

# STATIONARY DISTRIBUTIONS FOR THE NONSEMIMARTINGALE REFLECTED BROWNIAN MOTION WITH DRIFT IN A WEDGE: EXACT LOGARITHMIC TAILS

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ABSTRACT. We develop the quantitative tail asymptotics of the stationary distribution for the nonsemimartingale reflected Brownian motion with constant drift in a convex planar wedge, defined by the submartingale problem (Definition 2.1 in [9]). We give the exact logarithmic tails for each drift  $\mu$  such that the stationary distribution exists, throughout  $1 \leq \alpha < 2$  (the case when the RBM solution to the submartingale problem is nonsemimartingale). (The existence phase diagram for the drift  $\mu$  is identified in the companion paper [8].) For every stationary law we prove explicit exponential-moment estimates, locally uniform moment bounds, analyticity of the Laplace transform on the interior of its convergence domain, and logarithmic upper bounds for projections, radial tails, angular sectors, closed sets, and Laplace functionals. At  $\alpha = 1$ , the stationary marginal in the distinguished normal direction is exactly exponential.

The main theorem determines the complete closed exponential-moment domain. Let  $\mathbb{B}_\mu = \overline{B}(-\mu, |\mu|)$  be the kernel disk, and let  $\mathbb{C}_{12}(\mu)$  be its intersection with the two half-spaces determined by the one-face reflection costs. For every stationary drift and every stationary law  $\pi$ ,

$$D_\pi := \left\{ \theta : \int_S e^{\theta \cdot z} \pi(dz) < \infty \right\} = \mathbb{C}_{12}(\mu) - S^\vee, \quad 1 \leq \alpha < 2.$$

Every interior point of this set is a genuine moment vector. Finite sums of exponentials provide the matching Lyapunov functions, while killed-wedge, one-face, and broken-path estimates provide matching stationary lower bounds. Consequently, every linear functional strictly positive on the wedge, the radial tail, and every nondegenerate closed angular sector have exact logarithmic asymptotics. The domain and all resulting rates depend only on the drift and reflection geometry and not on the choice of stationary law.

## CONTENTS

|  |     |
|--|-----|
| 1. Introduction and Main Results                                 | 2   |
| 2. Geometry and basic properties of the reflected diffusion      | 3   |
| 3. Stationary identities and interior regularity                 | 6   |
| 4. Auxiliary geometric and truncation lemmas                     | 11  |
| 5. Quantitative estimates and exponential-moment lower bounds    | 13  |
| 5.1. Finite-exponential Lyapunov extensions of the moment region | 16  |
| 6. The closed exponential-moment domain and logarithmic tails    | 48  |
| 6.1. Full scalar abscissae and exact logarithmic tail rates      | 49  |
| 6.2. Kinematic outer bounds from the killed wedge kernel         | 65  |
| 6.3. Reflection-assisted one-face kinematic bounds               | 73  |
| 6.4. One-face broken paths and a sharper outer frontier          | 86  |
| 7. Discussion and further directions                             | 119 |
| 7.1. Boundary moments and transform singularities                | 119 |
| 7.2. Sharp one-term asymptotics                                  | 119 |
| 7.3. Raywise density and reflecting-face boundary layers         | 120 |

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|--|-----|
| 7.4. The vertex, uniqueness, and long-time convergence | 120 |
| 7.5. Extensions of the model                           | 120 |
| References   | 120 |

## 1. INTRODUCTION AND MAIN RESULTS

Let  $S$  be a convex planar wedge, and consider the reflected Brownian motion with constant drift  $\mu$  and oblique reflection directions  $v_1, v_2$  defined by the Lakner–Liu–Reed submartingale problem. The companion paper [8] determines the stationary-existence region for  $1 \leq \alpha < 2$ , establishes the stationary inequality, and develops its weak elliptic formulation. The present paper determines the stationary distribution at logarithmic scale throughout that entire region: first through the full exponential-moment domain, and then through exact projected, radial, and sector tail exponents.

For a stationary law  $\pi$ , write

$$\mathcal{D}_{\text{exp}}(\pi) := \left\{ \theta \in \mathbb{R}^2 : \int_S e^{\theta \cdot z} \pi(dz) < \infty \right\}, \quad D_\pi := \overline{\mathcal{D}_{\text{exp}}(\pi)}.$$

The set  $D_\pi$  is the closed exponential-moment domain. Since uniqueness of the stationary distribution is not assumed, this notation is attached to the chosen stationary law. A principal conclusion below is that, in the classified regimes, the set is in fact determined by the drift and reflection geometry alone.

