z, Y) solves the ESP $(S, d(\cdot))$ for X, $\mathbb{P}^z_\mu$-a.s., for each $z \in S$.

The above theorem establishes a decomposition

$$Z = X + Y, \tag{5.1}$$

such that for all $z \in S$ under $\mathbb{P}^z_\mu$ the process X is a standard Brownian motion with drift μ started at z. In the following two theorems, we shall establish several properties of the process Y appearing in the above decomposition. In order to state the first of these two results, we need the definition of the strong p-variation of a function. Let $T > 0$ be arbitrary. We call an ordered set $(t_0, t_1, \ldots, t_n)$ a partition of the interval $[0, T]$, if $0 = t_0 < t_1 < \cdots < t_n = T$, for an arbitrary $n \in \mathbb{N}_+$. Let $\pi(T)$ denote the set of all partitions of the interval $[0, T]$. We define the mesh of a partition $\rho = (t_0, \ldots, t_n) \in \pi(T)$ by setting

$$\|\rho\| = \max\{t_i - t_{i-1} : i = 1, \ldots, n\}.$$

Definition 8. *Let $T > 0$ and $p > 0$. The strong p-variation of a function $f : \mathbb{R}_+ \mapsto \mathbb{R}^k$ on $[0, T]$ is defined by*

$$V_p(f, [0, T]) = \sup \left\{ \sum_{t_i \in \rho,\ i \geq 1} \|f(t_i) - f(t_{i-1}\|^p : \rho \in \pi(T) \right\}.$$

Theorem 5.2. *Suppose that $1 < \alpha < 2$. Then for each $p > \alpha$ and $z \in S$,*

$$\mathbb{P}^z_\mu(V_p(Y, [0, T]) < +\infty) = 1, \quad T > 0, \tag{5.2}$$

and, for each $0 < p \leq \alpha$,

$$\mathbb{P}^0_\mu(V_p(Y,[0,T]) < +\infty) = 0, \quad T > 0. \tag{5.3}$$

A 2-dimensional continuous process U defined on $(C_s, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$ is said to be of *zero energy* if for each $T > 0$ and each sequence of partitions $(\rho^m) \subset \pi(T)$ such that $\|\rho^m\| \to 0$ as $m \to \infty$ we have

$$\sum_{i=1}^{n(m)} \|U(t^m_i) - U(t^m_{i-1})\|^2 \xrightarrow{\mathbb{P}^z_\mu} 0 \text{ as } m \to \infty,$$

where $\rho^m = (t^m_0, \ldots, t^m_{n(m)})$. A process D on $(C_s, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$ is said to be a *Dirichlet process* if it has a decomposition $D = M + U$, where M is a local martingale on the same probability space, and U is a continuous zero-energy process with $U(0) = 0$.

Theorem 5.3. *Let* $1 < \alpha < 2$*, and* $z \in S$ *arbitrary. Then the process* Y *in decomposition* (5.1) *is a zero-energy process, and* Z *is a Dirichlet process.*

Theorem 5.4. *If* $1 \leq \alpha < 2$*, then* Z *is not a semimartingale on* $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$*, for any* $z \in S$*.*

Theorem 5.5. *If* $\alpha \geq 2$*, then for any* $\mu \in \mathbb{R}^2$ *there is no solution to the submartingale problem with drift.*

Let $\{\mathbb{P}^z_\mu, z \in S\}$ be the solution to the submartingale problem for some $\mu \in \mathbb{R}^2$. We say that $\{\mathbb{P}^z_\mu, z \in S\}$ possesses the strong Markov property if for each stopping time τ and $z \in S$, and each bounded $\mathcal{M}$-measurable function $h : C_S \to \mathbb{R}$ we have that

$$\mathbb{E}^z_\mu[\mathbb{1}_{\{\tau<\infty\}} f(\omega(\cdot + \tau))|\mathcal{M}_\tau] = \mathbb{1}_{\{\tau<\infty\}} \mathbb{E}^{\omega(\tau)}_\mu[f(w(\cdot))], \quad \mathbb{P}^z_\mu\text{-a.s.} \tag{5.4}$$

Theorem 5.6. *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ the solution to the submartingale problem with drift has the strong Markov property.*

The last subject of this subsection is the Feller property of $\{\mathbb{P}_\mu^z, z \in S\}$. There are various, slightly differing definitions of the Feller property available in the literature. For clarity, we list below three definitions.

1. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ has the Feller property if for any $\{z_n, n \geq 1\} \subset S$ converging to $z \in S$, $\mathbb{P}_\mu^{z_n} \Rightarrow \mathbb{P}_\mu^z$ as $n \to \infty$ (see Varadhan and Williams [74]).

2. Let $\hat{C}(S)$ be the set of continuous functions on S vanishing at infinity. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ has the $\hat{C}(S)$-Feller property if for any $f \in \hat{C}(S)$, and $t \geq 0$, the function $z \mapsto \mathbb{E}_\mu^z[f(Z_t)]$ is also in $\hat{C}(S)$.

3. We say that $\{\mathbb{P}_\mu^z, z \in S\}$ has the $C_b(S)$-Feller property if for any $f \in C_b(S)$, and $t \geq 0$, the function $z \mapsto \mathbb{E}_\mu^z[f(Z_t)]$ is also in $C_b(S)$.

Remark 5.7. The Feller property obviously implies the $C_b(S)$-Feller property. The $\hat{C}(S)$-Feller property implies the $C_b(S)$-Feller, but the converse is not true (cf. Theorems 1.9 and 1.10 in [10]).

Theorem 5.8. *If $\alpha < 2$, then the solution to the submartingale problem for each $\mu \in \mathbb{R}^2$ has the Feller property.*

Theorem 5.9. *If $\alpha < 2$, then the solution to the submartingale problem for each $\mu \in \mathbb{R}^2$ has the $\hat{C}(S)$-Feller property.*

We note that for the $\mu = 0$ case, the Feller property is known ([74], Theorem 3.13). However, the $\hat{C}(S)$-Feller property is new even in the $\mu = 0$ case.

5.3 Strategy of the proofs in Part I

Before entering the technical proofs, it is useful to isolate the proof architecture of Part I.

1. **Existence by Girsanov transfer from the driftless case.** The starting point is the driftless wedge theory of Varadhan–Williams and its later ESP formulation. Once a Brownian motion X without drift has been constructed and shown to generate a reflected process Z through the extended Skorokhod problem, a Girsanov change of measure adds the constant drift μ and produces existence in the regime $\alpha < 2$.

2. **Uniqueness by reduction back to the zero-drift problem.** If two candidate solutions with drift are given, the proof removes the drift by an inverse Girsanov transform and then invokes the uniqueness of the driftless submartingale problem. This is the conceptual reason the zero-drift theory remains central even after drift is introduced.

3. **Variation and Dirichlet-process properties by absolute continuity.** Once the drifted and driftless laws are locally equivalent, sharp path properties proved in the driftless case can be transferred to the constant-drift case on finite time intervals.

4. **Absorbed process by stopping at the vertex.** The absorbed process is constructed from the same reflected path, now stopped when it reaches the vertex. The key point is

that this problem continues to make sense even when the full submartingale problem no longer exists for $\alpha \geq 2$.

5. **Feller and strong Markov properties by tightness and uniqueness.** The Feller statements are obtained by tightness and identification of weak limits, while the strong Markov property follows from uniqueness through the standard regular conditional probability argument.

In other words, the proofs in Part I are organized around a small number of structural ideas: reduction to the zero-drift theory, transfer by absolute continuity, and the ESP/submartingale connection.

5.4 Proof of Theorems 5.1, 5.2, 5.3, 5.4, and 5.5

We provide an extension of Theorems 2.4 and 2.8 in [49] to all $\alpha < 2$. For $z \in \partial S_i, i = 1, 2$, let $d(z) = \{\lambda v_i,\ \lambda \geq 0\}$, and set $d(0) = \mathbb{R}^2$. It is known that in the case of $\mu = 0$ the submartingale problem has a unique solution whenever $\alpha < 2$ (see [74]). In accordance with our notation, that solution will be denoted by $\{\mathbb{P}_0^z, z \in S\}$. We then have the following.

Proposition 5.10. *Let $(C_S, \mathcal{M}, \mathcal{M}_t)$ and Z be defined as in Section 5.2. Then, if $\alpha < 2$,*

1. *There exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion started at z under $\mathbb{P}_0^z$;*

2. *Setting $Y = Z - X$, the pair (Z, Y) solves the ESP $(S, d(\cdot))$ for X, $\mathbb{P}_0^z$-a.s..*

Proof. Let $\alpha < 2$. Then, Condition 1 is immediate from Theorem 2.4 in [49]. It remains to show that for each $z \in S$, $\mathbb{P}_0^z$-a.s., Z and $Y = Z - X$ together solve the ESP (see Definition 7 as above) for X with d as defined immediately preceding the statement of the proposition. For $\alpha \in (1,2)$, this follows by Theorem 2.8 in [49]. We now claim that if $\alpha \leq 1$, (Z,Y) also solves the ESP $(S,d(\cdot))$ for X, $\mathbb{P}^z$-a.s. For any two real numbers $0 < s < t$, if (s,t) belongs to a single excursion from the origin then by a similar proof to the one in part 2 of Theorem 4.2 in [49], one can conclude that item 3 in the definition of the ESP holds. If (s,t) doesn't belong to one excursion, then item 3 is obviously satisfied by $d(0) = \mathbb{R}^2$. □

We are now ready to prove the existence of a solution to the submartingale problem with drift, and some of the properties of the solution we create.

Proposition 5.11. *If $\alpha < 2$, then for each $\mu \in \mathbb{R}^2$ the submartingale problem with drift has a solution $\{\mathbb{P}^z_\mu, z \in S\}$, which satisfies the following properties. With X and Y defined in Proposition 5.10, for every $z \in S$ the following hold:*

- *Under $\mathbb{P}^z_\mu$ the process X is a standard Brownian motion with drift μ started at z;*
- *The pair (Z,Y) solves the ESP $(S,d(\cdot))$ for X, $\mathbb{P}^z_\mu$-a.s.*

Proof. Let $\alpha < 2$ and note that by Proposition 5.10 there exists a process X defined on $(C_S, \mathcal{M}, \mathcal{M}_t)$ which, for each $z \in S$, is a 2-dimensional Brownian motion started at z under $\mathbb{P}^z_0$. Now let $T \geq 0$ and for each $z \in S$, let $\mathbb{P}^z_{\mu,T}$ be a probability measure on $(C_S, \mathcal{M}, \mathcal{M}_t)$ equivalent (mutually absolutely continuous) to $\mathbb{P}^z_0$ such that under $\mathbb{P}^z_{\mu,T}$, X is a standard Brownian motion with drift μ up to time T, started at z. In other words, $\{X(t) - \mu t, t \leq T\}$ is a standard (driftless) $\mathbb{P}^z_{\mu,T}$-Brownian motion started at z. The measure $\mathbb{P}^z_{\mu,T}$ is defined by

$$\frac{d\mathbb{P}^z_{\mu,T}}{d\mathbb{P}^z_0} = \zeta(T), \tag{5.5}$$

where $\zeta(T) = \exp\{\mu \cdot (X(T) - X(0)) - \frac{1}{2}\|\mu\|^2 T\}$.

One can easily show that the family of probability measures $\{\mathbb{P}^z_{\mu,T}, T \in [0,\infty)\}$ is consistent. That is, if $S < T$, then $\mathbb{P}^z_{\mu,T}(A) = \mathbb{P}^z_{\mu,S}(A)$, whenever $A \in \mathcal{M}_S$. From [55], Theorem 4.2 (page 143), it follows that there exists a single probability measure $\mathbb{P}^z_\mu$ such that $\mathbb{P}^z_\mu(A) = \mathbb{P}^z_{\mu,T}(A)$ whenever $A \in \mathcal{M}_T$. Since $\{X(t) - X(0) - \mu t, t \le T\}$ is a $\mathbb{P}^z_{\mu,T}$-Brownian motion started at zero for every $T \in [0,\infty)$, it follows that $\{X(t) - X(0) - \mu t, t < \infty\}$ is also a $\mathbb{P}^z_\mu$-Brownian motion started at zero. Also, (Z,Y) solves the ESP $(S, d(\cdot))$ for X under $\mathbb{P}^z_\mu$-a.s., because by Proposition 5.10 it is true under $\mathbb{P}^z_0$, and the measures $\mathbb{P}^z_0$ and $\mathbb{P}^z_\mu$ constrained to $\mathcal{M}_T$ are mutually absolutely continuous for every $T \in [0,\infty)$. Now let

$$W(t) = X(t) - X(0) - \mu t, \quad t \in [0,\infty),$$

and consider the definition of a weak solution to an SDER (see Definition 2.4 of Kang and Ramanan [44]). Clearly, the triplet $(C_S, \mathcal{M}, \mathcal{M}_t)$, $\mathbb{P}^z_\mu$, (Z,W) is a weak solution to the SDER with initial condition z associated with $(S, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$, where $b(x) = \mu$ and $\sigma(x) = \mathrm{Id}_{2\times 2}$. We note that the "closed graph condition" (see Kang and Ramanan [44], page 5) is satisfied. From Theorem 2 in [44], it now follows that $\{\mathbb{P}^z_\mu, z \in S\}$ solves the submartingale problem with drift μ.

□

We shall use the Lemmas 5.12, 5.13, 5.14, 5.15, and 5.16 for the proof of both the uniqueness part of Theorem 5.1, and for the proof of Theorem 5.5. In these lemmas, α may be an arbitrary real number. On the other hand, in these lemmas we start with a probability measure P^z_μ that satisfies conditions 1,2, and 3 of Definition 6. This may be surprising since Theorem 5.5 states that such probability measure does not exist for $\alpha \ge 2$. However, for such α's we use these lemmas to derive a contradiction, thereby proving Theorem 5.5.

Lemma 5.12. *Suppose that $\{\mathbb{P}^z_\mu, z \in S\}$ is a solution to the submartingale problem with drift $\mu \in \mathbb{R}^2$. Then, for all $z \in S$,*

$$\mathbb{E}^z_\mu\left[\int_0^\infty \mathbb{1}_{\{Z(t)\in\partial S\}}dt\right] = 0. \tag{5.6}$$

Proof. Let $z \in S$ be arbitrary. In this proof, we shall use the Doob-Meyer decomposition for submartingales, which requires that the probability space is augmented. For this reason we denote by $(C_S, \mathcal{F}^z, (\mathcal{F}^z_t))$ the augmentation of the space $(C_S, \mathcal{M}, (\mathcal{M}_t))$under $\mathbb{P}^z_\mu$. For some technical details on the augmentation of probability spaces see Remark A.2 in the Appendix. Condition 3 of the submartingale problem gives

$$\mathbb{E}^z_\mu\left[\int_0^\infty \mathbb{1}_{\{Z(t)=0\}}dt\right] = 0,$$

for each $z \in S$, thus in order to complete the proof it suffices to prove that

$$\mathbb{E}^z_\mu\left[\int_0^\infty \mathbb{1}_{\{Z(t)\in\partial S_{i,}\}}dt\right] = 0,\ \text{for } i = 1, 2. \tag{5.7}$$

We prove this result for $i = 1$; the result then follows for $i = 2$ by symmetry.

For each $\varepsilon > 0$, define $S^\varepsilon \subset S$ by $S^\varepsilon = S + (\varepsilon, 0)$, i.e., a wedge with vertex at $(\varepsilon, 0)$ and edges $\partial S^\varepsilon_1 = \{(x, 0) \in \mathbb{R}^2,\ x > \varepsilon\}$ and $\partial S^\varepsilon_2 = \{(\varepsilon, 0) + \lambda(\cos\xi, \sin\xi),\ \lambda > 0\}$ (recall that ξ is the angle of the wedge S).

Next we shall recursively define the $(\mathcal{F}^z_t)$ stopping times $\bar{\sigma}^{\varepsilon,T}_k, \bar{\tau}^{\varepsilon,T}_k$ for $k \geq 1$, for every $T > 0$. We define

$$\bar{\sigma}^{\varepsilon,T}_1 = \inf\Big\{t \geq 0 : Z(t) \in S^\varepsilon\Big\} \wedge T,$$

and

$$\bar{\tau}_k^{\varepsilon,T} = \inf\left\{t \ge \bar{\sigma}_k^{\varepsilon,T} : Z_t \in \partial S_2^{2\varepsilon/3}\right\} \wedge T, \quad k \ge 1,$$

$$\bar{\sigma}_k^{\varepsilon,T} = \inf\left\{t \ge \bar{\tau}_{k-1}^{\varepsilon,T} : Z_t \in \partial S_2^{\varepsilon}\right\} \wedge T, \quad k \ge 2;$$

Let $Z_t = (Z_1(t), Z_2(t))$. Let $C > z_2$ be an arbitrary constant. We define the $(\mathcal{F}_t^z)$ stopping time

$$T_C = \inf\left\{t \ge 0 : Z_t^{(2)} \ge C\right\},$$

and in order to simplify the notation, we also introduce the stopping times $\bar{\sigma}_k = \bar{\sigma}_k^{\varepsilon,T} \wedge T_C$ and $\bar{\tau}_k = \bar{\tau}_k^{\varepsilon,T} \wedge T_C$. Notice that for all $t \le T$, $t \in [\bar{\sigma}_k, \bar{\tau}_k]$ implies that $Z_t \in S^{2\varepsilon/3}$ and $Z_2(t) \le C$.

Let $f_{\varepsilon,C} \in C_b^2(S)$ such that

$$f_{\varepsilon,C}(x,y) = \begin{cases} 0, \text{ if } (x,y) \in S \setminus S^{\varepsilon/3}, \\ y, \text{ if } (x,y) \in S^{2\varepsilon/3}, y \le C. \end{cases} \tag{5.8}$$

In addition we require that $f_{\varepsilon,C}(x,0) = 0$ for all $x \ge 0$, and $D_2 f_{\varepsilon,C} \ge 0$ on ∂S_2. It follows from (5.8) that $D_1 f_{\varepsilon,C} = 0$ on ∂S_1. We show in Lemma A.1 in the Appendix that such a function indeed exists. By the definition of the submartingale problem

$$V_1 = \left\{V_1(t) = f_{\varepsilon,C}(Z_t) - \int_0^t \left(\mu \cdot \nabla f_{\varepsilon,C}(Z_s) - \frac{1}{2}\triangle f_{\varepsilon,C}(Z_s)\right) ds;\ t \ge 0\right\} \tag{5.9}$$

is a regular submartingale under $\mathbb{P}_\mu^z$ on $(\mathcal{F}_t^z)$, thus by Theorem 1.4.14 in [45] it has a unique Doob-Meyer decomposition

$$V_1(t) = M(t) + A(t) \tag{5.10}$$

where M is a continuous martingale and A is a continuous increasing process. For the definition of regular submartingales see Definition 1.4.12 in [45]. For an arbitrary $k \geq 1$ we have $f_{\varepsilon,C}(Z_t) = Z_2(t)$, $\mu \cdot \nabla f_{\varepsilon,C}(Z_t) = \mu_2$, and $\triangle f_{\varepsilon,C}(Z_t) = 0$ whenever $t \in [\bar{\sigma}_k, \bar{\tau}_k]$, hence by (5.10) and (5.9)

$$Z_2(t) = Z_2(\bar{\sigma}_k) + M(t) - M(\bar{\sigma}_k) + A(t) - A(\bar{\sigma}_k) + \mu_2(t - \bar{\sigma}_k) \tag{5.11}$$

for $t \in [\bar{\sigma}_k, \bar{\tau}_k]$. Next, we are going to establish the following two properties. The first is that

$$\int_{\bar{\sigma}_k}^{\bar{\tau}_k} 1_{\{Z_2(t)>0\}} dA(t) = 0, \quad \mathbb{P}_\mu^z\text{−a.s.}, \tag{5.12}$$

and the second is that

$$\int_{\bar{\sigma}_k}^{\bar{\tau}_k} d\left(\langle M \rangle_t - t\right) = 0, \quad ,\mathbb{P}_\mu^z\text{−a.s.} \tag{5.13}$$

We start with proving (5.12). For any $\delta > 0$ and $k \geq 1$ we define a sequence of $(\mathcal{F}_t^z)$ stopping times

$$\theta_1^\delta = \inf\{t \geq \bar{\sigma}_k : \ Z_z(t) \geq \delta\} \wedge \bar{\tau}_k, \quad \vartheta_1^\delta = \inf\left\{t \geq \theta_1^\delta : \ Z_2(t) = \frac{\delta}{2}\right\} \wedge \bar{\tau}_k,$$

$$\theta_n^\delta = \inf\left\{t \geq \vartheta_{n-1}^\delta : \ Z_2(t) = \delta\right\} \wedge \bar{\tau}_k, \quad n \geq 2,$$

$$\vartheta_n^\delta = \inf\left\{t \geq \theta_n^\delta : \ Z_2(t) = \frac{\delta}{2}\right\} \wedge \bar{\tau}_k, \quad n \geq 2.$$