**Theorem 1.1** (qualitative stationary classification from [8]). *Assume  $0 < \xi < \pi$ . If  $\alpha = 1$ , a stationary distribution exists if and only if*

$$n_{\mathcal{L}} \cdot \mu < 0.$$

*If  $1 < \alpha < 2$ , a stationary distribution exists if and only if*

$$\mu \notin K_{\text{str}} := \text{cone}\{-v_1, -v_2\}.$$

*The stationary inequality, the zero-boundary-mass property, and the neutral-test identity are recalled below from [8].*

We first establish quantitative estimates that hold for every stationary law in this region.

**Theorem 1.2** (quantitative stationary estimates). *Let  $\pi$  be a stationary distribution in either classified regime. Then  $\pi$  possesses an explicit nonempty region of exponential moments. On compact subsets of this region the moments are locally bounded, and the Laplace transform is analytic. The same estimates yield directional, radial, angular-sector, compact-set, large-deviation, and Laplace upper bounds. If  $\alpha = 1$ , the marginal in the direction  $n_{\mathcal{L}}$  is exponential with parameter  $-2n_{\mathcal{L}} \cdot \mu$ .*

The central result identifies the exact logarithmic frontier. Let

$$\mathbb{B}_\mu := \overline{B}(-\mu, |\mu|)$$

be the kernel disk. For each face, a one-face variational problem defines a clipping half-space; their intersection with  $\mathbb{B}_\mu$  is denoted by  $\mathbb{C}_{12}(\mu)$ . The associated lower hull is

$$\mathcal{K}_{12}(\mu) := \mathbb{C}_{12}(\mu) - S^\vee.$$

These objects are defined explicitly in [section 6.4](#).

**Theorem 1.3** (closed moment domain and exact logarithmic tails). *Assume  $0 < \xi < \pi$ ,  $1 \leq \alpha < 2$ , and let  $\mu$  be a stationary drift. For every stationary law  $\pi$  at drift  $\mu$ ,*

$$D_\pi = \mathcal{K}_{12}(\mu), \quad \text{int } \mathcal{D}_{\text{exp}}(\pi) = \text{int } \mathcal{K}_{12}(\mu),$$

and

$$\text{int } \mathcal{K}_{12}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi) \subset \mathcal{K}_{12}(\mu).$$

Thus the interior and the closure of the moment domain are independent of the choice of stationary law.

For every  $c \in S_{\circ}^{\vee}$ , let

$$\alpha_c(\mu) := \sup\{\lambda \geq 0 : \lambda c \in \mathcal{K}_{12}(\mu)\}.$$

Then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z \in S : c \cdot z \geq x\} = -\alpha_c(\mu).$$

If  $C \subset S \cap \mathbb{S}^1$  is a nondegenerate closed angular interval, then

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in C \right\} = -\min_{u \in C} h_{\mathcal{K}_{12}(\mu)}(u).$$

Taking  $C = S \cap \mathbb{S}^1$  gives the exact radial logarithmic tail.

The proof has two complementary parts. Finite sums of exponentials, chosen so that the leading oblique derivative on each face is negative, fill the interior of  $\mathcal{K}_{12}(\mu)$  with genuine moment vectors. Conversely, killed-wedge estimates, one-face motion, and one-face motion followed by an interior segment give matching lower-tail bounds. A two-dimensional clipping argument shows that the resulting pathwise lower hull agrees with  $\mathcal{K}_{12}(\mu)$ . In the classified geometry at most one one-face clip is active, which upgrades the raywise identification to the whole closed domain. These arguments also supply the lower bounds needed for the existence of the full logarithmic limits; the moment abscissa alone would give only a liminf identity.

For semimartingale reflected Brownian motion in wedges and quadrants, transform methods, boundary measures, and complex-analytic techniques yield convergence domains, exact tail asymptotics, integral representations, and, in special geometries, finite sums of exponential densities; see [1, 2, 3, 4, 5]. The process in Lakner–Liu–Reed [9] need not possess a global boundary-local-time decomposition in the strict nonsemimartingale regime. Our argument therefore proceeds through the submartingale formulation, bounded Lyapunov tests, the Brownian–extended-Skorokhod decomposition away from the vertex, and finite-horizon path localization.