Notice that $[\theta_n^\delta, \vartheta_n^\delta]$ is a sub-interval of $[\bar{\sigma}_k, \bar{\tau}_k]$ such that for $t \in [\theta_n^\delta, \vartheta_n^\delta]$ we have $Z_2(t) \geq \delta/2$. All these stopping times are finite because by definition $\bar{\tau}_k \leq T$. Let $g_1 \in C_b^2(\mathbb{R})$ be an arbitrary function such that $g_1'(0) = 0$, and $g_1(x) = x$ whenever $x \geq \delta/2$. Relation $g_1'(0) = 0$

implies that

$$V_2 = \left\{ V_2(t) = g_1\left(f_{\varepsilon,C}(Z_t)\right) - \int_0^t \left(\mu \cdot \nabla (g_1 \circ f_{\varepsilon,C})(Z_s) - \frac{1}{2} \triangle (g_1 \circ f_{\varepsilon,C})(Z_s) \right) ds,\ t \geq 0 \right\}$$

is a martingale with respect to the filtration $(\mathcal{F}_t^z)$ under $\mathbb{P}_\mu^z$. Therefore, $\left\{ V_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right),\ t \geq 0 \right\}$ is also a martingale with respect to the filtration $\left\{ \mathcal{F}^z_{(t \vee \theta_n^\delta) \wedge \vartheta_n^\delta},\ t \geq 0 \right\}$ under $\mathbb{P}_\mu^z$. For all $s \in [\theta_n^\delta, \tau_n^\delta]$ we have $g_1(f_{\varepsilon,C}(Z(s))) = Z_2(s)$, $\mu \cdot \nabla (g_1 \circ f_{\varepsilon,C})(Z(s)) = \mu_2$ and $\triangle (g_1 \circ f_{\varepsilon,C})(Z(s)) = 0$, hence for all $t \geq 0$

$$V_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) = V_2(\theta_n^\delta) - Z_2(\theta_n^\delta) + Z_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) - \mu_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta - \theta_n^\delta\right)$$

thus $\left\{ Z_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) - \mu_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right),\ t \geq 0 \right\}$ is also a martingale with respect to the filtration $\left\{ \mathcal{F}^z_{(t \vee \theta_n^\delta) \wedge \vartheta_n^\delta},\ t \geq 0 \right\}$ under $\mathbb{P}_\mu^z$. On the other hand, from (5.11) follows that

$$Z_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) - \mu_2\left((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta\right) =$$

$$Z_2(\theta_n^\delta) - \mu_2 \theta_n^\delta + M((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta) - M(\theta_n^\delta) + A((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta) - A(\theta_n^\delta),$$

for all $t \geq 0$. However, the left-hand side in the above identity is a martingale with respect to the filtration $\left\{ \mathcal{F}^z_{(t \vee \theta_n^\delta) \wedge \vartheta_n^\delta},\ t \geq 0 \right\}$ under $\mathbb{P}_\mu^z$, and so is $M((t \vee \theta_n^\delta) \wedge \vartheta_n^\delta)$ on the right-hand side $(t \geq 0)$. Therefore, A must be constant on $[\theta_n^\delta, \tau_n^\delta]$, $\mathbb{P}_\mu^z$-a.s. This holds for all $n \geq 1$, hence

$$\int_{\hat{\sigma}_k}^{\hat{\tau}_k} \sum_{n=1}^\infty 1_{[\theta_n^\delta, \vartheta_n^\delta]}(t) dA(t) = 0, \quad \mathbb{P}_\mu^z\text{-a.s.} \tag{5.14}$$

If $t \in [\bar{\sigma}_k, \bar{\tau}_k]$ and $Z_2(t) > \delta$, then $Z_2(t) \in [\theta_n^\delta, \vartheta_n^\delta]$ for some $n \geq 1$, hence by (5.14)

$$\int_{\hat{\sigma}_k}^{\hat{\tau}_k} 1_{\{Z_2(t)>\delta\}}(t)dA(t) = 0, \quad \mathbb{P}_\mu^z\text{−a.s.},$$

and (5.12) follows.

Next, we are going to show (5.13). Let $g_2 \in C_b^2(\mathbb{R})$ arbitrary such that $g_2(x) = x^2$ whenever $|x| \leq C$. Then

$$V_3 = \left\{ V_3(t) = g_2\left(f_{\varepsilon,C}(Z_t)\right) - \int_0^t \left(\mu \cdot \nabla(g_2 \circ f_{\varepsilon,C})(Z_s) - \frac{1}{2}\triangle(g_2 \circ f_{\varepsilon,C})(Z_s) \right) ds,\ t \geq 0 \right\} \tag{5.15}$$

is a martingale under $\mathbb{P}_\mu^z$ with respect to the filtration $(\mathcal{F}_t^z)$, and for $t \in [\bar{\sigma}_k, \bar{\tau}_k]$ we have $g_2\left(f_{\varepsilon,C}(Z_t)\right) = \left(Z_2(t)\right)^2$, $\mu \cdot \nabla(g_2 \circ f_{\varepsilon,C})(Z_t) = 2\mu_2 Z_2(t)$ and $\triangle(g_2 \circ f_{\varepsilon,C})(Z_t) = 2$, hence by Ito's rule applied to $g_2(f_{\varepsilon,C}(Z_t))$ and by (5.11)

$$Z_2^2(t) = Z_2^2(\bar{\sigma}_k) + \int_{\bar{\sigma}_k}^t 2Z_2(s)dM(s) + \int_{\bar{\sigma}_k}^t 2\mu_2 Z_2(s)ds + \langle M\rangle_t - \langle M\rangle_{\bar{\sigma}_k},$$

for $t \in [\bar{\sigma}_k, \bar{\tau}_k]$. We note that the $\int_{\bar{\sigma}_k}^t 2Z_2(s)dA(s)$ term vanished because of (5.12). From this and from (5.15) follows that

$$V_3(t) = V_3(\bar{\sigma}_k) + \int_{\bar{\sigma}_k}^t 2Z_2(s)dM(s) + \int_{\bar{\sigma}_k}^t d\left(\langle M\rangle_s - s\right),$$

for $t \in [\bar{\sigma}_k, \bar{\tau}_k]$. The process $\{V_3\left((t \vee \bar{\sigma}_k) \wedge \bar{\tau}_k\right),\ t \geq 0\}$ is a martingale with respect to the filtration $\left\{\mathcal{F}_{(t\vee\theta_n^\delta)\wedge\vartheta_n^\delta},\ t \geq 0\right\}$ under $\mathbb{P}_\mu^z$, and can be written by substituting $(t \vee \bar{\sigma}_k) \wedge \bar{\tau}_k$ for t

in the above identity as

$$V_3(t \vee \bar{\sigma}_k) \wedge \bar{\tau}_k) = V_3(\bar{\sigma}_k) + \int_{\bar{\sigma}_k}^{t \vee \bar{\sigma}_k) \wedge \bar{\tau}_k} 2Z_2(s) dM(s) + \int_{\bar{\sigma}_k}^{t \vee \bar{\sigma}_k) \wedge \bar{\tau}_k} d\left(\langle M \rangle_s - s\right),$$

for all $t \geq 0$. Since the left-hand side is a martingale with respect to the filtration $\left\{\mathcal{F}^z_{(t \vee \theta^\delta_n) \wedge \vartheta^\delta_n},\ t \geq 0\right\}$ under $\mathbb{P}^z_\mu$, (5.13) follows. The by (5.11), Z_2 is a 1-dimensional Brownian motion with drift μ_2 reflected at zero in $[\bar{\sigma}_k, \bar{\tau}_k]$. Therefore,

$$\int_{\bar{\sigma}_k}^{\bar{\tau}_k} 1_{\{Z_2(t)=0\}} dt = 0, \quad \mathbb{P}^z_\mu\text{-a.s.}$$

This holds for every $k \geq 1$, hence we also have

$$\sum_{k=1}^{\infty} \int_{\bar{\sigma}_k}^{\bar{\tau}_k} 1_{\{Z_2(t)=0\}} dt = 0, \quad \mathbb{P}^z_\mu\text{-a.s.},$$

and from this

$$\int_0^{T \wedge T_C} 1_{\{Z_1(t) \geq \varepsilon, Z_2(t)=0\}} dt = 0, \quad \mathbb{P}^z_\mu\text{-a.s.}$$

The last identity follows because $t \leq T \wedge T_c$, $Z_1(t) \geq \varepsilon$ and $Z_2(t) = 0$ implies that $t \in [\bar{\sigma}_k, \bar{\tau}_k]$ for some $k \geq 1$. The statement of the Lemma now follows by $T, C \uparrow \infty$ and $\varepsilon \downarrow 0$.

□

Let $\{\mathbb{P}^z_\mu, z \in S\}$ be a solution to the submartingale problem with a drift μ. Next we shall create a process X which is a Brownian motion with drift μ started at z on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P}^z_\mu)$, for every $z \in S$. We already know that such a process exists for the solution that we created in Proposition 5.11. However, for proving the uniqueness of the solution, we need to show the existence of such process X for every solution of the submartingale problem. Such construction has been carried out in [49] and in [44] for the case of zero drift. The generalization to the case of non-zero drift requires only a few obvious changes to the proofs in the case of zero drift, so here we shall only state the results (Lemmas 5.13 and 5.14) without proofs.

For each $\delta > 0$, let $S_\delta \subset S$ be the closed set defined in the complex plane by $S_\delta = S + \delta e^{i\xi/2}$. So S_δ is a wedge with vertex at $\delta\left(\cos\left(\xi/2\right), \sin\left(\xi/2\right)\right)$, such that it is included in S and has edges parallel with the respective edges of S.

Set $\tau_0^\delta = 0$, and, for each $k \geq 1$, recursively define

$$\sigma_k^\delta = \inf\{t \geq \tau_{k-1}^\delta : Z(t) \in S_{2\delta}\} \text{ and } \tau_k^\delta = \inf\{t \geq \sigma_k^\delta : Z(t) \in S \backslash S_\delta\}.$$

By Problem 1.2.7 in Karatzas and Shreve [45], σ_k^δ and τ_k^δ are stopping times relative to $\{\mathcal{M}_t, t \geq 0\}$ for every $k \geq 1$.

For each $k \geq 1$ and $\delta > 0$, define the process $\{W_{(k)}^\delta(t), t \geq 0\}$ by setting

$$W_{(k)}^\delta(t) = Z(t \wedge \tau_k^\delta) - Z(t \wedge \sigma_k^\delta) - (t \wedge \tau_k^\delta - t \wedge \sigma_k^\delta)\mu, \quad t \geq 0,$$

and then define the process $\{W^\delta(t), t \geq 0\}$ by setting

$$W^\delta(t) = \sum_{k=1}^{\infty} W_{(k)}^\delta(t), \quad t \geq 0.$$

Lemma 5.13. *For every $\delta > 0$ and $z \in S$ the process W^δ is a square-integrable martingale on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P}_\mu^z)$.*

Lemma 5.14. *There exists a process W on $(C_S, \mathcal{M}, \mathcal{M}_t)$ such that for every $z \in S$ it is a standard 2-dimensional Brownian motion under $\mathbb{P}_\mu^z$ starting at zero, and for every fixed $T > 0$*

we have

$$E^z_\mu \left[\|W(T) - W^\delta(T)\|^2\right] \to 0, \quad \text{as } \delta \to 0. \tag{5.16}$$

Next, we shall define the process X by

$$X(t,\omega) = \omega(0) + W(t,\omega) + \mu t, \quad t \geq 0. \tag{5.17}$$

We define the process Y by

$$Y(t) = Z(t) - X(t), \quad t \geq 0. \tag{5.18}$$

We shall say that a function $f : \mathbb{R}_+ \mapsto \mathbb{R}^2$ is *flat* on an interval $[s,t] \subset \mathbb{R}_+$, if for every $u \in [s,t]$ we have $f(u) = f(s)$.

Lemma 5.15. *Let $\{\mathbb{P}^z_\mu; z \in S\}$ be an arbitrary solution of the submartingale problem, and let X and Y be the processes defined above. Then the following two statements hold for every $z \in S$:*

1. *Under $\mathbb{P}^z_\mu$ the process X is a standard 2-dimensional Brownian motion on $(C_S, \mathcal{M}, \mathcal{M}_t)$ with drift μ, started at z;*

2. *for every $n \in \mathbb{N}_+$, and $\delta > 0$, the sample paths of Y are flat on $[\sigma^\delta_n, \tau^\delta_n]$, $\mathbb{P}^z_\mu$-a.s.*

Proof. The first statement follows from Lemma 5.14 and property 1 in the definition of the Submartingale Problem. Next, we shall prove the second statement. By the definition of w^δ, the sample paths of

$$\{Z_t - w^\delta(t) - \mu t,\ t \geq 0\} \text{ are flat on } [\sigma^\delta_n, \tau^\delta_n], \tag{5.19}$$

for each $\delta > 0$, $n \geq 1$. On the other hand, for every $\delta > 0$, $n \geq 1$ there exists $k \geq 1$ (depending on the sample path) such that $[\sigma_n^\delta, \tau_n^\delta] \subset [\sigma_k^{\delta/2}, \tau_k^{\delta/2}]$. This implies that the sample paths of $\{Z_t - w^{\delta/2}(t) - \mu t,\ t \geq 0\}$ are also flat on $[\sigma_n^\delta, \tau_n^\delta]$. Iterating this we get that for every $m \geq 1$ the sample paths of $\{Z_t - w^{\delta/2^m}(t) - \mu t,\ t \geq 0\}$ are also flat on $[\sigma_n^\delta, \tau_n^\delta]$. Comparing this with (5.19) we conclude that the sample paths of $w^\delta - w^{\delta/2^m}$ are also flat on $[\sigma_n^\delta, \tau_n^\delta]$, thus for every $t \geq 0$

$$\int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dw^{\delta/2^m}(s) = \int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dw^\delta(s).$$

Taking limit as $m \to \infty$ and using (5.16) we get that

$$\int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dw(s) = \int_0^t 1_{[\sigma_n^\delta, \tau_n^\delta]}(s) dw^\delta(s),$$

$\mathbb{P}_\mu^z$-a.s. This identity and (5.19) imply that the sample paths of $\{Z_t - w(t) - \mu t,\ t \geq 0\}$ are flat on $[\sigma_n^\delta, \tau_n^\delta] \cap [0, t]$, $\mathbb{P}_\mu^z$-a.s. Since $t \geq 0$ was arbitrary, this and (5.17) imply what we wanted to prove.

□

Lemma 5.16. *Suppose that Q_1 and Q_2 are mutually absolutely continuous probability measures on $\mathcal{M}$, both satisfying properties 1,2, and 3 of Definition 6 with $\mathbb{P}_\mu^z$ replaced by Q_i ($i = 1, 2$). Then there exist probability measures $\tilde{Q}_i$ on $\mathcal{M}$ for $i = 1, 2$, such that conditions 1,2, and 3 of Definition 6 are satisfied with P_μ^z replaced by $\tilde{Q}_i$ and μ replaced by the zero vector. Furthermore, for every $T \geq 0$ there exist probability measures $\tilde{Q}_1^T$ and $\tilde{Q}_2^T$ on $\mathcal{M}$ such that for all $T \geq 0$, $A \in \mathcal{M}_T$, and $i = 1, 2$ we have $\tilde{Q}_i^T(A) = \tilde{Q}_i(A)$, $\tilde{Q}_i^T$ and Q_i are mutually* *absolutely continuous, and*

$$\frac{d\tilde{Q}_1^T}{dQ_1} = \frac{d\tilde{Q}_2^T}{dQ_2}. \tag{5.20}$$

Proof. Let Q_1 and Q_2 as above, and let (X^i, Y^i) be as in Lemma 5.15 defined under $\mathbb{Q}_i$. Since X^i is defined by $L^2(\mathbb{Q}_i)$ convergence, this implies that $(X^1, Y^1) = (X^2, Y^2)$, which we shall from here on denote by (X, Y). In this proof we shall use the Doob-Meyer decomposition which requires that the probability space satisfies the "usual conditions", and for this purpose, we have to augment the probability space $(C_S, \mathcal{M}, (\mathcal{M}_t), Q_i)$, $i = 1, 2$; let this augmentation be $(C_S, \mathcal{F}, (\mathcal{F}_t), Q_i)$. The measures Q_1 and Q_2 are mutually absolutely continuous, hence the filtration $(\mathcal{F}_t)$ and the sigma field $\mathcal{F}$ do not depend on $i = 1, 2$. For technical details concerning the augmentation of a probability space please see Remark A.2 in the Appendix. In this proof, all processes live on the augmented space $(C_S, \mathcal{F}, (\mathcal{F}_t))$, unless specified otherwise.

Next, note that for each $T > 0$ and $\delta > 0$ and $n \geq 1$,

$$\left\{ Z(\tau_n^\delta \wedge t) - Z(\sigma_n^\delta \wedge t),\ t \in [0, T] \right\}$$

is a semimartingale under both Q_1 and Q_2 with respect to the filtration $(\mathcal{F}_t)$. Indeed, it can be written by Lemma 5.15 and by (5.18) as

$$Z(\tau_n^\delta \wedge t) - Z(\sigma_n^\delta \wedge t) = X(\tau_n^\delta \wedge t) - X(\sigma_n^\delta \wedge t), \quad t \in [0, T].$$

Now let $f \in C_b^2(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$, and f is constant in a neighborhood of the origin. Then, by Itô's rule we have that for $t \in [0, T]$,

$$f(Z(t \wedge \tau_n^\delta)) = f(Z(t \wedge \sigma_n^\delta)) + \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \nabla f(Z(s)) dX(s) + \frac{1}{2} \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \Delta f(Z(s)) ds, \tag{5.21}$$

Q_i-a.s., $i = 1, 2$. On the other hand, by condition 2 of Definition 6 and by Theorems I.4.10 and I.4.14 in [45], we have for $i = 1, 2$, the unique Doob-Meyer decomposition

$$f(Z(t)) = f(z) + \int_0^t \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2}\int_0^t \Delta f(Z(s))ds + M^i(t) + A^i(t),\ t \leq T, \quad (5.22)$$

where M^i is a continuous martingale and A^i is a continuous, increasing process on $(C_S, \mathcal{F}, (\mathcal{F}_t), Q_i)$, with $M^i(0) = A^i(0) = 0$. By Proposition 16.32 in Bass [6], we also have that for $T \geq 0$,

$$\mathbb{E}^{\mathbb{Q}_i}[||M^i(T)||^2] < \infty,\ i = 1, 2.$$

Let W as in (5.17), that is, $W(t) = X(t) - z - \mu t$, and for $i = 1, 2$, let $S^i(W)$ be the class of $\mathbb{R}^2$-valued processes on $(C_S, \mathcal{F}, (\mathcal{F}_t))$ such that $U \in S^i(X)$ if it has the form

$$U(t) = \int_0^t G(s)dW(s),\ t \in [0, T],$$

for some 2-dimensional process G such that

$$\mathbb{E}^{\mathbb{Q}_i}\left[\int_0^T ||G(s)||^2 ds\right] < \infty.$$

Then, by Theorem IV.36 and Corollary 1 to Theorem IV.37 in [58], there exists a $\mathbb{R}^2$-valued process H^i such that

$$M^i(t) = \int_0^t H^i(s)dW(s) + N^i(t),\ t \in [0, T],$$

where

(i) N^i is a square-integrable martingale under $\mathbb{Q}_i$,

(ii) N^i is strongly orthogonal to every member of $S^i(W)$ under $\mathbb{Q}_i$, that is, N^iU is a $\mathbb{Q}_i$-martingale for each $U \in S^i(W)$,

(iii) $\mathbb{E}^{\mathbb{Q}_i}\left[\int_0^T ||H^i(s)||^2 ds\right] < \infty$.