The paper is organized as follows. Section 2 fixes the geometry and the reflected diffusion. Section 3 records the stationary identities and interior regularity used below, and section 4 collects two auxiliary lemmas. Section 5 proves the quantitative stationary estimates and constructs finite-exponential Lyapunov functions. Section 6 identifies the closed moment domain and proves the projected, radial, and sector logarithmic asymptotics. The final section delineates the finer questions that remain beyond the logarithmic scale.

## 2. GEOMETRY AND BASIC PROPERTIES OF THE REFLECTED DIFFUSION

Throughout the paper,

$$\text{cone}\{w_1, \dots, w_k\} := \left\{ \sum_{j=1}^k a_j w_j : a_j \geq 0 \right\}$$

denotes the closed conical hull generated by its arguments. In particular, it is not the compact convex hull. We use this notation consistently because the phase diagram below is organized by conical subsets and their complements.

Let

$$S = \{(r, \theta) : r \geq 0, 0 \leq \theta \leq \xi\} \subset \mathbb{R}^2, \quad 0 < \xi < \pi,$$

be the convex wedge. We write its two unit generators as

$$u_1 = (1, 0), \quad u_2 = (\cos \xi, \sin \xi),$$

so that  $S = \text{cone}\{u_1, u_2\}$ . Its two open boundary rays are denoted by  $\partial S_1$  and  $\partial S_2$ , excluding the vertex. Let  $v_1, v_2 \in \mathbb{R}^2$  be the reflection directions, normalized by

$$v_i \cdot n_i = 1, \quad i = 1, 2,$$

where  $n_i$  is the inward unit normal on  $\partial S_i$ . We write

$$D_i = v_i \cdot \nabla, \quad i = 1, 2.$$

Throughout,  $C_b^2(S)$  denotes functions whose values and first two Euclidean derivatives are bounded on  $S$ , with derivatives understood on the interior and by continuous extension up to each open face. Every admissible test used below is also constant on a neighborhood of the vertex, so its derivatives extend through the corner. Generator expressions  $\mathcal{L}f$  are the bounded open-wedge expressions, with arbitrary bounded values assigned on the boundary when an integral over  $S$  is written; the zero-boundary-mass result in [proposition 3.2](#) shows that these boundary choices never affect stationary integrals. For gauges that are introduced before localization and are not inserted directly into the submartingale problem, the following regularity convention is also used: a function  $W$  is *facewise  $C^2$  away from the vertex* if it is continuous on  $S$ , is  $C^2$  in  $S^\circ$ , and its first and second Euclidean derivatives admit continuous limits to compact subsets of each open face. A facewise  $C^2$  gauge is called *locally  $C^2$ -extendable away from the vertex* if, for every compact set  $K \subset S \setminus \{0\}$ , its restriction to  $K$  is the restriction of a  $C^2$  function on an open neighborhood of  $K$  in  $\mathbb{R}^2$ . Boundary directional derivatives such as  $D_i W$  are interpreted through the open-face limits. Whenever a localized gauge is used as an admissible test, the statement includes this local extendability, or verifies it directly. All functions used as admissible tests below are bounded members of  $C_b^2(S)$ , typically after localization.

We also fix the semigroup and stationarity conventions. For drift  $\mu$ , write

$$P_t^\mu g(z) := \mathbb{E}_z^\mu[g(Z(t))].$$

A probability measure  $\pi$  on  $S$  is stationary if  $\pi P_t^\mu = \pi$  for every  $t \geq 0$ . The state space  $S$  is locally compact and Polish. For  $\alpha < 2$ , Lakner–Liu–Reed prove the  $\widehat{C}(S)$ -Feller property [[9](#), Theorem 2.12]; their space  $\widehat{C}(S)$  is the usual space  $C_0(S)$  of continuous functions vanishing at infinity. This  $C_0(S)$ -Feller property, together with tightness of occupation measures, is the only semigroup regularity used in the Krylov–Bogoliubov argument below. The superscript  $\mu$  is suppressed when the drift is fixed.

As in Lakner–Liu–Reed [[9](#)], let  $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$  be the reflection angles and define

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

Throughout the paper we use the following sign convention for these angles. Let  $t_i$  be the unit tangent on  $\partial S_i$  pointing away from the vertex. The normalization  $v_i \cdot n_i = 1$  is then equivalent to

$$v_i = n_i - (\tan \theta_i)t_i, \quad i = 1, 2.$$

Thus a positive reflection angle means that the tangential component of the reflection direction points toward the vertex. This convention is the one used in the Varadhan–Williams harmonic calculation in [[8](#)].

The process studied in this paper is specified by the following submartingale problem.