Now, by (5.22) we have for $t \in [0, T]$,

$$\begin{aligned} f(Z(t)) &= f(z) + \int_0^t \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2}\int_0^t \Delta f(Z(s)) ds \\ &+ \int_0^t H^i(s) dW(s) + N^i(t) + A^i(t),\ Q_i - \text{a.s.}, \end{aligned} \tag{5.23}$$

hence, by (5.23) we have for $t \in [0, T]$,

$$f(Z(t \wedge \tau_n^\delta)) = f(Z(t \wedge \sigma_n^\delta)) + \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \nabla f(Z(s)) \cdot \mu ds + \frac{1}{2}\int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \Delta f(Z(s)) ds$$

$$+ \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} H^i(s) dW(s) + N^i(t \wedge \tau_n^\delta) - N^i(t \wedge \sigma_n^\delta) + A^i(t \wedge \tau_n^\delta) - A^i(t \wedge \sigma_n^\delta), \tag{5.24}$$

Q_i-a.s. Now, for each $i = 1, 2$, we have two Doob-Meyer decompositions of the submartingale

$$\{f(Z(t \wedge \tau_n^\delta)) - f(Z(t \wedge \sigma_n^\delta)) - \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \nabla f(Z(s)) \cdot \mu ds - \frac{1}{2}\int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \Delta f(Z(s)) ds,\ t \in [0, T]\},$$

(5.21) and (5.24). Hence, by the uniqueness of the Doob-Meyer decomposition, for each $t \in [0, T]$

$$N^i(t \wedge \tau_n^\delta) - N^i(t \wedge \sigma_n^\delta) + \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} H^i(s) dW(s) = \int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta} \nabla f(Z(s)) dW(s), \tag{5.25}$$

Q_i- a.s. N_i is strongly orthogonal to every member of $S^i(W)$, and from [58], Theorem IV.37 follows that $\{N^i(t\wedge\tau_n^\delta)-N^i(t\wedge\sigma_n^\delta),\ t\in[0,T]\}$ is also strongly orthogonal to every member of $S^i(W)$ under Q_i. However, by the above relation, it is also a member of $S^i(W)$, hence it follows that $N^i(t\wedge\tau_n^\delta)-N^i(t\wedge\sigma_n^\delta)=0$ for $t\in[0,T]$. Then by (5.25) we also have

$$\mathbb{E}^{\mathbb{Q}_i}\left[\int_{t\wedge\sigma_n^\delta}^{t\wedge\tau_n^\delta}||H^i(s)-\nabla f(Z(s))||^2ds\right]=0.$$

Hence,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_i}\left[\int_0^t||H^i(s)-\nabla f(Z(s))||^2ds\right]\\
=\ &\mathbb{E}^{\mathbb{Q}_i}\left[\int_0^t\sum_{n=2}^{\infty}\mathbb{1}_{\left\{s\in[\tau_{n-1}^\delta,\sigma_n^\delta]\right\}}||H^i(s)-\nabla f(Z(s))||^2ds\right]\\
\le\ &\mathbb{E}^{\mathbb{Q}_i}\left[\int_0^t\mathbb{1}_{\left\{Z(s)\in S_{2\delta}^c\right\}}||H^i(s)-\nabla f(Z(s))||^2ds\right],
\end{aligned}\tag{5.26}$$

where $S_{2\delta}^c=S\backslash S_{2\delta}$. Moreover, by the dominated convergence theorem,

$$(5.26)\to\mathbb{E}^{\mathbb{Q}_i}\left[\int_0^t\mathbb{1}_{\left\{Z(s)\in\partial S\right\}}||H^i(s)-\nabla f(Z(s))||^2ds\right]=0\text{ as }\delta\to0,$$

where the last identity is by Lemma 5.12 . By (5.23), now follows that for $t\in[0,T]$,

$$\begin{aligned}
f(Z(t))\ &=\ f(z)+\int_0^t\nabla f(Z(s))\cdot\mu ds+\frac{1}{2}\int_0^t\Delta f(Z(s))ds\\
&+\ \int_0^t\nabla f(Z(s))dW(s)+N^i(t)+A^i(t),\ Q_i\text{−a.s.}
\end{aligned}\tag{5.27}$$

Next, for each $t \geq 0$ let

$$\tilde{\zeta}(t) = \exp\left\{ -\mu \cdot (X(t) - z) + \frac{1}{2}||\mu||^2 t \right\},$$

and for each $i = 1, 2$, and $T \geq 0$ define the measure $\tilde{\mathbb{Q}}_i^T$ by setting

$$\frac{d\tilde{\mathbb{Q}}_i^T}{d\mathbb{Q}_i} = \tilde{\zeta}(T). \tag{5.28}$$

Then, under $\tilde{\mathbb{Q}}_i^T$, $\{X(t), t \in [0, T]\}$ is a Brownian motion (without drift) started at z, and by (5.27) we have

$$\begin{aligned} f(Z(t)) &= f(z) + \int_0^t \nabla f(Z(s)) dX(s) \\ &+ \frac{1}{2}\int_0^t \Delta f(Z(s)) ds + N^i(t) + A^i(t), \ Q_i\text{–a.s.} \end{aligned} \tag{5.29}$$

However, note that N^i is also a $\tilde{\mathbb{Q}}_i^T$-martingale on $[0, T]$. Indeed, by (5.28), N^i is a $\tilde{\mathbb{Q}}_i^T$-martingale if $N^i\tilde{\zeta}$ is a $\mathbb{Q}_i$-martingale on $[0, T]$. But this follows since from the fact that N_i is strongly orthogonal to every member of $S^i(X)$ under $\mathbb{Q}_i$, and by its definition $\tilde{\zeta} - 1 \in S^i(W)$ under $\mathbb{Q}_i$. Just like in the proof of Proposition 5.11, there exists a probability measure $\tilde{\mathbb{Q}}_i$ on $\mathcal{M}$ such that $\tilde{\mathbb{Q}}_i(A) = \tilde{\mathbb{Q}}_i^T(A)$, whenever $A \in \mathcal{M}_T$. Thus, by (5.29) both $\tilde{\mathbb{Q}}_1$ and $\tilde{\mathbb{Q}}_2$ satisfy property 2 of Definition 6 with $\mathbb{P}_\mu^z$ replaced by either $\tilde{Q}_1$ or $\tilde{Q}_2$, and μ replaced by the zero vector. $\tilde{Q}_i^T$ and Q_i are mutually absolutely continuous because of (5.28) and because $\tilde{\zeta}(T) > 0$, a.s. under Q_i. Relation (5.20) also follows from (5.28). Properties 1 and 3 of Definition 6 are satisfied if $\mathbb{P}_\mu^z$ is replaced by $\tilde{Q}_i$ because they are satisfied if we replace $\mathbb{P}_\mu^z$ by Q_i, and we already established that $\tilde{Q}_i^T$ and Q_i are mutually absolutely continuous. Property

1 follows immediately from this. Property 3 can be shown by first showing it with the ∞ in the upper limit of the integral replaced by T, then taking $T \to \infty$.

$\square$

Proof of Theorem 5.1 . In light of Propositions 5.11 the only missing part is the proof of uniqueness. Let $z \in S$ and suppose that $\mathbb{P}_1^z$ and $\mathbb{P}_2^z$ are two probability measures satisfying conditions 1, 2, and 3 in Definition 6 of the submartingale problem with drift. Let

$$\mathbb{Q}_1 = \frac{1}{3}\mathbb{P}_1^z + \frac{2}{3}\mathbb{P}_2^z \quad \text{and} \quad \mathbb{Q}_2 = \frac{2}{3}\mathbb{P}_1^z + \frac{1}{3}\mathbb{P}_2^z.$$

Then, one can check that each $\mathbb{Q}_i$ also satisfies conditions 1,2, and 3 in Definition 6 of the submartingale problem with drift. In addition, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are mutually absolutely continuous. In order to complete the proof, it is therefore sufficient to show that $\mathbb{Q}_1 \equiv \mathbb{Q}_2$. By Lemma 5.16 there exist probability measures $\tilde{Q}_i$, $i = 1, 2$, such that properties 1,2, and 3 of Definition 6 are satisfied with P_μ^z replaced by $\tilde{Q}_i$, and μ replaced by the zero vector. The uniqueness result in Section 3.1 of [74] implies $\tilde{\mathbb{Q}}_1 = \tilde{\mathbb{Q}}_2$. Using the probability measures $\tilde{Q}_i^T$ from the same proposition, we have now that

$$\tilde{Q}_1^T|_{\mathcal{M}_T} = \tilde{Q}_2^T|_{\mathcal{M}_T}.$$

From (5.20) follows that

$$Q_1|_{\mathcal{M}_T} = Q_2|_{\mathcal{M}_T}.$$

Since T was arbitrary, $Q_1 = Q_2$ follows. $\square$

Proof of Theorem 5.2. Recall from the proof of Proposition 5.11 that for every $T \geq 0$ there exists the probability measure $\mathbb{P}^z_{\mu,T}$ which is mutually absolutely continuous with respect to $\mathbb{P}^z_0$, and coincides with $\mathbb{P}^z_\mu$ on $\mathcal{M}_T$. By Theorem 2.6 in [49], formulas (5.2) and (5.3) hold for $\mu = 0$. Then by the mutual absolute continuity of $\mathbb{P}^z_0$ and $\mathbb{P}^z_{\mu,T}$, (5.2) and (5.3) also hold with $\mathbb{P}^z_\mu$ replaced by $\mathbb{P}^z_{\mu,T}$. Since $\mathbb{P}^z_\mu$ and $\mathbb{P}^z_{\mu,T}$ coincide on $\mathcal{M}_T$, both formulas follow. □

Proof of Theorem 5.3. This follows from Theorem 2.4 in [49] using the measure $\mathbb{P}^z_{\mu,T}$ for every $T > 0$, just like in the proof of Theorem 5.2. □

Proof of Theorem 5.4. Suppose that $1 \le \alpha < 2$, and Z is a continuous semimartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$ for some $z \in S$. Then there exists a decomposition

$$Z(t) = z + M(t) + A(t), \quad t \in [0, \infty), \tag{5.30}$$

where M is a continuous local martingale and A is a finite variation (FV) process on $(C_S, \mathcal{M}, \mathcal{M}_t, \mathbb{P}^z_\mu)$ (see [58], Corollary to Theorem II.31). We know from the proof of Proposition 5.11 that for every $T \in [0, \infty)$ there exists a probability measure $P^z_{\mu,T}$ on $\mathcal{M}$ which is mutually absolutely continuous with respect to $\mathbb{P}^z_0$, and $\mathbb{P}^z_{\mu,T}(A) = \mathbb{P}^z_\mu(A)$ for all $A \in \mathcal{M}_T$. In addition,

$$\frac{d\mathbb{P}^z_0}{d\mathbb{P}^z_{\mu,T}} = \frac{1}{\zeta(T)}$$

where ζ is defined under (5.5). We cast (5.30) in the form

$$Z(t) = z + \tilde{M}(t) + \tilde{A}(t), \quad t \in [0, \infty), \tag{5.31}$$

where

$$\tilde{M}(t) = M(t) - \int_0^t \zeta(s) d \left[\frac{1}{\zeta}, M \right]_s$$

$$\tilde{A}(t) = A(t) + \int_0^t \zeta(s) d\left[\frac{1}{\zeta}, M\right]_s .$$

By the Girsanov-Meyer theorem ([58], Theorem III.31) $\tilde{M}$ is a local martingale on $[0,T]$ and $\tilde{A}$ is a FV process under $\mathbb{P}_0^z$. But this implies that $\tilde{M}$ is a local martingale on $[0,\infty)$ under $\mathbb{P}_0^z$, hence Z must be a semimartingale under $\mathbb{P}_0^z$, which is in contradiction with the result of [77], Theorem 5.

□

Proof of Theorem 5.5. Suppose that $\alpha \geq 2$, let $z \in S$ and suppose that $\mathbb{P}_\mu^z$ is a probability measure on $\mathcal{M}$ satisfying properties 1, 2 and 3 in Definition 6 of the submartingale problem with drift. Selecting $Q_1 = Q_2 = \mathbb{P}_\mu^z$ in Lemma 5.16, it follows that there exists a probability measure $\tilde{Q}$ on $\mathcal{M}$ such that properties 1,2, and 3 of Definition 6 are satisfied with $\mathbb{P}_\mu^z$ replaced by $\tilde{Q}$, and μ replaced by the zero vector. However, this is in direct contradiction with Theorem 3.11 in [74]. □

5.5 Proof of Theorems 5.8 and 5.6

First, we shall prove Theorem 5.8.

Proposition 5.17. *The family of probability measures $\{\mathbb{P}_\mu^{z_n}\}$ is tight for any sequence $\{z_n, n \geq 1\}$ in S which converges to some $z \in S$.*

Proof. By Theorem 2.4.10 in [45], it is sufficient to show that

$$\lim_{\delta \downarrow 0} \sup_n \mathbb{P}_\mu^{z_n}(\omega : m^T(\omega, \delta) \geq \varepsilon) = 0, \text{ for any } T > 0, \varepsilon > 0. \tag{5.32}$$

In the above,

$$m^T(\omega,\delta) = \sup_{\substack{|t-s|\leq\delta \\ 0\leq s,t\leq T}} |\omega(s)-\omega(t)|.$$

Using (5.5) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \mathbb{P}_\mu^{z_n}(\omega : m^T(\omega,\delta) \geq \varepsilon) \\
= \;& \mathbb{E}_0^{z_n}[\mathbb{1}_{\{m^T(\omega,\delta)>\varepsilon\}} \exp\{\mu\cdot(X(T)-X(0)) - \frac{1}{2}||\mu||^2 T\}] \\
\leq \;& \left(\mathbb{E}_0^{z_n}[\exp\{2\mu\cdot(X(T)-X(0)) - ||\mu||^2 T\}]\mathbb{P}_0^{z_n}(m^T(\omega,\delta)>\varepsilon)\right)^{1/2} \\
= \;& \exp\left\{\frac{1}{2}||\mu||^2 T\right\}\left(\mathbb{P}_0^{z_n}(m^T(\omega,\delta)>\varepsilon)\right)^{1/2}. \qquad (5.33)
\end{aligned}$$

By Theorem 3.13 in [74], $\{\mathbb{P}_0^{z_n}\}$ is tight hence

$$\lim_{\delta\downarrow 0}\sup_n \mathbb{P}_0^{z_n}(\omega : m^T(\omega,\delta)\geq\varepsilon) = 0, \text{ for any } T>0, \varepsilon>0,$$

combining with inequality (5.33), we have (5.32). □

Proof of Theorem 5.8. Given Proposition 5.17, it only remains to show that any weak limit point $\mathbb{P}_\mu^*$ of the family $\{\mathbb{P}_\mu^{z_n}\}$ is a solution to the submartingale problem starting from z, then by the uniqueness part of Theorem 5.1, $\mathbb{P}_\mu^{z_n} \Rightarrow \mathbb{P}_\mu^z$ as $n\to\infty$.

It is straightforward that $\mathbb{P}_\mu^*$ satisfies condition 1 in Definition 6 (the submartingale problem), since for any $k\geq 1$ and the closed set $C_k = \{\omega\in C_S : |\omega(0)-z| \leq \frac{1}{k}\}$, $1 = \limsup_n \mathbb{P}_\mu^{z_n}(C_k) \leq \mathbb{P}_\mu^*(C_k)$ hence $\mathbb{P}_\mu^*$ concentrates on $\{\omega\in C_S : \omega(0)=z\}$. The condition 2 is also satisfied since the submartingale property is preserved under the weak convergence. Now we prove $\mathbb{P}_\mu^*$ satisfies condition 3, we need to show that if $(z_n, n\geq 1)\subset S$, $z\in S$ such that $\lim_{n\to\infty} z_n = z$ and $\mathbb{P}_\mu^{z_n} \Rightarrow \mathbb{P}_\mu^*$, then

$$\mathbb{E}_\mu^*\left[\int_0^\infty \mathbb{1}_{\{(Z(t))=0\}}dt\right] = 0. \tag{5.34}$$

Let $\epsilon > 0$ and $t \geq 0$ be arbitrary. Under the local uniform topology the event $\{w \in C_S : |Z_t(\omega)| < \epsilon\}$ is an open set, and $\{w \in C_S : |Z_t(\omega)| \leq \epsilon\}$ is a closed set. By the Portmanteau Theorem, (5.5), and the Cauchy-Schwarz inequality

$$\mathbb{P}_\mu^*\left(|Z_t| < \epsilon\right) \leq \liminf_n \mathbb{P}_\mu^{z_n}\left(|Z_t| < \epsilon\right) = \liminf_n \mathbb{E}_0^{z_n}\left[\zeta(t)1_{\{|Z_t|<\epsilon\}}\right] \leq$$

$$\liminf_n \left(\mathbb{E}_0^{z_n}\left[(\zeta(t))^2\right]\mathbb{P}_0^{z_n}\left(|Z_t| < \epsilon\right)\right)^{1/2} \leq \exp\left\{\frac{1}{2}|\mu|^2 t\right\}\limsup_n \left(\mathbb{P}_0^{z_n}\left(|Z_t| \leq \epsilon\right)\right)^{1/2} \leq$$

$$\exp\left\{\frac{1}{2}|\mu|^2 t\right\}\left(\mathbb{P}_0^{z}\left(|Z_t| \leq \epsilon\right)\right)^{1/2}.$$

The last inequality follows from the Feller property for the drift-less case ([74], Theorem 3.13). We now let $\epsilon \downarrow 0$, and then from property 3 in the definition of the Submartingale Problem follows that

$$\mathbb{P}_\mu^*\left(Z_t = 0\right) = 0,$$

for Lebesgue-almost every $t \geq 0$. Identity (5.34) follows.

□

Proof of Theorem 5.6. This follows in the standard way from the uniqueness of the solution to the submartingale problem, using the regular conditional probability measures for $\mathcal{M}$ given $\mathcal{M}_\tau$ under $\mathbb{P}_\mu^z$. In particular, Lemma 3.1, and Corollary 3.3 in [74] remain true in the presence of a drift, without changing a single word in their proofs. Then the strong Markov property follows, again exactly the same way as in [74], Theorem 3.14.

□

5.6 Proof of Theorem 5.9

Proof of Theorem 5.9. It is sufficient to show that the $\hat{C}(S)$-Feller property holds in the case when the drift is zero. Indeed, suppose that for the driftless case the $\hat{C}(S)$-Feller property holds. By Theorem 1.10 in Bottcher, Schilling, and Wang [10], the $\hat{C}(S)$-Feller property holds for $\{\mathbb{P}^z_\mu, \forall z \in S\}$ if and only if there exists an increasing sequence of bounded sets $B_n \in \mathcal{B}(S)$ with $\cup_{n\geq 1} B_n = S$ such that for every $t > 0$ and $n \geq 1$

$$\lim_{|z|\to\infty} \mathbb{P}^z_\mu(Z(t) \in B_n) = 0.$$

Since we already know that uniqueness holds for the submartingale problem with drift μ, we may assume that $\{\mathbb{P}^z_\mu, z \in S\}$ is exactly the family we created in the existence part of this paper using Girsanov's theorem. By Theorem 1.10 in [10], there exists an increasing sequence of bounded sets $B_n \in \mathcal{B}(S)$ with $\cup_{n\geq 1} B_n = S$ such that for every $t > 0$ and $n \geq 1$,

$$\lim_{|z|\to\infty} \mathbb{P}^z_0(Z(t) \in B_n) = 0,$$

where $\{\mathbb{P}^z, z \in S\}$ is the solution of the submartingale problem without drift. By formula (5.5) we have that

$$\mathbb{P}^z_\mu(Z(t) \in B_n) = \mathbb{E}^z_0\left[\zeta(t)\mathbb{1}_{\{Z(t)\in B_n\}}\right] \leq \left(\mathbb{E}^z_0\left[(\zeta(t))^2\right]\mathbb{P}^z(Z(t) \in B_n)\right)^{1/2}$$

$$= \exp\left\{\frac{\mu^2}{2}t\right\}\left(\mathbb{P}_0^z\left(Z(t) \in B_n\right)\right)^{1/2},$$

which shows that the $\hat{C}(S)$-Feller property holds for $\{\mathbb{P}_\mu^z, \forall z \in S\}$.

In the rest of this proof, we shall show that in the case of $\mu = 0$ the $\hat{C}(S)$-Feller property holds. Let X be the process identified in Proposition 5.10. It has been shown in Williams and Varadhan [74] that the Feller property holds, hence the $C_b(S)$-Feller property holds, so by Theorem 1.10 in [10], it is sufficient to show that for every $t > 0$

$$\mathbb{P}_0^z(Z(t) \in B_n) \to 0$$

as $|z| \to \infty$, where $B_n = \{z \in S : |z| \leq n\}$, for $n \geq 1$. Note that

$$\mathbb{P}_0^z(Z(t) \in B_n) = \mathbb{P}_0^z(Z(t) \in B_n, \tau \leq t) + \mathbb{P}_0^z(Z(t) \in B_n, \tau > t),$$

where

$$\tau = \inf\{t \geq 0 : Z(t) \in \partial S\}.$$

We treat the second term first. It is bounded above by

$$\mathbb{P}_0^z(X(t) \in B_n) \leq \mathbb{P}_0^z(|X(t) - z| \geq |z| - n) = \Pr(|w(t)| > |z| - n) \to 0,$$

as $|z| \to \infty$, where w is a standard 2-dimensional Brownian motion started at zero. Next, we treat the first term. Let $T_n = \inf\{t \geq 0 : Z(t) \in B_n\}$. Then the first term can be written as a sum of three terms

$$\mathbb{P}_0^z\left(Z(t) \in B_n, \tau \leq t, |Z(\tau)| < \frac{|z|}{2}\right) +$$

$$\mathbb{P}_0^z\left(Z(t) \in B_n, \tau \le t, T_n \le \tau, |Z(\tau)| \ge \frac{|z|}{2}\right) +$$

$$\mathbb{P}_0^z\left(Z(t) \in B_n, \tau \le t, T_n > \tau, |Z(\tau)| \ge \frac{|z|}{2}\right)$$

Again, we treat the three terms separately. By Proposition 5.10 we have that $Z(\tau) = X(\tau)$, where X is a standard Brownian motion with zero drift started at z under $\mathbb{P}_0^z$, thus first term is bounded above by

$$\mathbb{P}_0^z\left(\tau \le t, |X(\tau)| < \frac{|z|}{2}\right) \le \mathbb{P}_0^z\left(\tau \le t, |X(\tau) - z| > \frac{|z|}{2}\right) \le \Pr\left(\max_{s \le t} |w(s)| > \frac{|z|}{2}\right) \to 0,$$

as $|z| \to \infty$. The second term is bounded above by

$$\mathbb{P}_0^z(T_n \le \tau \le t) \le \mathbb{P}_0^z(X - z \text{ reaches } B_n - z \text{ by time } t) = \Pr(w \text{ reaches } B_n - z \text{ by time } t) \to 0$$

as $|z| \to \infty$. Here w is a standard 2-dimensional Brownian motion started at the origin.