**Definition 2.1** (Submartingale problem with drift). For a fixed drift vector  $\mu \in \mathbb{R}^2$ , a family of probability measures  $\{P_z^\mu : z \in S\}$  on  $C([0, \infty), S)$  is said to solve the submartingale problem with drift if the coordinate process  $Z$  satisfies the three conditions of Definition 2.1 in Lakner–Liu–Reed [[9](#)]: it starts from  $z$ , it makes

$$\left\{ f(Z(t)) - \int_0^t \mu \cdot \nabla f(Z(s)) ds - \frac{1}{2} \int_0^t \Delta f(Z(s)) ds : t \geq 0 \right\}$$

a submartingale for every admissible test function  $f \in C_b^2(S)$  that is constant near the vertex and satisfies  $D_i f \geq 0$  on  $\partial S_i$ , and it spends zero occupation time at the vertex.

The following results of Lakner–Liu–Reed [9] will be used.

- (i) If  $\alpha < 2$ , the submartingale problem with drift has a unique solution; this is [9, Theorem 2.3].
- (ii) More precisely, on the canonical space there is a Brownian motion with drift  $X$ , started at the initial state, such that  $Y := Z - X$  and  $(Z, Y)$  solve the extended Skorokhod problem for  $X$ ; this is the pathwise decomposition in [9, Theorem 2.3]. Although  $Y$  need not have bounded variation globally, this decomposition becomes the ordinary one-face Skorokhod representation after stopping in a region that meets only one open face; see lemma 6.41 below.
- (iii) If  $\alpha < 2$ , the transition semigroup has the  $C_0(S)$ -Feller property; in the notation of [9], this is the  $\widehat{C}(S)$ -Feller property of [9, Theorem 2.12].
- (iv) For the solution to the submartingale problem, the whole boundary has zero occupation time; this is [9, Lemma 3.3].
- (v) If  $0 < \xi < \pi$  and  $\alpha \geq 1$ , then there exists  $b \neq 0$  such that

$$b \cdot z < 0 \quad (z \in S \setminus \{0\}), \quad b \cdot v_i \geq 0 \quad (i = 1, 2).$$

This follows from the Lakner–Liu–Reed reflection-cone geometry, in particular the statement that  $\text{cone}\{v_1, v_2\} \cap S = \{0\}$  in the convex  $\alpha \geq 1$  case.

- (vi) If  $0 < \xi < \pi$  and  $\alpha = 1$ , then  $\text{cone}\{v_1, v_2\}$  is a line and  $\text{cone}\{v_1, v_2\} \cap S = \{0\}$ .

We refer to this vector  $b$  as the *geometric separator*. The term *strict nonsemimartingale regime* below refers only to  $1 < \alpha < 2$ ; the boundary case  $\alpha = 1$  is treated separately.

The same angle convention also gives the following strict reflection-cone geometry, which will be used in the one-dimensional and Lyapunov arguments.

**Lemma 2.2** (strict reflection cone contains the wedge). *Assume*

$$0 < \xi < \pi, \quad 1 < \alpha < 2.$$

Then

$$S = \text{cone}\{u_1, u_2\} \subset \text{cone}\{-v_1, -v_2\}.$$

More explicitly, if the boundary generators have polar angles 0 and  $\xi$ , then the ray generated by  $-v_1$  has angle

$$\theta_1 - \frac{\pi}{2} < 0,$$

and the ray generated by  $-v_2$  has angle

$$\xi + \frac{\pi}{2} - \theta_2 > \xi.$$

The counterclockwise angle from  $-v_1$  to  $-v_2$  is

$$\pi + \xi - (\theta_1 + \theta_2) = \pi - (\alpha - 1)\xi,$$

which lies in  $(\xi, \pi)$ . Hence the proper closed cone generated by  $-v_1$  and  $-v_2$  contains the whole wedge sector.

*Proof.* With the convention  $v_i = n_i - (\tan \theta_i)t_i$ , the first face has  $t_1 = e_r(0)$  and  $n_1 = e_\theta(0)$ . Thus

$$v_1 = e_\theta(0) - (\tan \theta_1)e_r(0),$$

so  $v_1$  has angle  $\pi/2 + \theta_1$  and  $-v_1$  has angle  $\theta_1 - \pi/2$ . Since  $\theta_1 < \pi/2$ , this angle is strictly below the first boundary ray.