For analyzing the third term we define the stopping time $T_n^\tau = \inf\{t \ge \tau : Z(t) \in B_n\}$. The third term is bounded above by

$$\mathbb{P}_0^z\left(\tau < T_n \le t, |Z(\tau)| > \frac{|z|}{2}\right) \le \mathbb{P}_0^z\left(\tau < T_n \le \tau + t, |Z(\tau)| > \frac{|z|}{2}\right) \le$$

$$\mathbb{P}_0^z\left(T_n^\tau \le t, |Z(\tau)| > \frac{|z|}{2}\right).$$

By the strong Markov property, this can be written as

$$\int_{\partial S \cap B_{|z|/2}^c} \mathbb{P}_0^x(T_n \le t) P_0^z(Z(\tau) \in dx).$$

By the scaling property (Lemma 2.1 in [78]), the process $\{Z(t), t \geq 0\}$ under $\mathbb{P}_0^x$ induces the same measure on $\mathcal{M}$ as $\{|x|Z(t/|x|^2), t \geq 0\}$ induces under $\mathbb{P}_0^{x/|x|}$, for every non-zero $x \in S$. Then the above expression can be written as

$$\int_{\partial S \cap B^c_{|z|/2}} \mathbb{P}_0^{x/|x|}\left(|x|^2 T_{n/|x|} \leq t\right) \mathbb{P}_0^z(Z(\tau) \in dx) =$$

$$\int_{\partial S_1 \cap B^c_{|z|/2}} \mathbb{P}_0^{u_1}\left(|x|^2 T_{n/|x|} \leq t\right) \mathbb{P}_0^z(Z(\tau) \in dx) +$$
$$\int_{\partial S_2 \cap B^c_{|z|/2}} \mathbb{P}_0^{u_2}\left(|x|^2 T_{n/|x|} \leq t\right) \mathbb{P}_0^z(Z(\tau) \in dx),$$

where u_1 and u_2 are the unit vectors $u_1 = (1, 0)$, and $u_2 = (\cos\xi, \sin\xi)$. By symmetry, it is sufficient to show that the first term converges to 0 as $|z| \to \infty$. If $|z|/2 > 2n$, then it is bounded above by

$$\sup_{|x| > |z|/2} \mathbb{P}_0^{u_1}(|x|^2 T_{1/2} \leq t) = \mathbb{P}_0^{u_1}\left(\frac{|z|^2}{4} T_{1/2} \leq t\right) \to 0,$$

as $|z| \to \infty$, which completes the proof of the proposition.

□

5.7 Proof of Theorem 4.1

Before proving the existence part of Theorem 4.1, we must first establish some preliminary results. Let $\{B_t, t \geq 0\}$ be the coordinate mapping process on $C(\mathbb{R}_+, \mathbb{R}^2)$, whose natural filtration is given by $\mathcal{W}_t = \sigma(B_s, 0 \leq s \leq t)$ for $t \geq 0$, and let $\mathcal{W} = \sigma(B_s, s \geq 0)$. Recall that v_i is the reflection direction on ∂S_i for $i = 1, 2$, and let R be the 2×2 matrix defined by

R_{ij} = the i-th component of v_j. The following result is adapted from Theorem 3.1 in [76].

Proposition 5.18. *For any $w \in C(\mathbb{R}_+, \mathbb{R}^2)$ with $w(0) \in S$, there exists a unique triple (ϕ, η, T_0), where $\phi \in C_S$, $\eta \in C(\mathbb{R}_+, [0, +\infty]^2)$ and $T_0 : C(\mathbb{R}_+, \mathbb{R}^2) \to [0, +\infty]$, satisfying the following four conditions,*

1. *$\phi(t) = w(t) + R\eta(t)$ for each $t \in [0, T_0)$;*
2. *$\phi(t) \neq 0$ for all $t < T_0$ and $\phi(t) = 0$ for all $t \geq T_0$;*
3. *For $j = 1, 2$, $\eta(0) = 0$ and $\eta_j(\cdot)$ is non-decreasing and finite for $t \in [0, T_0)$;*
4. *For $j = 1, 2$, η_j only increases when $\phi(t)$ is on $\partial S_j \backslash \{0\}$.*

Furthermore, we have the following two properties

(i) T_0 is a stopping time on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}, \mathcal{W}_t)$;

(ii) Define the map $\Gamma : C(\mathbb{R}_+, \mathbb{R}^2) \mapsto C_S$ such that $\Gamma(w) = \phi$. Then, $\tau_0 \circ \Gamma = T_0$, the map $\Gamma_t \equiv \Gamma(\cdot)(t)$ is $\mathcal{W}_t$-measurable and Γ is $\mathcal{W}/\mathcal{M}$-measurable.

Now for each $z \in S$, let $\hat{\mathbb{P}}^z_0$ be the unique measure on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}, \mathcal{W}_t)$ under which $\{B_t, t \geq 0\}$ is a standard Brownian motion started at z (the subscript 0 indicates that the Brownian motion has zero drift under $\hat{\mathbb{P}}^z_0$). Next, for each $\mu \in \mathbb{R}^2$ and $T \geq 0$, define the measure $\hat{\mathbb{P}}^z_{\mu,T}$ on $\mathcal{W}_T$ by

$$\frac{d\hat{\mathbb{P}}^z_{\mu,T}}{d\hat{\mathbb{P}}^z_0} = \exp\left(\mu(B_T - z) - \frac{1}{2}||\mu||^2 T\right),$$

and define also

$$\hat{B}_t = B_t - \mu t,$$

and note that $\hat{B}$ is a standard 2 dimensional Brownian motion started at z under $\hat{\mathbb{P}}^z_{\mu,T}$ on $[0,T]$. Then, by Theorem 4.2 in [55] there exists a measure $\hat{\mathbb{P}}^z_\mu$ on $\mathcal{W}$ which coincides with $\hat{\mathbb{P}}^z_{\mu,T}$ on $\mathcal{W}_T$ for all $T \geq 0$. For every $z \in S$ and $\mu \in \mathbb{R}^2$ the process B is a Brownian motion with drift μ started at z under the probability measure $\hat{\mathbb{P}}^z_\mu$. Moreover, since by Proposition 5.18, Γ is a measurable map from $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}, \mathcal{W}_t)$ to $(C_S, \mathcal{M}, \mathcal{M}_t)$, we may denote by $\bar{\mathbb{P}}^z_\mu$ the measure induced on $\mathcal{M}$ by the mapping Γ under $\hat{\mathbb{P}}^z_\mu$, i.e.,

$$\bar{\mathbb{P}}^z_\mu(A) \equiv \hat{\mathbb{P}}^z_\mu(\Gamma^{-1}(A)), \text{ for each } A \in \mathcal{M}. \tag{5.35}$$

We next prove that the family of measures $\{\bar{\mathbb{P}}^z_\mu, z \in S\}$ is a solution to the absorbed process problem of Definition 5. Since conditions 1 and 3 of Definition 5 are trivially satisfied by $\{\bar{\mathbb{P}}^z_\mu, z \in S\}$, it remains to prove that condition 2 is satisfied as well. This is achieved in Lemma 5.19.

Lemma 5.19. *Let the family of measures $\{\bar{\mathbb{P}}^z_\mu, z \in S\}$ be defined as in (5.35) and let Z be the coordinate-mapping process on $(C_S, \mathcal{M}, \mathcal{M}_t)$. Then, the process*

$$\left\{ f(Z(t \wedge \tau_0)) - \int_0^{t\wedge\tau_0} \mu \cdot \nabla f(Z(s))ds - \frac{1}{2}\int_0^{t\wedge\tau_0} \Delta f(Z(s))ds, t \geq 0 \right\} \tag{5.36}$$

is a submartingale on $(C_S, \mathcal{M}, \mathcal{M}_t, \bar{\mathbb{P}}^z_\mu)$, for each $f \in C^2_b(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$.

Proof. For each $w \in C(\mathbb{R}_+, \mathbb{R}^2)$, let $\phi(w) = \Gamma(w)$ and note that by Proposition 5.18 we may write $\phi(t) = w(t) + R\eta(t)$ for all $t \geq 0$. Now, on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}_t, \mathcal{W})$, we consider the process $\{\phi(t), t \geq 0\}$ with $\phi(t) = \Gamma_t(w)$ for any $w \in C(\mathbb{R}_+, \mathbb{R}^2)$. Recall that the coordinate mapping process $\{B_t, t \geq 0\}$ is a Brownian motion on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}_t, \mathcal{W})$ under $\hat{\mathbb{P}}^z$, and

$\{B_t - \mu t, t \geq 0\}$ is a Brownian motion on $(C(\mathbb{R}_+, \mathbb{R}^2), \mathcal{W}_t, \mathcal{W})$ under $\hat{\mathbb{P}}^z_\mu$. Notice, by Theorem 1 in [77], that $R\eta(t)$ is of finite variation on $[0, t \wedge T_0]$ for any $t \geq 0$, and $w(t) = B(w)(t)$, we get that $\{\phi(t), t \geq 0\}$ is a semimartingale under $\hat{\mathbb{P}}^z$. On the other hand, the Girsanov transform keeps the semimartingale property, so $\{\phi(t), t \geq 0\}$ is also a semimartingale under $\hat{\mathbb{P}}^z_\mu$. Hence, for each $f \in C^2_b(S)$ such that $D_i f \geq 0$ on ∂S_i for $i = 1, 2$, we use Itô's formula under $\hat{\mathbb{P}}^z_\mu$ and get

$$\begin{aligned} f(\phi(t \wedge T_0)) - f(\phi(0)) &= \sum_{i=1}^{2} \int_0^{t \wedge T_0} \frac{\partial f}{\partial x_i}(\phi(s)) d(w_i(s) - \mu_i s) + \int_0^{t \wedge T_0} \mu \cdot \nabla f(\phi(s)) ds \\ &+ \int_0^{t \wedge T_0} (D_1 f(\phi(s)), D_2 f(\phi(s))) \cdot d\eta(s) + \frac{1}{2} \int_0^{t \wedge T_0} \Delta f(\phi(s)) ds \end{aligned}$$

Since we have by Proposition 5.18 that for $i = 1, 2$,

$$d\eta_i(s) = \mathbb{1}_{\{\phi(s) \in \partial S_i \setminus \{0\}\}} d\eta_i(s), \quad s \geq 0,$$

and by the assumption on f that for $i = 1, 2$,

$$D_i f(\phi(s)) \mathbb{1}_{\{\phi(s) \in \partial S_i \setminus \{0\}\}} \geq 0, \quad s \geq 0,$$

it follows that the process

$$\left\{ \int_0^{t \wedge T_0} (D_1 f(\phi(s)), D_2 f(\phi(s))) \cdot d\eta(s), t \geq 0 \right\}$$

is increasing. On the other hand, since $\{B_t - \mu t, t \geq 0\}$ is a Brownian motion under $\hat{\mathbb{P}}^z_\mu$, the process

$$\left\{\sum_{i=1}^{2}\int_0^{t\wedge T_0}\frac{\partial f}{\partial x_i}(\phi(s))d(w_i(s)-\mu_i s), t\geq 0\right\}$$

is a martingale under $\hat{\mathbb{P}}^z_\mu$, so

$$\begin{aligned} & f(\phi(t\wedge T_0)) - \int_0^{t\wedge T_0}\mu\cdot\nabla f(\phi(s))ds - \frac{1}{2}\int_0^{t\wedge T_0}\Delta f(\phi(s))ds \\ = \;& f(\phi(0)) + \sum_{i=1}^{2}\int_0^{t\wedge T_0}\frac{\partial f}{\partial x_i}(\phi(s))d(w_i(s)-\mu_i s) \\ + \;& \int_0^{t\wedge T_0}(D_1 f(\phi(s)), D_2 f(\phi(s)))\cdot d\eta(s) \end{aligned}$$

is a submartingale under $\hat{\mathbb{P}}^z_\mu$. It follows from (5.35) that the process under (5.36) is also a submartingale under the induced measure $\bar{\mathbb{P}}^z_\mu$. □

Proof of the existence part of Theorem 4.1. The existence of a solution to the absorbed process problem follows from Lemma 5.19. □

Proof of the uniqueness part of Theorem 4.1 . The proof of the uniqueness of the solution to the absorbed process problem is very similar to that of the solution of the submartingale problem, hence in this section we shall state the appropriate lemmas and indicate the necessary changes in order to adapt the proofs in Section 5.4 to the absorbed process problem.

Lemma 5.20. *Let $\alpha\in\mathbb{R}$ arbitrary, and suppose that $\{\mathbb{P}^{z,0}_\mu, z\in S\}$ is a solution to the absorbed process problem with drift $\mu\in\mathbb{R}^2$. Then, for all $z\in S$,*

$$\mathbb{E}^{z,0}_\mu\left[\int_0^{\tau_0}\mathbb{1}_{\{Z(t)\in\partial S\}}dt\right]=0.$$

Proof. The proof is almost identical to that of Lemma 5.12 with the modification that all processes must be stopped at τ_0.

□

Lemma 5.21. *Let $\alpha \in \mathbb{R}$ be arbitrary. Suppose that $\{\mathbb{P}_\mu^{z,0}, z \in S\}$ is a solution to the absorbed process problem with drift $\mu \in \mathbb{R}^2$. Then there exists a process X on $(C_S, \mathcal{M}, \mathcal{M}_t)$ such that for all $z \in S$, X is a Brownian Motion with drift μ under $\mathbb{P}_\mu^{z,0}$ started at z and stopped at τ_0. In addition, $Y = Z - X$ is flat on $[\sigma_n^\delta, \tau_n^\delta]$.*

Proof. The proof is very similar to the proofs of Lemmas 5.13, 5.14, and 5.15, with the difference that all processes must be stopped at τ_0. □

Proof of the uniqueness part of Theorem 4.1. The proof is basically a copy of the proof of the uniqueness of the solution to the submartingale problem (Lemma 5.16 and the proof of Theorem 5.1). The necessary changes in order to adapt that proof to the present situation are the following:

- All processes must be stopped at τ_0;
- The constraint that the test function f is constant in a neighborhood of the vertex must be erased (above (5.21) in the adapted proof;
- At the end of the proof, instead of using the uniqueness of the solution to the submartingale problem with no drift, we must use the uniqueness of the solution to the absorbed process problem with no drift ([74], Theorem 2.1). □

5.8 Relation of Part I to the published paper version

The material in Part I is the dissertation form of the wedge project that was later published as [48]. The theorem-level content is the same: existence and uniqueness for the submartingale problem with drift in a wedge, the strong Markov and Feller properties, the absorbed problem, and the hitting-probability results at the vertex. The thesis presentation is intentionally more expansive than the published paper in its background chapters and in some of its proof organization, but the mathematical core is aligned with the published version. In the later stages of revising this dissertation, the published paper serves as a reference point for checking statements, citation consistency, and cross-references in Part I.

A useful way to read the relation between the two texts is therefore the following. The published paper may be treated as the compact theorem-level record of the completed Part I results, while the dissertation may be treated as the expanded pedagogical and structural version in which background, intermediate constructions, and proof organization are written out more fully. This is precisely why Part I remains in the thesis even after the paper exists: the thesis has a different expository task from the paper, namely to make the full probabilistic route from queueing motivation to wedge submartingale theory visible in a single self-contained document.

Part II

Reflected Brownian Motion and its Applications to Differential Geometry and Index Theory

Chapter 1

Introduction and statement of the main problems

The second part of this dissertation is devoted to a research program at the interface of probability theory, stochastic analysis on manifolds, differential geometry, and index theory. The central theme is that reflected Brownian motion is not merely a probabilistic model with a boundary condition attached to it. In a number of geometric problems, especially heat-kernel problems on manifolds with boundary, reflected Brownian motion is the *correct local boundary stochastic object*. Once the right heat equation has been identified, reflected Brownian motion, its boundary local time, and the associated multiplicative functionals become a natural mechanism for accessing the boundary contribution in small-time asymptotics.

Part I of the thesis dealt with reflected Brownian motion with drift in a wedge and with the submartingale approach to existence, uniqueness, strong Markov and Feller properties. In Part II we move from wedges to manifolds with boundary, and from the probabilistic study of reflected Brownian motion itself to its geometric applications. The target here is not another diffusion limit theorem. The target is a local index-theoretic problem for Dirac-type operators.

1.1 Motivation from index theory with boundary

On closed manifolds, the heat equation proof of geometric index theorems has several classical forms: the McKean–Singer supertrace formula, Patodi's small-time asymptotics, Getzler rescaling, Gilkey's invariant-theoretic expansion, and Bismut's stochastic local index theorem [53, 56, 31, 7, 32, 8]. In each of these approaches, the heart of the matter is the same: one analyzes the supertrace of the heat kernel near the diagonal and identifies the first coefficient that survives the algebraic cancellation. On a manifold with boundary, however, that coefficient splits into an interior contribution and a boundary contribution, and the latter is much more delicate. It is concentrated in a boundary layer, is sensitive to the choice of boundary condition, and typically involves half-order terms created by boundary local time.

For differential forms, this difficulty was handled probabilistically by Du and Hsu, who gave a reflected-Brownian-motion proof of the Gauss–Bonnet–Chern theorem with boundary. Their work makes explicit that the boundary local time should be counted with half-order scaling, and that the critical boundary contribution satisfies the weight law $2p + q = d - 1$ [25]. This thesis is motivated by the hope that an analogous reflected-diffusion strategy can be developed for local boundary problems associated with Dirac-type operators.

1.2 The central problem of Part II

The main problem guiding the second part of the thesis is the following.

Problem 1.1 (Central problem of Part II)**.** Let M be a compact even-dimensional Riemannian manifold with boundary, and let D be a Dirac-type operator acting on a Hermitian Clifford module over M. Choose a *local elliptic boundary condition* for D. Is it possible to derive a local boundary index formula by a probabilistic method based on reflected Brownian motion, boundary local time, mirror/interface constructions, and supertrace cancellation?

This question is intentionally more specific than the broad phrase "index theorem with boundary". The purpose is not to treat arbitrary boundary conditions. The reflected-Brownian-motion method is inherently local at the boundary, and therefore the natural first target is a *local* boundary condition. In particular, nonlocal Atiyah–Patodi–Singer boundary conditions are not the immediate entrance point for the present probabilistic method, even though they remain a major analytic benchmark [1, 2, 3]. One of the conceptual points of this part of the thesis is precisely that local probabilistic methods and local elliptic boundary conditions should be matched at the outset.

1.3 Why reflected Brownian motion should enter

Once a first-order Dirac problem is squared, the resulting operator is of generalized Laplace type. For generalized Laplacians with local mixed boundary conditions there is a well-developed probabilistic intuition: one expects a reflected diffusion in the normal direction, stochastic parallel transport in the bundle, and a multiplicative functional carrying curvature and boundary endomorphism terms. This structure appears in several rigorous forms in the work of Hsu, Ouyang, and de Lima [42, 41, 54, 21, 22]. Thus the first question is not how to write down a stochastic formula for the first-order Dirac operator itself, but how to transform the first-order local boundary problem into a mixed heat problem to which the reflected-diffusion machinery genuinely applies.

This transformation is what we shall call the *square-to-mixed-heat bridge*. Once one is on the heat side, several things happen at once. First, the boundary condition becomes mixed Dirichlet–Robin data on a second-order operator. Second, reflected Brownian motion becomes analytically natural. Third, the boundary contribution is expected to appear through a boundary-layer scaling regime in which the boundary local time has half-order weight. This is the place where the probabilistic theory and the geometric theory begin to meet.

1.4 Local boundary conditions and the Gromov–Lawson point of view

A further reason to insist on local boundary conditions is conceptual. In the geometric literature there are a number of local boundary conditions for Dirac-type operators that are compatible with the Clifford structure and are substantially more local than APS. Among the motivating viewpoints in the background of this thesis is the idea that the Gromov–Lawson perspective on local boundary conditions should be more compatible with reflected Brownian motion than nonlocal spectral projections. At the present stage of the thesis, however, we deliberately work in the more general setting of local elliptic boundary projectors rather than fixing a single preferred boundary condition from the outset. This allows the probabilistic mechanism to be analyzed first, before one specializes to a narrower geometric class.

1.5 Methodological roadmap

The overall method of Part II is organized in the following stages.

1. Formulate the relevant local first-order Dirac boundary problem and place it in the modern elliptic-boundary framework.
2. Pass from the first-order problem to a second-order mixed heat problem by squaring.
3. Introduce reflected Brownian motion, stochastic parallel transport, boundary local time, and multiplicative functionals as the probabilistic layer for the mixed heat equation.
4. Show that naive smooth doubling is not the right model and replace it by a transmission or interface picture.
5. Analyze the boundary-layer asymptotics of the resulting kernels, first in frozen models and then in variable-geometry perturbations.
6. Determine which coefficients vanish after supertrace and identify the first slot that can contribute to the boundary index density.

1.6 Research objectives and thesis-level deliverables

The present dissertation does not claim that the entire reflected-diffusion program for local boundary index theory has already been finished. What it does claim is more disciplined and, for a thesis at this stage, more useful. The thesis-level objectives of Part II are:

1. to formulate the central boundary Dirac problem in a way that is genuinely compatible with reflected Brownian motion;

2. to explain carefully why local boundary conditions are the natural starting point for a probabilistic method, and why APS-type conditions are not the first model to attack;

3. to review, in enough detail to be genuinely useful, the existing machinery from heat-kernel index theory, stochastic analysis on manifolds with boundary, and the Du–Hsu reflected-diffusion method;

4. to record, with a clear distinction between established theorems, working reduction principles, and open directions, the theorem-level progress already achieved in the current project.

This chapter should therefore be read as an introduction to a research program with a completed first part and a partially completed second part. The goal is not to present Part II as more closed than it currently is, but rather to make the current mathematical status transparent enough that the document can serve both as a dissertation and as a public research text.

1.7 What is already achieved and what is not

At the time of writing, this program is only partially complete. The thesis does *not* claim a final local boundary index theorem for Dirac-type operators. What it does claim, and what will be systematically recorded in the later chapters of Part II, is that several pieces of the probabilistic-geometric mechanism are already in place:

- the first-order-to-second-order bridge has been isolated at the level of local boundary problems;

- the reflected-diffusion layer and the mirror/interface mechanism have been identified;
- explicit frozen boundary-layer coefficients have been computed;
- the universal critical boundary slot has been identified by scaling;
- first supertrace cancellations have been proved in simplified Clifford-degree regimes;
- the first unresolved localized coefficient has been isolated as a residual distributional profile.

In short, the problem is no longer to guess which mechanism is responsible for the boundary contribution. The mechanism is already visible. What remains is to identify the truly surviving spinorial coefficient and to show that it is the correct local boundary density.

1.8 A convention on mathematical status

Because Part II records a research program in progress, it is important to distinguish carefully between three types of statements.

1. **Classical background results.** These are statements proved in the literature and used here with citation.
2. **Theorem-level results established in the current project.** These are the statements that are presented as theorems, propositions, or lemmas in later chapters and whose proofs, or proof sketches at the level appropriate for a thesis draft, are already stable.

3. **Working reductions and model computations.** These are steps that organize the program and are strongly supported by the current manuscript, but are not yet presented as final stand-alone theorems of the dissertation. They will be described as reduction principles, model calculations, or scope conventions rather than as settled final theorems.

This convention is followed throughout the remainder of Part II. In particular, when a proof from Du–Hsu or from another source is not being rederived here, it will be cited directly rather than rewritten; and when a mechanism is part of the current working infrastructure of the project but not yet thesis-final in its strongest form, it will be identified as such.

1.9 Organization of Part II

Chapter 2 gives a detailed background and literature review. Chapter 3 develops the geometric and analytic preliminaries for the boundary Dirac program. Chapter 5 reviews reflected Brownian motion, Feynman–Kac formulae, and the role of boundary local time. Chapter 6 is devoted to a detailed review of the probabilistic proof of Du and Hsu; its aim is not to copy their proof, but to distill the specific structural lessons needed in the present thesis. Chapter 7 states the main theorem-level results already obtained in the current project, and Chapter 8 gives detailed proofs of selected foundational results. The final chapter summarizes the current frontier and lists the open problems left by the present dissertation.

Chapter 2

Background and literature review

This chapter is intentionally longer than a standard introduction. The reason is that Part II is not just applying a ready-made piece of stochastic calculus to a familiar geometric theorem; it sits at the meeting point of several previously separate literatures. The purpose of this chapter is therefore to review these lines carefully enough that the reader can see why the problem is natural, why it is difficult, and exactly where the present thesis enters.

2.1 Heat-kernel proofs of index theorems on closed manifolds

The modern local theory begins with the heat equation. McKean and Singer observed that the index of an elliptic complex can be represented as the supertrace of the corresponding heat semigroup, independently of time [53]. This identity already contains the local index problem in compressed form: the global index is equal to the integral of a local heat-kernel supertrace density, so one must understand the small-time asymptotics of that density. Patodi carried out the decisive analysis for the de Rham and signature operators, showing how the local coefficients encode curvature data [56]. Getzler later introduced a rescaling argument that makes the Clifford cancellation transparent and is now standard in local index theory [31]. Gilkey systematized the invariant-theoretic side of the subject, while Berline–Getzler–Vergne provided the most convenient synthesis of heat kernels, superconnections, and Clifford-module techniques [32, 7].

For the reader of this thesis, the essential lesson of that closed-manifold story is not any single formula, but the structure of the argument. One isolates the heat kernel, rescales it near the diagonal, identifies the order at which Clifford supertrace can first survive, and then computes that coefficient. Part II follows this template, but the boundary creates two decisive changes: there is a new boundary layer, and the relevant weight counting is no longer purely integer-valued.

2.2 Probabilistic local index theory on closed manifolds

There is also a stochastic version of this story. Bismut showed that the local index theorem can be proved through stochastic differential equations, stochastic parallel transport, and Malliavin-type ideas, thereby replacing part of the PDE machinery by stochastic analysis [8]. Hsu's book remains the most accessible general reference for the stochastic side of heat-kernel asymptotics on manifolds [42]. In the closed case, the probabilistic method already makes clear that curvature enters through multiplicative functionals carried by the diffusion path. Thus the stochastic approach is not merely an alternative proof; it reorganizes the mechanism of the local index theorem in a way that is especially sensitive to geometric transport along paths.

For a thesis centered on reflected Brownian motion, this stochastic reformulation is essential. It suggests from the outset that a boundary version of local index theory should not be sought by forcing the entire interior heat-kernel method to survive unchanged, but by identifying the correct boundary stochastic mechanism and then asking which part of the classical local index argument it replaces.

2.3 Dirac operators, spin geometry, and boundary value problems

Dirac-type operators on manifolds with boundary have a large and rich literature. For the geometric side, the standard references are Lawson–Michelsohn and Booß–Wojciechowski [50, 9]. The key point for the present thesis is that boundary conditions for first-order elliptic operators are fundamentally subtler than for second-order scalar equations. In particular, the Atiyah–Patodi–Singer condition is spectral and nonlocal [1, 2, 3]. By contrast, local elliptic boundary conditions involve fiberwise projectors or subbundles on the boundary and are therefore much closer in spirit to reflected-diffusion mechanisms.

The broad modern analytic framework for first-order elliptic boundary value problems used in this thesis is that of B"ar and Ballmann [4, 5]. Their work makes explicit the difference between local and nonlocal boundary data, clarifies the role of adapted boundary operators, and provides a clean way to formulate local elliptic conditions for Dirac-type operators. This framework is particularly important here because it tells us what kind of boundary problem is even compatible with a probabilistic method based on reflection.

2.4 Local boundary conditions in the Dirac literature

The local boundary-value-problem side of the subject deserves separate emphasis. For Dirac operators on manifolds with boundary, the older monograph of Booß–Bavnbek and Wojciechowski remains a standard reference for the analytic structure of elliptic boundary problems, Calderón projectors, and the difference between local and nonlocal conditions [9]. Together with the later expository work of Bär and Ballmann [4, 5], it clarifies a point that is decisive for the present thesis:

> **a reflected-diffusion method should start from local elliptic boundary data, not from APS-type spectral boundary conditions.**

This observation is not merely philosophical. A nonlocal APS-type projector is naturally adapted to spectral and K-theoretic index theory, but it is not the kind of local boundary mechanism that one expects a reflected Brownian motion to detect pathwise. By contrast, local projector boundary conditions, local chirality conditions, and bag-type conditions are all stated fiberwise along the boundary and therefore fit much better with a collarwise reflected process. For this reason the present dissertation treats the local elliptic framework as the primary analytic setting for the probabilistic program.

It is also useful to keep in view the broader geometric context created by Gromov–Lawson and subsequent work on Dirac operators with geometric boundary conditions [33]. In this thesis, however, the phrase "Gromov–Lawson viewpoint" is used cautiously: it refers to the broad local-geometric philosophy surrounding Dirac operators and positive scalar curvature, not to any claim that a final Gromov–Lawson-style boundary condition has already been proved here to be the unique or definitive probabilistic model. A useful modern complement is the recent systematic study of local smooth self-adjoint boundary conditions for Dirac-type operators by Große, Uribe, and van den Bosch, which reinforces the need to distinguish carefully between a general local-projector framework and any one preferred geometric test case [34].

2.5 Reflected Brownian motion on manifolds with boundary and multiplicative functionals

The next strand of literature concerns boundary stochastic analysis. In the scalar case, reflected Brownian motion on domains and manifolds with boundary is classical. But for the geometric applications relevant here one needs more: stochastic parallel transport on vector bundles, multiplicative functionals carrying curvature terms, and careful control of boundary local time. Hsu developed multiplicative functionals for the heat equation on manifolds with boundary, and Ouyang extended this line in a way especially convenient for differential forms and bundle-valued equations [41, 54]. The role of the boundary local time is already visible there: it acts as the stochastic carrier of the boundary interaction.

A crucial step for the present thesis came from de Lima, who established Feynman–Kac formulae for differential forms on manifolds with boundary and then used them in a probabilistic approach to Gauss–Bonnet-type formulas [21, 22]. In those works, the reflected-diffusion machinery is not merely heuristic; it becomes a rigorous representation formula for heat semigroups with local boundary conditions. This is one of the strongest reasons to believe that a Dirac-boundary program based on reflected Brownian motion is viable.

2.6 The probabilistic Gauss–Bonnet–Chern theorem with boundary

The most direct predecessor of the present thesis is the work of Du and Hsu [25]. They consider the Gauss–Bonnet–Chern theorem for manifolds with boundary and prove it probabilistically using reflected Brownian motion. From the perspective of the present thesis, their contribution has at least four distinct aspects.

First, they identify reflected Brownian motion as the correct boundary stochastic object for the problem. Second, they make it clear that the boundary local time contributes with half-order weight in the short-time asymptotics. Third, they show how a mirror/doubling picture can be used to separate interior and boundary contributions without losing control of the geometry. Fourth, they derive the critical weight law $2p + q = d - 1$, which says precisely which mixed bulk/local-time terms can survive globally after boundary-layer integration.

These four ingredients are exactly the ones that Part II attempts to transport from differential forms to Dirac-type operators under local boundary conditions. The purpose of the Du–Hsu chapter below is therefore not to reprove their theorem, but to extract and reorganize these structural ingredients so that they can be imported into the Dirac program.

2.7 Reflected Brownian motion in wedges, sticky models, and why Part I matters

The first part of the thesis may at first seem far away from the boundary Dirac problem. In fact it is not. The wedge problem provides a rigorous probability-theoretic training ground for several mechanisms that reappear in Part II: reflection, oblique boundary data, singular behavior at corners, submartingale formulations, and regimes in which the process is not a semimartingale. Beyond wedges, sticky reflected models such as those studied by Rácz and Shkolnikov show how local-time-driven slowdown and singular boundary behavior can arise in scaling limits [59]. The present thesis does not claim that these wedge and sticky models solve the geometric problem directly. Their role is more subtle: they sharpen probabilistic intuition for what reflected processes can do at boundaries, especially when naive smooth representations fail.

2.8 Boundary heat asymptotics, transgression terms, and why the boundary is harder

The boundary case is not merely the closed-manifold story plus a correction term. In every classical heat-kernel proof, the interior local coefficient is extracted from a near-diagonal asymptotic expansion with integer heat-kernel weights. With a boundary present, however, there is an additional geometric scale: the boundary layer of thickness $\sqrt{t}$. The boundary contribution is not visible in the same way as the interior term. It comes from a rescaled normal variable, from reflection or mirroring, and from the interaction between the heat kernel and the boundary local time. This is why the boundary formulas in the index-theory literature are usually described in terms of transgression or Chern–Simons type expressions rather than as a naive restriction of the interior density.

From the probabilistic viewpoint this difference is even sharper. The reflected process sees the boundary through its local time, and the local time enters with half-order scaling.

Thus the boundary asymptotics necessarily involve mixed integer and half-integer slots. This is the conceptual reason that the present thesis spends so much time on boundary-layer bookkeeping: before one can identify a boundary index density, one must know *which slots are even capable of surviving after global integration and supertrace.* In that sense the boundary problem is not a small perturbation of the interior local index problem; it is a different asymptotic geometry with its own critical scaling law.

2.9 Why a long literature review is necessary here

In a finished monograph one might compress the background substantially. In the present thesis, however, Part II documents an active transition from a completed probability project to a developing geometric-index-theory program. For that reason the literature review is not ornamental. It has three specific functions. First, it identifies the exact technical inputs from the existing literature that the thesis relies on. Second, it explains why local boundary conditions, reflected diffusion, and boundary local time belong together as a single methodological package. Third, it makes clear which statements in the present part are reworked reviews of existing methods and which are new theorem-level steps achieved in the current project.

The chapter on Du–Hsu below should be read in this spirit. It is neither a compressed summary nor a reproduction of their proof. Its purpose is to isolate the pieces of their method that must later be transplanted into the Dirac boundary problem.

2.10 Where the present thesis enters

The present thesis enters precisely between these literatures. It takes from the closed-manifold local index theorem the heat-kernel and supertrace viewpoint, from B"ar–Ballmann the local first-order boundary framework, from Hsu/Ouyang/de Lima the reflected-diffusion and multiplicative-functional machinery, and from Du–Hsu the half-order local-time scaling law and the mirror/interface viewpoint. Its distinctive contribution is to combine these ingredients into a single Dirac-boundary program and to record the theorem-level progress already obtained in that direction.

2.11 How this literature is used in the present thesis

The literature reviewed in this chapter plays three distinct roles in the present dissertation.

1. Some results are used as *background infrastructure*: for example the classical heat-kernel proof of the local index theorem on closed manifolds, the basic stochastic analysis on manifolds developed by Bismut and Hsu, and the general first-order elliptic boundary framework of B"ar and Ballmann.

2. Some results are used as *methodological models.* The clearest instance is the Du–Hsu theorem, whose full proof is not reproduced here but whose architecture is extracted and used as a guide for the Dirac-boundary program.

3. Some results are used as *direct comparison points.* This is especially true for Part I, whose theorem-level content is already aligned with the published wedge paper, and for Part II where de Lima's and Ouyang's boundary Feynman–Kac formulae provide the nearest rigorous stochastic analogues currently available in the literature.

This distinction is important because the present thesis is not simply a survey. It is a hybrid document: part review, part record of completed theorem-level work, and part mathematically controlled research roadmap. The aim is to make those roles legible rather than blur them.

Chapter 3

Geometric and analytic preliminaries for the boundary Dirac program

3.1 Dirac-type operators and Clifford modules

Let M be a compact even-dimensional oriented Riemannian manifold with smooth boundary ∂M. Let $F \to M$ be a Hermitian Clifford module endowed with a compatible connection. The associated Dirac-type operator is a first-order formally self-adjoint elliptic operator

$$D : C^\infty(M, F) \to C^\infty(M, F).$$

If F is $\mathbb{Z}_2$-graded, then one may write

$$F = F^+ \oplus F^-, \qquad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Near the boundary we choose a product-type collar $[0,\varepsilon)_r \times \partial M$ with inward normal coordinate r. In such a collar one may write

$$D = G(\partial_r + A + r\mathcal{E}_1 + \mathcal{E}_0),$$

where $G = c(\nu)$ is Clifford multiplication by the inward unit normal, A is the adapted tangential boundary operator, and $\mathcal{E}_1, \mathcal{E}_0$ collect lower-order terms.

3.2 Twisted spin bundles as the running geometric model

Although Part II is stated for a general Hermitian Dirac bundle F, the geometric model kept in the background throughout is the twisted spin case. Thus, whenever M is spin and $V \to M$ is a Hermitian vector bundle with compatible connection, we take

$$F = \Sigma M \otimes V, \qquad F^{\pm} = \Sigma^{\pm} M \otimes V, \qquad D_V = \sum_{j=1}^{d} c(e_j)\nabla_{e_j}^{\Sigma\otimes V}. \tag{3.1}$$

Here $\Sigma M = \Sigma^{+}M \oplus \Sigma^{-}M$ is the spinor bundle in even dimension, and D_V is the twisted spin Dirac operator. In this model, a local boundary projector acts fiberwise on the restriction

$$F|_{\partial M} = (\Sigma M \otimes V)|_{\partial M},$$

so the boundary condition is imposed on an honest geometric bundle rather than on an abstract auxiliary space.

This clarification is important for the thesis because the eventual probabilistic index theorem is intended for Dirac operators in the Gromov–Lawson/Lawson–Michelsohn sense, and not only for the bare spin operator. In particular, the twisting bundle V is not an inessential decoration: it is part of the natural geometric class already built into the Dirac-bundle formalism.

3.3 Local boundary data

The first-order problem relevant to reflected Brownian motion is not an arbitrary boundary problem but a local elliptic one. Thus we consider a local projector on the boundary bundle,

$$B^+ : F^+|_{\partial M} \to F^+|_{\partial M},$$

and the corresponding realization

$$D^+_{B^+} : \mathrm{dom}(D^+_{B^+}) \subset L^2(F^+) \to L^2(F^-), \qquad \mathrm{dom}(D^+_{B^+}) = \{u \in H^1(F^+) : B^+(u|_{\partial M}) = 0\}.$$

The complementary local boundary condition for the adjoint problem will be denoted by B^-. In the present thesis we keep the local boundary condition flexible, because the probabilistic mechanism should first be understood at the general local level before one specializes to narrower geometric classes.

3.4 Green's formula and complementary local adjoints

One of the key formal facts behind local boundary problems for Dirac-type operators is the boundary Green identity. In the collar notation used above, if $u, v \in C^\infty(M, F)$ are smooth sections, then one has the formal integration-by-parts relation

$$\langle Du, v\rangle_{L^2(M)} - \langle u, Dv\rangle_{L^2(M)} = -\int_{\partial M} \langle c(\nu)u|_{\partial M}, v|_{\partial M}\rangle \, d\sigma. \tag{3.2}$$

Thus the obstruction to symmetry of a first-order realization is entirely concentrated in the boundary pairing.

This formula immediately explains why complementary local projector conditions should appear. If a boundary involution Γ_∂ satisfies

$$\Gamma_\partial c(\nu) = -c(\nu)\Gamma_\partial, \qquad P_\partial^\pm = \frac{1}{2}(I \pm \Gamma_\partial),$$

then one has the algebraic relation

$$c(\nu)P_\partial^\pm = P_\partial^\mp c(\nu). \tag{3.3}$$

Hence P_∂^- on one side naturally pairs with P_∂^+ on the adjoint side. This is the local analytic reason why complementary projector conditions are the correct first-order boundary pair for the product chirality model developed later in Part II.

3.5 Square-to-mixed-heat bridge

The fundamental analytic bridge is that the first-order problem naturally produces a second-order one. Formally one sets

$$\Delta_+ = D^-_{B^-} D^+_{B^+}, \qquad \Delta_- = D^+_{B^+} D^-_{B^-}.$$

In the interior these operators are of generalized Laplace type,

$$\Delta_+ = \nabla^*\nabla + \mathcal{R}_+, \qquad \Delta_- = \nabla^*\nabla + \mathcal{R}_-,$$

and the boundary condition becomes mixed. More precisely, the local projector splits the boundary bundle into a Dirichlet sector and a complementary Robin/Neumann sector, with a boundary endomorphism S determined by the collar decomposition of D.

Proposition 3.1 (Square-to-mixed-heat bridge, formal version)**.** *Under the collar decomposition and local complementary projectors B^+, B^-, the heat problem associated with $D^-_{B^-} D^+_{B^+}$ is a generalized mixed heat problem: Dirichlet on one boundary subbundle and Robin/Neumann on its complement.*

Proof sketch. Writing D in collar form and applying Green's formula yields the adjoint complementary condition on F^-. Squaring and separating the projected sectors shows that the boundary trace vanishes on the Dirichlet sector, while on the complementary sector the normal derivative satisfies a condition of the form $\nabla_\nu u + Su = 0$. Thus the second-order operator is of mixed Dirichlet–Robin type. $\square$

3.6 Why this bridge is decisive

The reflected-diffusion machinery acts naturally on generalized Laplacians with local mixed boundary conditions. In other words, reflected Brownian motion does not attack the first-order Dirac problem directly; it attacks the second-order mixed heat problem produced by the square. The bridge therefore marks the exact analytic entrance point of the probabilistic method.

Chapter 4

Local boundary conditions and the choice of the geometric model

The purpose of this chapter is to make one structural point completely explicit: the probabilistic method proposed in Part II is not aimed at arbitrary boundary conditions. It is aimed at *local* boundary conditions. This choice is not cosmetic. It is dictated by the nature of reflected Brownian motion itself.

4.1 Why APS is not the natural starting point for reflected diffusion

The Atiyah–Patodi–Singer boundary condition is one of the great achievements of boundary index theory, but from the point of view of reflected Brownian motion it is not the natural first target. APS is defined by a spectral projection of the tangential boundary operator; it is therefore global and nonlocal along the boundary. By contrast, reflected Brownian motion is local in the strongest possible sense: its interaction with the boundary is encoded by the normal reflection law and the accumulated boundary local time. There is no obvious mechanism by which an ordinary reflected diffusion should directly encode a global spectral projection.

For this reason the present thesis deliberately postpones APS-type conditions and starts instead with local elliptic boundary conditions. The philosophical stance is simple: one should first understand the boundary index mechanism in a setting whose boundary behavior is itself local, and only later ask whether the method can be extended or modified to treat nonlocal spectral data.

4.2 Local elliptic boundary conditions in the Dirac setting

Let D be a Dirac-type operator acting on a Clifford module $F = F^+ \oplus F^-$. In a collar one writes

$$D = G(\partial_r + A + \Psi),$$

with $G = c(\nu)$ the normal Clifford multiplication and A the tangential boundary operator. A local boundary condition is given by a fiberwise projector

$$B^+ : F^+|_{\partial M} \to F^+|_{\partial M},$$

or, equivalently, by a local boundary subbundle. The resulting domain

$$\operatorname{dom}(D^+_{B^+}) = \{u \in H^1(F^+) : B^+(u|_{\partial M}) = 0\}$$

is local along the boundary and therefore compatible, at least in principle, with a local reflected-diffusion analysis after squaring.

At the present stage of the thesis we keep the local boundary condition abstract, for two reasons. First, the reflected-diffusion mechanism should be developed at the broadest local level before specializing. Second, several local geometric conditions may turn out to be useful candidates, and it is helpful not to commit too early before the stochastic machinery has been fully clarified.

4.3 Boundary chirality, bag-type conditions, and the Gromov–Lawson viewpoint

Among local boundary conditions that interact naturally with Dirac operators, at least three geometric patterns are relevant for the present program.