On the second face,  $t_2 = e_r(\xi)$  and  $n_2 = -e_\theta(\xi)$ , hence

$$v_2 = -e_\theta(\xi) - (\tan \theta_2)e_r(\xi).$$

Therefore  $v_2$  has angle  $\xi - \pi/2 - \theta_2$ , and  $-v_2$  has angle  $\xi + \pi/2 - \theta_2$ . Since  $\theta_2 < \pi/2$ , this angle is strictly above the second boundary ray.

The counterclockwise aperture from  $-v_1$  to  $-v_2$  is

$$\left(\xi + \frac{\pi}{2} - \theta_2\right) - \left(\theta_1 - \frac{\pi}{2}\right) = \pi + \xi - (\theta_1 + \theta_2) = \pi - (\alpha - 1)\xi.$$

Because  $\alpha > 1$ , this aperture is less than  $\pi$ . Because  $\theta_1 + \theta_2 < \pi$ , it is larger than  $\xi$ . Thus  $-v_1$  and  $-v_2$  form a proper convex cone whose angular interval contains  $[0, \xi]$ , which is exactly the wedge sector in the convex case.  $\square$

**Lemma 2.3** (nonempty strict Lyapunov directions). *Assume*

$$0 < \xi < \pi, \quad 1 < \alpha < 2.$$

Let  $K_{\text{str}} := \text{cone}\{-v_1, -v_2\}$ . Then  $\mathfrak{A} \neq \emptyset$ . More precisely, if

$$K_{\text{str}}^\vee := \{a \in \mathbb{R}^2 : a \cdot x \geq 0 \text{ for every } x \in K_{\text{str}}\},$$

then every  $a \in \text{int } K_{\text{str}}^\vee$  belongs to  $\mathfrak{A}$ .

*Proof.* By lemma 2.2,  $K_{\text{str}}$  is a proper two-dimensional closed cone and contains  $S$ . Hence its dual cone has nonempty interior, and every  $a \in \text{int } K_{\text{str}}^\vee$  is strictly positive on  $K_{\text{str}} \setminus \{0\}$ . Since  $u_1, u_2, -v_1, -v_2$  are nonzero elements of  $K_{\text{str}}$ , this gives

$$a \cdot u_j > 0 \quad (j = 1, 2), \quad a \cdot (-v_i) > 0 \quad (i = 1, 2).$$

Thus  $a \cdot v_i < 0$  for  $i = 1, 2$ , and therefore  $a \in \mathfrak{A}$ . The nonemptiness of  $\text{int } K_{\text{str}}^\vee$  gives  $\mathfrak{A} \neq \emptyset$ .  $\square$

We use the dual cone notation

$$S^\vee := \{\vartheta \in \mathbb{R}^2 : \vartheta \cdot z \geq 0 \text{ for all } z \in S\}, \quad S_\circ^\vee := \{\vartheta \in \mathbb{R}^2 : \vartheta \cdot z > 0 \text{ for all } z \in S \setminus \{0\}\}.$$

In the convex regimes  $0 < \xi < \pi$ , we also fix the following Lyapunov cone notation at the outset.

**Definition 2.4** (strict oblique Lyapunov directions). Define

$$\mathfrak{A} := \{a \in S_\circ^\vee : a \cdot v_1 \leq 0, a \cdot v_2 \leq 0\}.$$

### 3. STATIONARY IDENTITIES AND INTERIOR REGULARITY

For a fixed drift  $\mu$ , write

$$\mathcal{L}_\mu = \mu \cdot \nabla + \frac{1}{2} \Delta,$$

and write  $\mathcal{L}$  when the drift is fixed by context.

**Lemma 3.1** (zero stationary mass from zero occupation times). *Assume  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the solution to the submartingale problem with drift  $\mu$ . Then*

$$(3.1) \quad \pi(\{0\}) = 0, \quad \pi(\partial S) = 0.$$

*Consequently, the stationary integral of any bounded open-wedge expression is independent of the bounded Borel values assigned on  $\partial S$ .*

*Proof.* Condition 3 in Definition 2.1 gives zero occupation time at the vertex, and [9, Lemma 3.3] gives zero occupation time on the whole boundary. Under the stationary initial law, the following expectation identities are finite because the integrands are bounded. For every  $t > 0$ ,

$$0 = \mathbb{E}_\pi \int_0^t \mathbf{1}_{\{0\}}(Z(s)) ds = t \pi(\{0\}),$$

The boundary zero-occupation identity gives separately

$$0 = \mathbb{E}_\pi \int_0^t \mathbf{1}_{\partial S}(Z(s)) ds = t \pi(\partial S).$$

The last assertion follows because two bounded Borel extensions of the same open-wedge expression differ only on  $\partial S$ , a  $\pi$ -null set.  $\square$