1. **Boundary chirality type conditions.** These arise when the boundary bundle carries a local involution that anti-commutes with normal Clifford multiplication. They are conceptually close to the local product geometry and often lead to complementary local conditions on F^+ and F^-.

2. **Bag-type conditions.** In these, the boundary projector is constructed directly from normal Clifford multiplication and an auxiliary involution. Their importance here is not physical but structural: they are explicit examples of local conditions with nontrivial spinorial content.

3. **The Gromov–Lawson viewpoint.** In the broad sense relevant to this thesis, the Gromov–Lawson perspective treats local geometric boundary conditions as legitimate

> and natural partners of Dirac-type operators. What matters for us is not a single finalized boundary theorem, but the guiding idea that a local geometric boundary condition should exist which is both analytically elliptic and conceptually compatible with a reflected-diffusion treatment.

The present thesis does not yet make a definitive final choice among these local conditions. What it does do is narrow the analytic field: the eventual probabilistic index theorem should be formulated under a local boundary condition of one of the geometric types above, not under a nonlocal spectral condition.

4.4 Gromov–Lawson style local conditions as a target model

The phrase "Gromov–Lawson" must be used carefully here. Historically, Gromov and Lawson introduced an important geometric class of generalized Dirac operators; they did *not* give a complete classification of local boundary conditions for all Dirac-type operators. In the language used later by Lawson–Michelsohn and Bär–Ballmann, the point is this: the relevant operator is defined on a *Dirac bundle*, and the spinorial model is only the basic geometric example.

Concretely, let (F, ∇^F, c) be a Hermitian Dirac bundle over a Riemannian manifold M. The associated Dirac operator is

$$D = \sum_{j=1}^{d} c(e_j)\nabla^F_{e_j}, \tag{4.1}$$

where $\{e_j\}_{j=1}^d$ is any local orthonormal frame. Thus the Gromov–Lawson/Lawson–Michelsohn framework is *not* restricted to the untwisted spinor bundle.

If M is spin and $V \to M$ is an auxiliary Hermitian vector bundle with connection, then the principal example relevant to the present thesis is the twisted spin model

$$F = \Sigma M \otimes V, \qquad D_V = \sum_{j=1}^{d} c(e_j)\nabla_{e_j}^{\Sigma\otimes V}. \tag{4.2}$$

This directly answers the main structural question for the current thesis: the boundary Dirac problem need not be formulated only on the bare spin bundle; the natural geometric setting already includes twisted spinors.

Near the boundary one fixes a collar $[0,\varepsilon)_r \times \partial M$ with inward unit normal ν. In a collar trivialization one writes

$$D = c(\nu)\big(\partial_r + A + \Psi\big), \tag{4.3}$$

where A is an adapted tangential operator on $F|_{\partial M}$ and Ψ is zeroth order. For local boundary problems one imposes a condition directly on the boundary bundle $F|_{\partial M}$.

A standard local projector model is obtained from a self-adjoint involution

$$\Gamma_\partial : F|_{\partial M} \to F|_{\partial M}, \qquad \Gamma_\partial^2 = I, \qquad \Gamma_\partial^* = \Gamma_\partial, \qquad \Gamma_\partial c(\nu) = -c(\nu)\Gamma_\partial. \tag{4.4}$$

Then the fiberwise orthogonal projectors

$$P_\partial^\pm := \frac{1}{2}\big(I \pm \Gamma_\partial\big) \tag{4.5}$$

define local complementary boundary conditions via

$$\mathrm{dom}(D_{P_\partial^\pm}) = \{u \in H^1(F) : P_\partial^\pm(u|_{\partial M}) = 0\}. \tag{4.6}$$

This is the structural pattern that matters here: a geometric Dirac bundle, a local involution on the boundary bundle, and a projector boundary condition built from that involution.

What the present thesis does *not* claim is that Gromov–Lawson's original work already provides the final probabilistic boundary condition we need. What is true, and all that is presently needed, is that the Gromov–Lawson/Lawson–Michelsohn viewpoint identifies the right bundle-theoretic setting: the operator lives on a Dirac bundle, and in the spin case the most relevant concrete model is the twisted operator on $\Sigma M \otimes V$.

4.5 MIT bag and boundary chirality as test cases

Two concrete local test cases deserve explicit formulas.

Boundary chirality. Suppose there exists a self-adjoint involution Γ_∂ satisfying (4.4). Then the projectors (4.5) yield local complementary boundary conditions. This is the cleanest model when one wants a genuinely local condition compatible with the Clifford structure and with the plus/minus splitting.

MIT bag. Under the Riemannian Clifford convention $c(\nu)^2 = -I$, the endomorphism $i\,c(\nu)$ is a self-adjoint involution. Hence one may form the local MIT bag projectors

$$P_{\mathrm{MIT}}^\pm := \frac{1}{2}\big(I \pm i\,c(\nu)\big), \tag{4.7}$$

and the corresponding domains

$$\mathrm{dom}(D^{\pm}_{\mathrm{MIT}}) = \{u \in H^1(F) : P^{\pm}_{\mathrm{MIT}}(u|_{\partial M}) = 0\}. \tag{4.8}$$

These conditions are useful here not because the thesis commits itself to the MIT bag model, but because they show explicitly that a local projector boundary condition can already contain nontrivial Clifford action on the spinor factor.

What is fixed and what is not fixed. The thesis still does not make a final theorem-level commitment to one distinguished geometric boundary condition. The current standpoint is narrower and more accurate: the probabilistic mechanism should first be developed for a stable class of local elliptic projector conditions, and only afterwards specialized to the most effective geometric choice. The formulas above show exactly what this means in practice: the eventual boundary condition should be of projector form (4.6), with the geometric Dirac-bundle model (4.1) and especially the twisted-spin model (4.2) kept in view.

4.6 A working convention for this thesis

Because the final geometric choice is still part of the ongoing development, the working convention in Part II is the following.

> **Working convention.** We formulate the theory for local elliptic boundary projectors in the sense of first-order elliptic boundary value problems, and we regard the later specialization to a more geometric local condition — in particular one in the spirit of Gromov–Lawson or boundary chirality — as a subsequent

stage of the program.

This convention is sufficient for the current goals of the thesis. It allows the square-to-mixed-heat bridge, the reflected-diffusion representation, the interface picture, the critical-slot theorem, and the first supertrace cancellations to be stated without forcing a premature choice of boundary model.

4.7 A concrete working class of local projector boundary conditions

For the remainder of Part II, the local boundary conditions considered in the theorem-level statements are understood in the following working sense. We choose a smooth fiberwise orthogonal projector

$$B : B(F|_{\partial M}) \to B(F|_{\partial M})$$

on the boundary bundle and require that the corresponding first-order realization be formally symmetric and elliptic in the local sense needed for the square-to-mixed-heat reduction. We do not claim in this dissertation that every such projector has already been classified in the precise geometric setting of interest. Rather, the point is that the reflected-diffusion construction should first be carried out for a stable local projector class before one specializes to a narrower geometric model.

This convention has two advantages. First, it keeps the analytic entrance broad enough to include the standard local geometric examples. Second, it prevents the thesis from over-claiming that a final and unique boundary condition has already been selected. In particular, the present text does *not* claim that the Gromov–Lawson-type choice has already been proved to be the definitive local model for the probabilistic index theorem. Instead, it is treated as a particularly important guiding example that should be revisited once the remaining probabilistic and Clifford-algebraic steps are complete.

4.8 An exact product-type chirality model

The general local-projector framework above is intentionally broad. For the purposes of explicit calculation, however, it is useful to isolate one exact model in which the square-to-mixed-heat bridge can be written down with complete formulas and no extra lower-order terms. This is the product-type chirality model.

Assume the collar is exactly product and that the Dirac operator takes the exact form

$$D = c(\nu)(\partial_r + A), \tag{4.9}$$

where the tangential operator A is independent of r. Assume also that there is a self-adjoint involution Γ_∂ on $F|_{\partial M}$ such that

$$\Gamma_\partial^2 = I, \qquad \Gamma_\partial^* = \Gamma_\partial, \qquad \Gamma_\partial c(\nu) = -c(\nu)\Gamma_\partial, \qquad [\Gamma_\partial, A] = 0. \tag{4.10}$$

Set

$$P_\partial^\pm := \frac{1}{2}(I \pm \Gamma_\partial). \tag{4.11}$$

Then the first-order realization

$$\mathrm{dom}(D_{P_\partial^-}) = \{u \in H^1(F) : P_\partial^- (u|_{\partial M}) = 0\} \tag{4.12}$$

leads, after squaring, to a completely explicit mixed second-order model.

Proposition 4.1 (Exact product-type square-to-mixed-heat model)**.** *Under assumptions* (4.9)–(4.11)*, one has*

$$D^2 = -\partial_r^2 + A^2. \tag{4.13}$$

Moreover, the domain of the squared realization $D^2_{P_\partial^-}$ *is described by the mixed boundary conditions*

$$P_\partial^- \left(u|_{\partial M}\right) = 0, \qquad P_\partial^+ \big((\partial_r + A)u\big)\big|_{\partial M} = 0. \tag{4.14}$$

Similarly, the square of the complementary realization $D_{P_\partial^+}$ *is governed by*

$$P_\partial^+ \left(u|_{\partial M}\right) = 0, \qquad P_\partial^- \big((\partial_r + A)u\big)\big|_{\partial M} = 0. \tag{4.15}$$

This proposition is important for the thesis for two reasons. First, it gives a completely explicit model in which the abstract square-to-mixed-heat bridge becomes a literal theorem. Second, it shows in a single formula why a local boundary projector gives rise to a Dirichlet condition on one boundary sector and a Robin-type condition on the complementary sector. That is exactly the structure to which reflected Brownian motion is naturally adapted.

Remark 4.2. The MIT bag projectors are still useful test cases, but they do not fit into Proposition 4.1 by the same involution mechanism, because the involution $i\,c(\nu)$ commutes with $c(\nu)$ rather than anti-commuting with it. Thus MIT bag remains a local first-order model, but its square must be analyzed separately from the boundary-chirality product model above.

4.9 How this chapter interacts with the rest of Part II

This chapter clarifies the logic of the remaining chapters. The next chapter on reflected Brownian motion and Feynman–Kac formulae should be read as the probabilistic layer attached to a *local* mixed heat problem. The Du–Hsu review chapter should be read as the model example showing how a local boundary stochastic mechanism can recover a boundary geometric theorem. The main-results and proofs chapters should then be read as the first theorem-level steps in carrying that same philosophy over to local boundary problems for Dirac-type operators.

A second role of the present chapter is negative: it prevents the later chapters from over-claiming. In particular, the thesis does *not* assert that a final choice of local boundary condition has already been made, nor that the Gromov–Lawson viewpoint has already been completely translated into the reflected-diffusion language. What is asserted is narrower and more accurate: the present program should be developed at the level of local elliptic boundary projectors first, and only afterwards specialized to the most geometrically effective local model.

Chapter 5

Reflected Brownian motion, Feynman–Kac formulae, and boundary local time

5.1 Reflected Brownian motion on manifolds with boundary

Let $L = \nabla^*\nabla + \mathcal{R}$ be a generalized Laplace-type operator over a bundle on M, equipped with local mixed boundary conditions. The reflected-diffusion principle predicts a representation of the heat semigroup in terms of reflected Brownian motion X_t and a multiplicative functional $\mathcal{M}_t$:

$$(e^{-tL/2}u)(x) = \mathbb{E}_x[\mathcal{M}_t u(X_t)].$$

In general, the multiplicative functional carries parallel transport, the Weitzenb"ock term, and the boundary endomorphism data. The precise form depends on the operator and the boundary condition, but the point relevant to this thesis is structural: the boundary contribution is encoded by the reflected path and its local time.

5.2 Reflected Brownian motion as a boundary-value diffusion

There are several equivalent ways to define reflected Brownian motion on a smooth compact manifold with boundary. For the present thesis the most useful viewpoint is the boundary-value characterization: reflected Brownian motion is the diffusion whose generator is $\frac{1}{2}\Delta$ in the interior and whose boundary interaction is encoded by reflection rather than absorption.

A convenient formulation is the following submartingale characterization. A continuous Markov process $X = (X_t)_{t\geq 0}$ on M is a reflected Brownian motion if, for every $f \in C^\infty(M)$ such that $\partial_\nu f \geq 0$ on ∂M, the process

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta f(X_s)\, ds \tag{5.1}$$

is a submartingale. This is the manifold-with-boundary analogue of the reflected diffusion/-submartingale problem already used in Part I for wedges.

For smooth manifolds with smooth boundary, one may refine this statement by introducing boundary local time. In the reflected-Brownian-motion literature this yields the decomposition

$$f(X_t) = f(X_0) + M_t^f + \frac{1}{2}\int_0^t \Delta f(X_s)\, ds + \int_0^t \partial_\nu f(X_s)\, dl_s, \tag{5.2}$$

where M_t^f is a local martingale and l_t is the boundary local time. Formula (5.2) is the basic place where the boundary enters the stochastic analysis: it enters neither as a singular correction term nor as an external forcing, but as the local time of the reflected path itself [42, 41].

5.3 Boundary local time

If X_t is reflected Brownian motion, then its boundary local time l_t measures the cumulative amount of reflection accumulated up to time t. In the boundary index problem this local time is not a secondary correction; it is one of the principal asymptotic carriers of the boundary geometry. The crucial scaling phenomenon is that the local time is naturally of half-order in the short-time expansion. This is the stochastic origin of the boundary-layer half-power counting used throughout the current project and made completely explicit by Du and Hsu [25].

5.4 Model scaling law for boundary local time

The half-order role of the boundary local time can already be seen in the one-dimensional model $R_t = |B_t|$, where B_t is standard Brownian motion. By Brownian scaling,

$$L_t^0(B) \overset{d}{=} \sqrt{t}\, L_1^0(B), \tag{5.3}$$

and therefore, whenever the moment exists,

$$\mathbb{E}\big[(L_t^0(B))^q\big] = t^{q/2}\,\mathbb{E}\big[(L_1^0(B))^q\big]. \tag{5.4}$$

This elementary model already explains why every factor of boundary local time contributes one half-power of t in a short-time expansion. In higher-dimensional reflected Brownian motion on manifolds with boundary the same scaling principle survives at the level relevant to boundary-layer asymptotics, and this is exactly the mechanism exploited in the Du–Hsu analysis.

5.5 Multiplicative functionals

Hsu, Ouyang, and de Lima show that the heat semigroup can be represented by multiplicative functionals adapted to the reflected path [41, 54, 21]. In the cases relevant to this thesis, the multiplicative functional has three schematic components:

1. stochastic parallel transport along the reflected path;
2. an interior curvature or Weitzenb"ock factor;
3. a boundary term coupled to the local time.

The current thesis uses this viewpoint as a conceptual framework even where the final explicit formula has not yet been derived for the full Dirac problem.

5.6 A schematic Feynman–Kac formula for mixed heat problems

In the simplest Neumann-type reflected setting one expects a formula of the schematic form

$$(e^{-tL/2}u)(x) = \mathbb{E}_x\big[\mathcal{M}_t\, u(X_t)\big], \tag{5.5}$$

where $L = \nabla^*\nabla + \mathcal{R}$ is the relevant generalized Laplacian, X_t is reflected Brownian motion, and $\mathcal{M}_t$ is the multiplicative functional carrying the bundle and boundary data. For mixed boundary conditions there is an additional subtlety: the Dirichlet sector must be enforced by killing or projection, while the Robin sector is encoded through the boundary part of the multiplicative functional. The exact implementation depends on the local boundary decomposition and is one of the main reasons that transmission/interface models become necessary later in Part II.

Thus formula (5.5) should be read as a guiding template rather than as a single final theorem valid in every local mixed setting considered in this thesis. What is stable across all the relevant versions is that the reflected path, the bundle transport, the boundary local time, and the boundary endomorphism data all appear in a single probabilistic representation.

5.7 Mirror/interface models

Near the boundary, one would like to replace reflection by a mirror construction. In the mixed local boundary setting, however, a naive smooth doubling fails. The correct replacement is an interface model, or equivalently a transmission model followed by a delta-interface reduction. This yields exact frozen one-dimensional kernels and, after parabolic scaling, a boundary-layer expansion with half-order and integer-order slots. These slots form the basis of the theorem-level results recorded later in this part.

Chapter 6

The Du–Hsu probabilistic proof of the Gauss–Bonnet–Chern theorem: a detailed review

This chapter serves a very specific purpose. It is not a substitute for the paper of Du and Hsu, and it does not reproduce their long proof inside the thesis. On the contrary, the guiding principle of this chapter is the opposite: *do not copy their proof; extract the method*. Whenever a long technical proof from [25] is required, we cite it directly. The aim here is to record the structural features of their argument that are genuinely needed for the present Dirac-boundary program.

6.1 What Du and Hsu prove

Du and Hsu prove a probabilistic form of the Gauss–Bonnet–Chern theorem on compact manifolds with boundary using reflected Brownian motion. The setting is the Hodge Laplacian acting on differential forms under absolute boundary conditions. Their theorem identifies the Euler characteristic with the small-time supertrace of the heat kernel and then recovers the interior Pfaffian term together with the correct boundary transgression term by reflected-Brownian-motion asymptotics [25].

From the viewpoint of the present thesis, three features are decisive.

1. The boundary term is generated by reflected Brownian motion and boundary local time, not by any ad hoc correction.
2. The reflected local time contributes with half-order scaling.
3. The globally relevant boundary contribution satisfies the critical law $2p + q = d - 1$.

6.2 The probabilistic ingredients in their method

The method of Du and Hsu combines several ingredients from stochastic analysis on manifolds with boundary.

- Reflected Brownian motion in a collar neighborhood of the boundary.
- Stochastic parallel transport and multiplicative functionals for the Hodge Laplacian.
- Boundary local time and its short-time moments.
- A mirror or doubling idea that separates interior and boundary sectors.

- A careful bookkeeping of the powers of t and the powers of the local time.

For the technical details of these constructions we refer directly to [25]. What matters for the present thesis is the conceptual pattern: the boundary contribution emerges from a separate boundary-layer asymptotic regime and must be treated with its own weight counting.

6.3 Why we do not reproduce their full proof here

The paper of Du and Hsu contains a substantial amount of machinery specific to differential forms, absolute boundary conditions, and the Gauss–Bonnet–Chern theorem. Reproducing the full proof inside this thesis would make Part II much longer without serving the main goal of the present dissertation. More importantly, it would risk obscuring the role the paper plays here. The function of the Du–Hsu theorem in this thesis is methodological, not expository: it supplies the correct boundary local-time paradigm and the correct critical scaling law.

For that reason, whenever one of their long arguments is needed, we cite the relevant result rather than rewrite it. Only those intermediate ideas that directly feed the current Dirac program are extracted and summarized below.

6.4 The critical weight law

The first structural output imported from Du–Hsu is the critical weight law

$$2p + q = d - 1.$$

Here p is the bulk heat-kernel order and q counts the power of the boundary local time. In the present thesis this law reappears in a more abstract scaling theorem: if a boundary-layer coefficient occupies slot j, then after integration against the boundary-layer measure it contributes at order $t^{(j-(d-1))/2}$. Thus the only slot that can contribute a finite nonzero global boundary term is $j = d - 1$. Interpreting j as $2p + q$ recovers the Du–Hsu weight law exactly.

6.5 Mirror constructions and what must change for Dirac operators

The second structural output imported from Du–Hsu is the mirror viewpoint. The reflected path near the boundary can be reorganized into an interior piece and a mirror/interface piece. For differential forms this can be implemented in a way adapted to the Hodge Laplacian. For Dirac-type operators, however, the situation is subtler. One must first square to a mixed heat problem, and then even on the heat side naive smooth doubling fails in the local Dirichlet–Robin setting. This is why the present thesis replaces smooth doubling by transmission doubling and then by a delta-interface reduction.

6.6 A thesis-level statement of the Du–Hsu theorem

For the purposes of this dissertation, the relevant content of Du–Hsu may be summarized informally as follows: for the Hodge Laplacian on differential forms with the appropriate local boundary condition, the McKean–Singer supertrace can be represented through reflected Brownian motion and a boundary-adapted multiplicative functional, and the small-time asymptotic analysis of that representation yields the full Gauss–Bonnet–Chern formula with boundary. The boundary term is not inserted by hand. It is produced by the reflected path itself, more precisely by its local time and the geometry encoded in the multiplicative functional [25].

This statement is already enough to explain why Du–Hsu matters for the present thesis. Their theorem does not solve the Dirac-boundary problem directly, but it proves beyond doubt that reflected Brownian motion is capable of carrying a nontrivial local boundary geometric theorem.

6.7 The architecture of the Du–Hsu proof

At a high level, their proof proceeds through the following layers.

1. A probabilistic representation of the heat semigroup on forms using reflected Brownian motion and a multiplicative functional.

2. A boundary-adapted asymptotic expansion in which the boundary local time is counted with half-order weight.

3. A separation of interior and boundary sectors through a mirror-type reorganization.

4. A moment analysis for the local time that identifies which mixed bulk/boundary terms are globally relevant.

5. A final geometric recognition step in which the surviving asymptotic coefficient is matched with the classical boundary Gauss–Bonnet–Chern density.

The present thesis uses this architecture as a template. What must be changed for Dirac-type operators is not the existence of a boundary stochastic mechanism, but the

Clifford-module algebra and the choice of local boundary condition.

6.8 What we cite directly and what we only extract

Because the Du–Hsu paper is already a long and highly organized proof, the thesis follows a simple rule. Whenever a long technical argument from their paper is needed only as background, we cite it directly rather than paraphrasing or reproducing it. Whenever a structural point is needed for the present program, we extract and reformulate it. In particular, the present chapter does *not* attempt to replicate their local-time moment estimates or the full proof of the geometric identification step. Instead, it records the four structural ingredients that are later imported into the Dirac setting.

6.9 How the Du–Hsu method interfaces with the present thesis

The present thesis uses the Du–Hsu paper in a deliberately asymmetric way. On the one hand, the geometric theorem they prove is not the theorem pursued here: they work with the Hodge Laplacian on differential forms and absolute boundary conditions, whereas the present program is aimed at Dirac-type operators with local elliptic boundary conditions. On the other hand, several of the most important *structural* features of their proof appear to survive intact when one passes from the form case to the Dirac case. Those features are precisely the ones emphasized throughout this thesis:

1. the boundary contribution must be isolated in a genuine boundary-layer scaling regime;

2. boundary local time contributes with half-order weight;
3. a mirror or interface sector must be analyzed separately from the bulk kernel;
4. the final difficulty is not the existence of boundary terms but their algebraic survival after supertrace.

The last point is especially important. In the differential-form setting of Du–Hsu, the surviving boundary term is eventually identified with the classical Gauss–Bonnet–Chern transgression form. In the Dirac setting considered here, the analogous step is not yet complete. Nevertheless, their work already tells us what the correct type of boundary stochastic mechanism must look like. This is why Part II repeatedly cites Du–Hsu at the level of method and asymptotic architecture, while not reproducing their technical proof machinery inside the thesis.

6.10 What is transferred into the present thesis

The specific lessons imported from [25] are therefore the following.

1. The boundary contribution should be studied in a boundary-layer scaling regime.
2. The boundary local time must be counted as a half-order object.
3. A mirror/interface sector, not just the interior kernel, must be analyzed.
4. The critical global boundary slot is the slot of weight $d - 1$.

These four points are now woven into the Dirac-side analysis carried out later in this part.

6.11 What the present thesis does *not* claim about Du–Hsu

For the sake of clarity, let us state explicitly what the present thesis is *not* doing. It is not claiming a new proof of the Du–Hsu theorem, it is not reproducing their long local-time asymptotic argument line by line, and it is not presenting their result as if it were an original theorem of this dissertation. Their theorem is cited as an established result in the literature and used as a methodological model. The original proof remains theirs; what is new here is the attempt to transplant its structural insights into a different geometric setting, namely local boundary problems for Dirac-type operators.

This distinction matters for the thesis as a whole. Part II should be readable both as a review of the methods on which the present program depends and as a record of the theorem-level progress already achieved in this program. The Du–Hsu chapter belongs mainly to the first category: it is a review-and-extraction chapter, not a claim of ownership over the original proof.

Chapter 7

Main results of the current project

The purpose of this chapter is to state clearly what has already reached theorem-level status in the present project, and to distinguish those results from the reduction principles and model computations that organize the broader program but are not yet being claimed here as final stand-alone theorems of the dissertation. In particular, whenever a completely closed exact product-type local model is available, we also record the corresponding index identity in genuine theorem form rather than only as informal motivation.

7.1 Two reduction principles that organize the program

Reduction principle A: the square-to-mixed-heat entrance. For a Dirac-type operator endowed with a local elliptic boundary projector, the square of the first-order realization produces a generalized Laplace-type operator with local mixed boundary conditions. This is the analytic doorway through which reflected Brownian motion enters the problem. In the present thesis this principle is used as a standing reduction, not as a final independent theorem with all analytic details written out in full.

Theorem 7.1 (Exact product-type local chirality bridge)**.** *Assume the exact product collar model* (4.9) *and the boundary chirality relations* (4.10)*. Then the square of the first-order realization with boundary condition* $P_\partial^- u|_{\partial M} = 0$ *is the mixed second-order operator*

$$D^2_{P_\partial^-} = -\partial_r^2 + A^2 \tag{7.1}$$

on the domain determined by

$$P_\partial^- (u|_{\partial M}) = 0, \qquad P_\partial^+ \big((\partial_r + A)u\big)\big|_{\partial M} = 0. \tag{7.2}$$

Likewise, the complementary projector P_∂^+ *gives the mixed condition*

$$P_\partial^+ (u|_{\partial M}) = 0, \qquad P_\partial^- \big((\partial_r + A)u\big)\big|_{\partial M} = 0. \tag{7.3}$$

In particular, the square-to-mixed-heat bridge is completely explicit in this product-type local chirality model.

Theorem 7.2 (Boundary Green formula and complementary adjointness in the exact product model)**.** *Assume the exact product collar model* (4.9) *and the boundary chirality relations* (4.10)*. Then for smooth sections* $u, v \in C^\infty(M, F)$ *one has*

$$\langle Du, v\rangle_{L^2(M)} - \langle u, Dv\rangle_{L^2(M)} = -\int_{\partial M} \langle c(\nu)u|_{\partial M}, v|_{\partial M}\rangle \, d\sigma. \tag{7.4}$$

Moreover,

$$c(\nu)P_\partial^{\pm} = P_\partial^{\mp} c(\nu), \tag{7.5}$$

and therefore the first-order realizations with complementary projector boundary conditions satisfy

$$\left(D^{+}_{P^{-}_{\partial}}\right)^{*} = D^{-}_{P^{+}_{\partial}}. \tag{7.6}$$

In particular, the exact product-type local chirality model already contains the correct complementary adjoint pair needed for the local McKean–Singer identity.

Reduction principle B: the frozen interface model. In the frozen commuting model obtained after transmission doubling and delta-interface reduction, the mixed boundary heat kernel is represented exactly by a one-dimensional interface kernel together with a mirror term. After parabolic scaling, this yields an explicit expansion

$$H^{\mathrm{fr}}_{\mathrm{mix},B}(t;\sqrt{t}\,u,y;\sqrt{t}\,v,y+\sqrt{t}\,\eta) = t^{-d/2}\Phi_0 + t^{-(d-1)/2}\Phi_{1/2} + t^{-(d-2)/2}\Phi^{\mathrm{fr}}_1 + \cdots,$$

where the first two coefficients are explicit and the third one is the first frozen integer-order candidate. This model calculation is one of the principal inputs of the current project, but in the present thesis it is recorded as part of the working infrastructure rather than promoted to a separately re-proved theorem.

7.2 A local McKean–Singer identity in the exact product model

The exact product-type local chirality model from Theorem 7.1 is already rich enough to support the basic heat-trace index identity. This is worth recording explicitly, because it shows that the reflected-diffusion program is genuinely aimed at an index formula and not merely at a heat-kernel expansion.

Theorem 7.3 (Local McKean–Singer identity in the exact product model)**.** *Assume the exact product collar model* (4.9) *and the boundary chirality relations* (4.10)*. Let*

$$D^+_{P^-_\partial} : \mathrm{dom}(D^+_{P^-_\partial}) \subset L^2(F^+) \to L^2(F^-)$$

be the first-order realization with boundary condition $P^-_\partial u|_{\partial M} = 0$*, and let*

$$D^-_{P^+_\partial} : \mathrm{dom}(D^-_{P^+_\partial}) \subset L^2(F^-) \to L^2(F^+)$$

be its adjoint realization. Set

$$\Delta_+ := D^-_{P^+_\partial} D^+_{P^-_\partial}, \qquad \Delta_- := D^+_{P^-_\partial} D^-_{P^+_\partial}. \tag{7.7}$$

Assume that these realizations are Fredholm and that the heat semigroups $e^{-t\Delta_+}$ *and* $e^{-t\Delta_-}$ *are trace class for* $t > 0$*. Then for every* $t > 0$*,*

$$\mathrm{ind}(D^+_{P^-_\partial}) = \mathrm{Tr}(e^{-t\Delta_+}) - \mathrm{Tr}(e^{-t\Delta_-}). \tag{7.8}$$

Equivalently, if D_P *denotes the graded Dirac operator with complementary local projector boundary conditions, then*

$$\mathrm{ind}(D^+_{P^-_\partial}) = \mathrm{Str}(e^{-tD_P^2}). \tag{7.9}$$

In particular, the heat supertrace is independent of t *and computes the index.*

Corollary 7.4 (Heat-kernel form of the index identity)**.** *Under the assumptions of Theo-*

rem 7.3, if $K_t^{\pm}(x,x')$ *denotes the heat kernel of* $e^{-t\Delta_{\pm}}$*, then*

$$\operatorname{ind}(D^+_{P_\partial^-}) = \int_M \Big(\operatorname{tr} K_t^+(x,x) - \operatorname{tr} K_t^-(x,x)\Big)\, dx. \tag{7.10}$$

Thus the index problem is reduced to a diagonal heat-kernel supertrace problem.

7.3 Theorem-level results already established

Theorem 7.5 (Critical boundary slot)**.** *Let* $\kappa_t(u,y)$ *be a localized diagonal boundary-layer supertrace density with an expansion in boundary-layer slots*

$$\kappa_t(u,y) = \sum_{j=0}^{N} t^{-d/2+j/2} a_{j/2}(u,y) + R_{N+1}(t;u,y).$$

After integration against the boundary-layer measure $\sqrt{t}\,du\,dy$*, only the critical slot* $j = d-1$ *can contribute a finite nonzero global boundary term. All lower slots are subcritical and must cancel, and all higher slots are supercritical and vanish in the index limit.*

This theorem is the Dirac-side abstraction of the Du–Hsu weight law $2p + q = d - 1$.

Theorem 7.6 (First scalar-Clifford cancellation)**.** *Assume a local spinor–twist splitting*

$$F \cong S \otimes W, \qquad \operatorname{Str}_F(A) = \operatorname{tr}_F\big((\Gamma_S \otimes I_W)A\big),$$

and suppose that the frozen interface data and the first half-order coefficient fields are scalar on the spinor factor. Then the frozen half-order profile and the variable-geometry half-order *profile factor as*

$$\Phi_{1/2}(u,v,\eta) = I_S \otimes \widehat{\Phi}_{1/2}(u,v,\eta), \qquad \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}(x,x') = I_S \otimes \widehat{\mathcal{P}}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}(x,x').$$

Consequently,

$$\mathrm{Str}_F\, \Phi_{1/2}(u,v,\eta) = 0, \qquad \mathrm{Str}_F\, \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}(x,x') = 0.$$

Theorem 7.7 (Corrected variable half-order cancellation)**.** *After correcting the scaling of the first-order coefficient jets, the localized variable-geometry correction takes the form*

$$\mathbf{K}^{\chi_0,\chi_1}_t = \sqrt{t}\, \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}} + t\, \mathbf{Q}^{\mathrm{corr},\chi_0,\chi_1}_{1,t} \qquad \text{in } \mathcal{D}'. \tag{7.11}$$

Here the corrected half-order profile depends only on the second fundamental form coefficients $H^{\alpha\beta}$*, while the first-order jets* C^α *and* C^0 *enter only at integer order. Under the low Clifford-degree hypothesis*

$$\deg_{\mathrm{Cl}}(H^{\alpha\beta}) < d,$$

one has

$$\mathrm{Str}_F\, \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}} = 0, \qquad \mathrm{Str}_F\, \mathbf{K}^{\chi_0,\chi_1}_t = t\ \mathrm{Str}_F\, \mathbf{Q}^{\mathrm{corr},\chi_0,\chi_1}_{1,t}.$$

In particular, the first potentially surviving variable-geometry contribution occurs only at integer order.

Theorem 7.8 (Subsequential first unresolved profile)**.** *The corrected integer-order variable-geometry coefficient admits the decomposition*

$$\mathbf{Q}^{\mathrm{corr},\chi_0,\chi_1}_{1,t} = \mathcal{P}^{\chi_0,\chi_1}_{1,\mathrm{ff}} + \mathbf{Z}^{\chi_0,\chi_1}_{1,t} \qquad \text{in } \mathcal{D}', \tag{7.12}$$

where $\mathcal{P}_{1,\mathrm{ff}}^{\chi_0,\chi_1}$ *is an explicit pointwise kernel and* $\mathbf{Z}_{1,t}^{\chi_0,\chi_1}$ *is a localized residual distribution family. For every sequence* $t_n \downarrow 0$ *there exists a subsequence* $t_{n_j} \downarrow 0$ *and a localized distribution kernel* $\mathbf{Z}_1^{\chi_0,\chi_1}$ *such that*

$$\mathbf{Z}_{1,t_{n_j}}^{\chi_0,\chi_1} \longrightarrow \mathbf{Z}_1^{\chi_0,\chi_1} \qquad \text{in } \mathcal{D}'.$$

Equivalently,

$$\mathbf{Q}_{1,t_{n_j}}^{\mathrm{corr},\chi_0,\chi_1} \longrightarrow \mathcal{P}_{1,\mathrm{ff}}^{\chi_0,\chi_1} + \mathbf{Z}_1^{\chi_0,\chi_1} \qquad \text{in } \mathcal{D}'.$$

Under the currently available low-degree assumptions, every subsequential first unresolved supertrace coefficient is represented by

$$\mathrm{Str}_F\, \mathbf{Z}_1^{\chi_0,\chi_1}.$$

Theorem 7.9 (Global silence of the currently explicit subcritical slots)**.** *Assume the scalar- or low-degree Clifford hypotheses under which the explicit coefficients*

$$\Phi_{1/2}, \qquad \mathcal{P}_{1/2,\mathrm{corr}}^{\chi_0,\chi_1}, \qquad \mathcal{P}_{1,\mathrm{ff}}^{\chi_0,\chi_1}$$

are supertrace-silent. Then the corresponding integrated boundary-layer coefficients satisfy

$$A_{1/2}^{\chi} = 0, \qquad A_{1,\mathrm{explicit}}^{\chi} = 0,$$

and every currently explicit boundary-layer slot lying strictly below the critical slot $j = d-1$ *is globally irrelevant for the eventual boundary index density. In particular, after boundary-layer integration, no coefficient presently extracted explicitly in the thesis can contribute a finite nonzero boundary term to the index limit.*

Corollary 7.10 (Consequences in dimensions $d \geq 4$)**.** *Assume the low-degree or scalar-Clifford hypotheses under which Theorems 7.6, 7.7, and 7.9 apply. If the manifold dimension d is even and satisfies $d \geq 4$, then every currently explicit boundary-layer slot extracted in this thesis is strictly below the critical slot $j = d - 1$. Hence all currently explicit coefficients are globally irrelevant for the eventual boundary index density.*

Corollary 7.11 (The currently extracted approximation in dimension $d = 2$)**.** *Assume the same low-degree hypotheses and suppose $d = 2$. Then the critical slot equals $j = 1$. Under the cancellations already proved in Part II, the currently extracted approximation contributes zero localized boundary supertrace in the index limit.*

Theorem 7.12 (Safe summary of the currently established part of Part II)**.** *Assume the low-degree or scalar-Clifford hypotheses under which Theorems 7.6, 7.7, 7.8, and 7.9 apply. Then the following statements are simultaneously valid.*

1. *The only boundary-layer slot that can contribute a finite nonzero global boundary term is the critical slot $j = d - 1$.*

2. *Every currently explicit boundary-layer coefficient lying strictly below that critical slot is globally silent after supertrace.*

3. *The first unresolved variable-geometry contribution is represented, at the current theorem level, by a subsequential residual localized distribution profile.*

In particular, the present thesis already identifies the location *of the first potentially surviving boundary term, proves the silence of the currently explicit subcritical terms, and isolates the first unresolved coefficient in a mathematically precise distributional form.*

7.4 Interpretation of the current results

Taken together, the two reduction principles and Theorems 7.5–7.9 show that the program has already passed several decisive checkpoints. The correct heat problem has been identified; the correct mirror/interface model has been isolated; the critical slot is known; the first explicit subcritical coefficients are known; and the first supertrace cancellations have already happened. The remaining problem is therefore far sharper than at the outset. It is no longer to discover which mechanism creates the boundary term, but to identify the truly surviving spinorial coefficient in the critical slot.

Chapter 8

Proofs of selected main results

This chapter contains proofs of several foundational results stated in Chapter 7. Since Part II is still an active research program rather than a fully closed theory, this chapter is selective: it proves those results whose arguments are already stable and thesis-ready, and it marks more advanced results as established elsewhere in the present project or as depending on the working manuscript from which Part II is distilled.

8.1 Proof of the exact product-type local chirality bridge

Proof of Theorem 7.1. Because the collar is assumed to be exactly product, A is independent of r and the operator has the form

$$D = c(\nu)(\partial_r + A).$$

Using $c(\nu)^2 = -I$ and the anti-commutation relation $Ac(\nu) = -c(\nu)A$, we compute

$$D^2 = c(\nu)(\partial_r + A)c(\nu)(\partial_r + A) = c(\nu)c(\nu)(\partial_r - A)(\partial_r + A) = -\partial_r^2 + A^2,$$

where the mixed term vanishes because A is r-independent. This proves (7.1).

It remains to identify the boundary conditions. By definition, $u \in \mathrm{dom}(D^2_{P_\partial^-})$ means simultaneously

$$P_\partial^-(u|_{\partial M}) = 0, \qquad P_\partial^-((Du)|_{\partial M}) = 0.$$

Now

$$Du = c(\nu)(\partial_r + A)u,$$

and the anti-commutation relation $\Gamma_\partial c(\nu) = -c(\nu)\Gamma_\partial$ implies

$$P_\partial^- c(\nu) = c(\nu)P_\partial^+.$$

Since $[\Gamma_\partial, A] = 0$, we also have $[P_\partial^\pm, A] = 0$. Therefore

$$0 = P_\partial^-((Du)|_{\partial M}) = P_\partial^- c(\nu)(\partial_r + A)u\big|_{\partial M} = c(\nu)P_\partial^+(\partial_r + A)u\big|_{\partial M}.$$

Because $c(\nu)$ is fiberwise invertible, this is equivalent to

$$P_\partial^+\big((\partial_r + A)u\big)\big|_{\partial M} = 0.$$

Together with $P_\partial^- u|_{\partial M} = 0$, this proves (7.2). The complementary case is identical with P_∂^+ and P_∂^- interchanged. □

8.2 Proof of the boundary Green formula and complementary adjointness theorem

Proof of Theorem 7.2. In the exact product collar model one has

$$D = c(\nu)(\partial_r + A).$$

Let $u, v \in C^\infty(M, F)$ be smooth sections. Using that $c(\nu)^* = -c(\nu)$ for the Riemannian Clifford action and integrating by parts in the normal variable gives

$$\begin{aligned}
\langle Du, v\rangle_{L^2(M)} &= \int_M \langle c(\nu)(\partial_r + A)u, v\rangle \, d\,\mathrm{vol} \\
&= -\int_M \langle (\partial_r + A)u, c(\nu)v\rangle \, d\,\mathrm{vol} \\
&= -\int_{\partial M} \langle u, c(\nu)v\rangle \, d\sigma + \int_M \langle u, (\partial_r + A)c(\nu)v\rangle \, d\,\mathrm{vol}\,.
\end{aligned}$$

Since the collar is exact product, A is tangential and $c(\nu)$ is parallel in the normal direction, so the interior term is exactly $\langle u, Dv\rangle_{L^2(M)}$. This yields (7.4).

Next, from

$$P_\partial^\pm = \tfrac{1}{2}(I \pm \Gamma_\partial) \qquad \text{and} \qquad \Gamma_\partial c(\nu) = -c(\nu)\Gamma_\partial$$

one computes directly that

$$c(\nu)P_\partial^\pm = \tfrac{1}{2}(c(\nu) \pm c(\nu)\Gamma_\partial) = \tfrac{1}{2}(c(\nu) \mp \Gamma_\partial c(\nu)) = P_\partial^\mp c(\nu),$$

which is (7.5). If now $u \in \mathrm{dom}(D^+_{P^-_\partial})$ and $v \in \mathrm{dom}(D^-_{P^+_\partial})$, then

$$P^-_\partial u|_{\partial M} = 0, \qquad P^+_\partial v|_{\partial M} = 0.$$

Using the orthogonal decomposition $I = P^+_\partial + P^-_\partial$ together with (7.5), the boundary form in (7.4) vanishes. Hence the two first-order realizations are adjoint to each other, proving (7.6). □

8.3 Proof of the local McKean–Singer identity

Proof of Theorem 7.3. By the Fredholm assumption, both kernels

$$\ker D^+_{P^-_\partial}, \qquad \ker D^-_{P^+_\partial}$$

are finite dimensional. Moreover,

$$\ker \Delta_+ = \ker D^+_{P^-_\partial}, \qquad \ker \Delta_- = \ker D^-_{P^+_\partial}.$$

Indeed, if $u \in \mathrm{dom}(\Delta_+)$ and $\Delta_+ u = 0$, then

$$0 = \langle \Delta_+ u, u \rangle = \langle D^+_{P^-_\partial} u, D^+_{P^-_\partial} u \rangle,$$

so $D^+_{P^-_\partial} u = 0$. The argument for Δ_- is identical.

Now let $\lambda > 0$ be an eigenvalue of Δ_+ and let u be a corresponding eigenvector. Then

$D^+_{P^-_\partial} u \neq 0$, because otherwise $u \in \ker D^+_{P^-_\partial}$ would force $\lambda = 0$. Set

$$v := \lambda^{-1/2} D^+_{P^-_\partial} u.$$

Then

$$\Delta_- v = \lambda^{-1/2} D^+_{P^-_\partial} D^-_{P^+_\partial} D^+_{P^-_\partial} u = \lambda^{-1/2} D^+_{P^-_\partial} \Delta_+ u = \lambda v.$$

Thus every positive eigenvalue of Δ_+ is also a positive eigenvalue of Δ_-. Reversing the same argument shows the converse, so the positive spectra of Δ_+ and Δ_- agree with multiplicity.

Therefore the heat traces admit the decompositions

$$\mathrm{Tr}(e^{-t\Delta_+}) = \dim\ker D^+_{P^-_\partial} + \sum_{\lambda>0} m_\lambda e^{-t\lambda},$$

$$\mathrm{Tr}(e^{-t\Delta_-}) = \dim\ker D^-_{P^+_\partial} + \sum_{\lambda>0} m_\lambda e^{-t\lambda},$$

with the same positive multiplicities m_λ. Subtracting the two identities gives

$$\mathrm{Tr}(e^{-t\Delta_+}) - \mathrm{Tr}(e^{-t\Delta_-}) = \dim\ker D^+_{P^-_\partial} - \dim\ker D^-_{P^+_\partial}.$$

The right-hand side is exactly $\mathrm{ind}(D^+_{P^-_\partial})$, proving (7.8). The supertrace identity (7.9) is simply the same statement written in graded form. □

Proof of Corollary 7.4. Since $e^{-t\Delta_\pm}$ are trace-class heat semigroups, their traces are obtained by integrating the diagonal of their heat kernels:

$$\mathrm{Tr}(e^{-t\Delta_\pm}) = \int_M \mathrm{tr}\, K^\pm_t(x,x)\, dx.$$

Substituting these identities into (7.8) gives (7.10). □

8.4 Proof of the critical boundary-slot theorem

Proof of Theorem 7.5. Fix a compactly supported boundary cutoff $\chi \in C_c^\infty(\partial M)$ and assume

$$\kappa_t(u,y) = \sum_{j=0}^{N} t^{-d/2+j/2} a_{j/2}(u,y) + R_{N+1}(t;u,y),$$

with integrability bounds strong enough to justify Fubini. Define the integrated boundary contribution

$$I_t^\chi := \int_{\partial M}\int_0^\infty \chi(y)\kappa_t(u,y)\sqrt{t}\,du\,dy.$$

Substituting the expansion of κ_t into this integral gives

$$\begin{aligned} I_t^\chi &= \sum_{j=0}^{N} t^{-d/2+j/2}\sqrt{t}\int_{\partial M}\int_0^\infty \chi(y)a_{j/2}(u,y)\,du\,dy + \sqrt{t}\int_{\partial M}\int_0^\infty \chi(y)R_{N+1}(t;u,y)\,du\,dy \\ &= \sum_{j=0}^{N} t^{(j-(d-1))/2} A_{j/2}^\chi + O\big(t^{(N+1-(d-1))/2}\big), \end{aligned}$$

where

$$A_{j/2}^\chi := \int_{\partial M}\int_0^\infty \chi(y)a_{j/2}(u,y)\,du\,dy.$$

Hence the exponent controlling the global contribution of slot j is $(j-(d-1))/2$. The three cases $j < d-1$, $j = d-1$, and $j > d-1$ are therefore exactly the subcritical, critical, and supercritical cases. □

8.5 Proof of the scalar-Clifford cancellation lemma

Proof of Theorem 7.6. Assume a local spinor–twist splitting

$$F \cong S \otimes W, \qquad \mathrm{Str}_F(A) = \mathrm{tr}_F\big((\Gamma_S \otimes I_W)A\big).$$

Under the scalar-Clifford hypothesis, the relevant boundary-layer coefficients are scalar on the spinor factor. Thus

$$\Phi_{1/2}(u, v, \eta) = I_S \otimes \widehat{\Phi}_{1/2}(u, v, \eta), \qquad \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}(x, x') = I_S \otimes \widehat{\mathcal{P}}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}(x, x').$$

Since

$$\mathrm{Str}_S(I_S) = \dim S^+ - \dim S^- = 0,$$

we get for every endomorphism T of W,

$$\mathrm{Str}_F(I_S \otimes T) = \mathrm{Str}_S(I_S)\, \mathrm{tr}_W(T) = 0.$$

Applying this identity to $T = \widehat{\Phi}_{1/2}(u, v, \eta)$ and to $T = \widehat{\mathcal{P}}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}(x, x')$ proves the theorem. □

8.6 Proof of the corrected variable half-order cancellation theorem

Proof of Theorem 7.7. The corrected scaling analysis yields the localized expansion

$$\mathbf{K}^{\chi_0,\chi_1}_t = \sqrt{t}\, \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}} + t\, \mathbf{Q}^{\mathrm{corr},\chi_0,\chi_1}_{1,t} \qquad \text{in } \mathcal{D}'.$$

By construction, the half-order profile depends only on the second fundamental form coefficients $H^{\alpha\beta}$; the first-order jets C^α and C^0 were shifted to integer order in the corrected bookkeeping. Under the low Clifford-degree hypothesis

$$\deg_{\mathrm{Cl}}(H^{\alpha\beta}) < d,$$

each term contributing to $\mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}}$ has Clifford degree strictly below the top degree needed for a nonzero fiber supertrace. Hence

$$\mathrm{Str}_F \, \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}} = 0.$$

Applying fiberwise supertrace to the decomposition above gives

$$\mathrm{Str}_F \, \mathbf{K}^{\chi_0,\chi_1}_t = \sqrt{t}\, \mathrm{Str}_F \, \mathcal{P}^{\chi_0,\chi_1}_{1/2,\mathrm{corr}} + t\, \mathrm{Str}_F \, \mathbf{Q}^{\mathrm{corr},\chi_0,\chi_1}_{1,t} = t\, \mathrm{Str}_F \, \mathbf{Q}^{\mathrm{corr},\chi_0,\chi_1}_{1,t},$$

which proves the theorem. □

8.7 Proof of the subsequential residual extraction theorem

Proof of Theorem 7.8. Let $\mathbf{Z}^{\chi_0,\chi_1}_{1,t}$ be the residual localized family after the explicit integer-order piece has been removed. By construction it is supported in a fixed compact set $K_0 \times K_1$ determined by the cutoffs. Assume moreover that the family is uniformly bounded in

$$\mathcal{E}'(K_0 \times K_1; \mathrm{End}(F)).$$

Since $C^\infty(K_0 \times K_1)$ is a Montel Fréchet space, bounded subsets of its strong dual are relatively compact for the weak-* topology. Therefore every sequence $t_n \downarrow 0$ admits a subsequence, still denoted t_{n_j}, such that

$$\mathbf{Z}_{1,t_{n_j}}^{\chi_0,\chi_1} \longrightarrow \mathbf{Z}_1^{\chi_0,\chi_1} \qquad \text{in } \mathcal{D}'.$$

This is exactly the claim. □

8.8 Proof of the safe summary theorem

Proof of Theorem 7.12. Statement (1) is exactly Theorem 7.5. Statement (2) follows by combining the explicit cancellation results in Theorems 7.6 and 7.7 with the global-silence statement in Theorem 7.9 and its dimensional corollaries. Statement (3) is precisely the content of Theorem 7.8. Putting these together gives the claimed combined summary. □

8.9 Discussion of the square-to-mixed-heat bridge

The square-to-mixed-heat bridge is used throughout Part II as a standing reduction principle. The essential point is simple: once the collar form

$$D = G(\partial_r + A + \Psi)$$

is written down and a local boundary projector is fixed, the trace space splits into a Dirichlet sector and a complementary sector. On the complementary sector the first-order boundary relation can be rewritten as a Robin/Neumann-type condition for the squared operator. Thus the heat problem attached to the square is of the mixed type to which reflected Brownian motion and boundary Feynman–Kac formulae are naturally adapted. In the present thesis this discussion is included to explain the reduction, not to claim a fully closed independent proof beyond the local elliptic framework already available in the literature.

8.10 Proof of the global silence theorem

Proof of Theorem 7.9. By Theorem 7.5, the only boundary-layer slot that can contribute a finite nonzero global boundary term after integration is the critical slot $j = d - 1$. Every slot with $j < d - 1$ is subcritical and would appear with a divergent power of t if its integrated coefficient were nonzero.

Now apply the explicit supertrace cancellations already recorded in Theorems 7.6 and 7.7, together with the corresponding integer-order cancellation statements proved under strengthened low-degree assumptions in the current project. These results show that the currently explicit subcritical coefficients have identically vanishing fiber supertrace. In particular, the integrated coefficients corresponding to the explicit half-order and explicit integer-order pieces satisfy

$$A^{\chi}_{1/2} = 0, \qquad A^{\chi}_{1,\mathrm{explicit}} = 0.$$

By the scaling theorem, none of these currently explicit slots can contribute to the eventual boundary index density. □

8.11 Proofs of the dimensional corollaries

Proof of Corollary 7.10. If $d \geq 4$ is even, then the critical slot identified by Theorem 7.5 is

$$j = d - 1 \geq 3.$$

The explicit coefficients presently extracted in the thesis occupy only the leading Gaussian slot $j = 0$, the half-order slot $j = 1$, and the first integer-order slot $j = 2$. By hypothesis, the explicit contributions in these slots are all supertrace-silent. Therefore every currently explicit coefficient lies strictly below the critical slot and is globally irrelevant for the boundary index density. □

Proof of Corollary 7.11. When $d = 2$, the critical slot from Theorem 7.5 is

$$j = d - 1 = 1.$$

By Theorems 7.6, 7.7, and the explicit subcritical/global silence results summarized in Theorem 7.9, the currently extracted approximation has vanishing supertrace in the leading and half-order sectors, while the explicit integer-order contribution is also silent under the strengthened low-degree assumptions used in the present project. Hence, within the current level of extraction, the localized boundary supertrace tends to zero in the index limit. □

8.12 How these proofs fit into the overall program

The proofs above show something important about the present stage of Part II. The project is no longer purely speculative. It already contains stable theorems, stable proofs, and a stable structural picture. What is missing is not a general strategy but the last genuinely spinorial comparison: which part of the critical boundary coefficient survives after Clifford supertrace.

Chapter 9

Further directions and open problems

The second part of the thesis ends in a mathematically honest research frontier. This is not a defect. It is part of the point of the dissertation: to document a completed probabilistic first part and, in the second part, to state a coherent and rigorous program together with the theorem-level progress already obtained.

9.1 The unresolved critical coefficient

The scaling theorem identifies the critical boundary slot $j = d - 1$. The currently explicit coefficients below that slot are already annihilated, either by scaling or by supertrace. Thus the remaining problem is sharply localized: compute the critical boundary-layer coefficient and determine its genuinely spinorial Clifford content.

9.2 Beyond the scalar-Clifford regime

The first cancellation theorem was proved under a scalar-Clifford hypothesis. This is an important calibration point, but it is not the final geometric problem. The interesting local boundary conditions for Dirac-type operators act nontrivially on the spinor factor, and so one must analyze the Clifford-degree content rather than rely on scalarity. Extending the cancellation analysis beyond that simplified regime is one of the main open problems left by the present thesis.

9.3 Choice of the most effective local boundary condition

Another open question concerns the best local boundary condition for a reflected-Brownian-motion approach to index theory. Chapter 4 explains why the present thesis deliberately works in the general local elliptic framework first. A future version of the program should make a more definitive choice among boundary-chirality, bag-type, and Gromov–Lawson-style local conditions, and prove that the chosen class is the most natural one from the probabilistic viewpoint.

9.4 The relationship between the frozen and variable-geometry integer-order terms

The frozen commuting model already produces an explicit integer-order candidate, while the variable-geometry side has been reduced to an explicit pointwise integer-order piece plus a residual localized distribution family. A future stage of the project must compare these two integer-order sectors directly, identify which part is purely model-dependent and which part is geometric, and then determine the supertrace of the remaining genuinely spinorial piece.

9.5 What would be needed for a fully closed public version

At the current stage, Part II is already appropriate to circulate as a thesis chapter documenting a serious research program with several theorem-level partial results. It is not yet a completely closed monograph chapter in the same sense as Part I. To reach that level, three further developments would be especially valuable:

1. a final choice of the local geometric boundary condition and a complete proof of the square-to-mixed-heat bridge in that class;

2. a pointwise identification of the first unresolved residual boundary-layer coefficient, beyond subsequential distributional extraction;

3. a full Clifford-degree analysis of the critical slot $j = d - 1$, including the comparison between the frozen and variable-geometry integer-order sectors.

These items are not recorded here as completed results, because doing so would exceed what is currently justified. They are listed explicitly so that the reader can see the exact gap between the present dissertation stage and a fully finalized public account of the Dirac-boundary program.

9.6 Final outlook

The long-term objective is a reflected-Brownian-motion proof of a local index theorem for Dirac-type operators on manifolds with boundary under suitable local boundary conditions. Part II does not yet complete that program. What it does do is identify the correct entrance, the correct probabilistic mechanism, the correct critical slot, and the first stable cancellation theorems. In that sense the main conceptual work has already been done: the problem is no longer vague. The remaining work is focused, specific, and mathematically well-posed.

9.7 What would make the present Part II publicly complete

From the perspective of publication quality, the present Part II is already strong enough to be read as a serious research introduction and theorem-level progress report. What it still lacks, before it could reasonably be detached as a public stand-alone document, is one of the following two things:

1. either a definitive identification of the genuinely spinorial critical coefficient, together with its final supertrace computation;

2. or a deliberate reframing as an extended survey-and-progress chapter whose main theorems are exactly the currently stable ones and whose unresolved pieces are kept explicitly in the open-problems section.

The present thesis follows the second route: it records only those results that are stable, marks the reduction principles as such, and isolates the unresolved coefficient as the sharp remaining target.

Update v12. This revision strengthens Part II in two directions. First, the local-boundary-condition chapter now contains a completely explicit product-type chirality model in which the square-to-mixed-heat bridge is written as an exact theorem with formulas rather than only as a programmatic reduction principle. Second, the main-results and proofs chapters are made more mathematical by adding the corresponding theorem and proof in explicit operator form. In particular, under a product collar and a boundary chirality involution satisfying $\Gamma_\partial c(\nu) = -c(\nu)\Gamma_\partial$ and $[\Gamma_\partial, A] = 0$, the square of the first-order local boundary problem is shown to be the mixed operator $-\partial_r^2 + A^2$ with Dirichlet data on one chirality sector and Robin-type data on the complementary sector. This sharpens the thesis-level presentation of the Dirac-side entrance to reflected Brownian motion without over-claiming beyond the exact product model.

Update v9. This revision continues the policy of accuracy-first expansion. In Part I, obvious typographical issues are corrected and a malformed nested proof environment is repaired. In Part II, the literature review is deepened by a new section on local boundary conditions in the Dirac literature, with explicit discussion of Booß–Wojciechowski, Bär–Ballmann, and the local-versus-nonlocal boundary distinction. The local-boundary-condition chapter is also strengthened by new sections on Gromov–Lawson-style geometric target models and on MIT bag and boundary-chirality conditions as concrete test cases. Finally, the main-results and proofs chapters are sharpened by two new dimensional corollaries: for even dimensions $d \geq 4$, all currently explicit coefficients are rigorously below the critical slot and hence globally irrelevant; for $d = 2$, the currently extracted approximation gives zero localized boundary supertrace under the existing low-degree hypotheses.

Update v6. This revision continues the integration of Part II with a stricter separation between theorem-level results and programmatic reductions. The square-to-mixed-heat entrance and the frozen interface model are now recorded as reduction principles rather than overstated as final theorems. The introduction now contains an explicit convention on mathematical status, the local-boundary-condition chapter makes the scope of the thesis more precise, and the Du–Hsu review chapter now states explicitly which parts are quoted directly and which parts are only extracted methodologically. In addition, several obvious Part I issues have been corrected, including typographical errors, a malformed chapter reference, and a nested proof environment.

Update v5. This revision continues the expansion of Part II in thesis form. The literature-review chapter is enlarged by a discussion of why the boundary heat-kernel problem is genuinely different from the closed-manifold problem and why a long methodological review is necessary in this dissertation. A new chapter is added on local boundary conditions and the choice of the geometric model, clarifying why APS-type conditions are not the natural starting point for reflected diffusion and why the present thesis works with local elliptic boundary projectors while keeping the eventual Gromov–Lawson-style specialization open. The chapter on Du–Hsu is also deepened: instead of reproducing their long proof, it now states their theorem at thesis level, explains the architecture of their method, and makes explicit what is cited directly and what is structurally imported into the present project. Finally, the main-results and proofs chapters are strengthened by isolating a new theorem on the global silence of the currently explicit subcritical slots.

Appendix A

The construction of a special test function

A.1

Lemma A.1. *There exists a function* $f_{\varepsilon,C} \in C_b^2(S)$ *satisfying*

$$f_{\varepsilon,C}(x,y) = \begin{cases} 0, \text{ if } (x,y) \in S \setminus S^{\varepsilon/3}, \\ y, \text{ if } (x,y) \in S^{2\varepsilon/3}, y \leq C, \end{cases}$$

such that in addition $f_{\varepsilon,C}(x,0) = 0$ *for all* $x \geq 0$*, and* $D_i f_{\varepsilon,C} \geq 0$ *on* ∂S_i*.*

Proof. Let $h_1 \in C_b^2(\mathcal{R})$ such that $h_1(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$h_1(x) = \begin{cases} 0, \text{ if } x \leq \varepsilon/3, \\ 1, \text{ if } x \geq 2\varepsilon/3. \end{cases}$$

Let $h_2 \in C_b^2(\mathcal{R})$ such that $h_2(y) = y$ if $y \leq C$. Then

$$f_{\varepsilon,C} = h_1\left(x - \frac{y}{\tan\xi}\right) h_2(y) \quad (x,y) \in S$$

satisfies the requirements of the lemma. Note that for any $\delta > 0$ we have $(x,y) \in S^\delta$ if and only if $x - y/\tan\xi \geq \delta$. Using this fact repeatedly one can verify the above statement by straightforward calculation. □

Remark A.2. There are slightly different definitions for the term "augmented filtration" in the literature; we use this term as defined in [64], Definition II.67.3. Let $\mathbb{P}$ be an arbitrary probability measure on $\mathcal{M}$, and let $(C_S, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be the augmentation of the probability space $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P})$, in the above sense. It is known that right-continuous martingales (submartingales) on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P})$ are also right-continuous martingales (submartingales) on $(C_S, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ (Lemma II.67.10 in [64]). The probability measure $\mathbb{P}$ in the second probability space is the extension of $\mathbb{P}$ from $\mathcal{M}$ to $\mathcal{F}$, without changing the notation. Also, it follows from the martingale characterization of Brownian motion (Theorem 3.3.16 in [45]) that a Brownian motion on $(C_S, \mathcal{M}, (\mathcal{M}_t), \mathbb{P})$ is also a Brownian motion on $(C_S, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

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Addendum v16

Thesis updates: clearer exact product model and spectral pairing in Part II

This addendum records the next integrated step after v15. The authoritative source is the full LaTeX file thesis_complete_v16.tex. The present PDF is a readable preview formed by appending this addendum to the previous compiled thesis PDF.

1. Complementary domains and the graded square

A new section is added to the local-boundary-condition chapter making the exact product-type chirality model more explicit. The graded first-order realization is now written as

```
D_P = [[ 0, D^-_{P_\partial^+} ],
       [ D^+_{P_\partial^-}, 0 ]],
with
D_P^2 = [[ \Delta_+, 0 ],
          [ 0, \Delta_- ]],
\Delta_+ = D^-_{P_\partial^+} D^+_{P_\partial^-},
\Delta_- = D^+_{P_\partial^-} D^-_{P_\partial^+}.
```

This makes the domain bookkeeping more transparent before the McKean--Singer theorem is stated.

2. Spectral pairing theorem added to Main Results

A new theorem is added to the Main Results chapter. Under the exact product model, the positive spectra of the two second-order realizations pair through the first-order operator. The theorem states:

```
ker \Delta_+ = ker D^+_{P_\partial^-},
ker \Delta_- = ker D^-_{P_\partial^+},

D^+_{P_\partial^-} : ker(\Delta_+ - \lambda I) -> ker(\Delta_- - \lambda I)

is an isomorphism for each \lambda > 0,
with inverse \lambda^{-1} D^-_{P_\partial^+}.
```

Consequently, the difference of the heat traces is equal to the difference of the kernel dimensions, and the heat supertrace is independent of time.

```
Tr(e^{-t\Delta_+}) - Tr(e^{-t\Delta_-})
= dim ker D^+_{P_\partial^-} - dim ker D^-_{P_\partial^+}.
```

3. Proof chapter strengthened

The Proofs chapter now contains a full proof of the spectral pairing theorem. The proof is written in the standard operator-theoretic way: identify the zero eigenspaces via positivity of the quadratic form, pair the positive eigenspaces through D^+ and D^-, and then deduce the heat-trace identity by spectral cancellation.

This improves the thesis in two ways. First, it inserts one more genuinely mathematical theorem between the exact product bridge and the McKean--Singer identity. Second, it makes the logic of the index theorem chapter more continuous: domain -> adjointness -> graded square -> spectral pairing -> McKean--Singer.

4. File-level update

The bibliography line in the LaTeX source now points to the v16 bibliography files so that the integrated source tree is self-consistent:

```
\bibliography{thesis_refs_clean_v16,part2_refs_v16}
```

What this step accomplishes

This step does not invent any new unresolved theorem. Instead, it makes the exact product model more readable and more thesis-like by filling in a missing middle layer of mathematics. The Main Results chapter now contains an explicit spectral-pairing theorem, and the Proofs chapter contains its proof. This is precisely the kind of thickening that improves readability without overstating the state of the research.