**Proposition 3.2** (stationary inequality and zero boundary mass). *Assume  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the solution to the submartingale problem with drift  $\mu$ . Then for every admissible test function  $f$ ,*

$$(3.2) \quad \int_S \mathcal{L}f(x) \pi(dx) \leq 0.$$

Moreover,

$$(3.3) \quad \pi(\{0\}) = 0, \quad \pi(\partial S) = 0.$$

*Proof.* The zero-mass statement is [lemma 3.1](#). Fix an admissible test function  $f$ . Definition 2.1 implies that

$$M_f(t) := f(Z(t)) - \int_0^t \mathcal{L}f(Z(s)) ds$$

is a submartingale. Since  $f \in C_b^2(S)$ , both  $f(Z(t))$  and the time integral of the bounded open-wedge generator are integrable after any bounded Borel extension to the boundary. Under a stationary initial law  $\pi$ ,

$$\mathbb{E}_\pi[f(Z(t))] = \mathbb{E}_\pi[f(Z(0))],$$

so

$$\mathbb{E}_\pi \left[ \int_0^t \mathcal{L}f(Z(s)) ds \right] \leq 0.$$

Because  $\mathcal{L}f$  is bounded in the open wedge and  $\pi(\partial S) = 0$ , Fubini's theorem and stationarity give

$$\mathbb{E}_\pi \int_0^t \mathcal{L}f(Z(s)) ds = \int_0^t \int_S \mathcal{L}f d\pi ds = t \int_S \mathcal{L}f d\pi,$$

where any bounded Borel extension of the open-wedge generator may be used on  $\partial S$ . Division by  $t$  yields (3.2).  $\square$

**Corollary 3.3** (stationary equality for neutral boundary tests). *Assume  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Suppose  $f \in C_b^2(S)$  is constant near the vertex and satisfies*

$$D_i f = 0 \quad \text{on } \partial S_i, \quad i = 1, 2.$$

*Then both  $f$  and  $-f$  are admissible test functions, the open-wedge generator  $\mathcal{L}f$  has a bounded Borel extension to  $S$ , and*

$$\int_S \mathcal{L}f d\pi = 0.$$

*The value assigned to  $\mathcal{L}f$  on  $\partial S$  is immaterial.*

*Proof.* The function  $-f$  is also in  $C_b^2(S)$  and is constant near the vertex. The boundary equalities imply  $D_i f \geq 0$  and  $D_i(-f) \geq 0$  on both faces, so both signs are admissible. Since  $f \in C_b^2(S)$ , the open-wedge generator  $\mathcal{L}f$  is bounded and therefore has bounded Borel extensions to  $S$ . By [proposition 3.2](#),  $\pi(\partial S) = 0$ , so the integral is independent of the chosen boundary extension. Applying [proposition 3.2](#) to  $f$  and to  $-f$  gives opposite inequalities for the same finite integral, hence equality.  $\square$

**Corollary 3.4** (boundary-null integration of generator bounds). *Assume  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Let  $f \in C_b^2(S)$ , and let  $G$  be a bounded Borel function on  $S$ . Interpret  $\mathcal{L}f$  as any bounded Borel extension to  $S$  of the open-wedge generator. If*

$$\mathcal{L}f \geq G \quad \text{on } S^\circ,$$

*then*

$$\int_S \mathcal{L}f d\pi \geq \int_S G d\pi.$$

The analogous statement holds for upper bounds and equalities. Consequently, when applying stationary inequalities, generator bounds that are proved in the open wedge and extended continuously to open faces may be integrated without specifying their values at the vertex or on the boundary.

*Proof.* By [proposition 3.2](#),  $\pi(\partial S) = 0$ . Hence the inequality  $\mathcal{L}f \geq G$ , which holds on  $S^\circ$ , holds  $\pi$ -almost surely after arbitrary choices of values on  $\partial S$ . Both functions are integrable because the chosen extension of  $\mathcal{L}f$  is bounded and  $G$  is bounded. Integration gives the claim. Changing the extension of  $\mathcal{L}f$  on  $\partial S$  changes neither side of any stationary integral.  $\square$

**Corollary 3.5** (pathwise integration of interior generator bounds). *Assume  $\alpha < 2$ , and fix an initial state  $z \in S$ . Let  $f \in C_b^2(S)$ , and let  $G$  be a bounded Borel function on  $S$ . Interpret  $\mathcal{L}f$  as any bounded Borel extension to  $S$  of the open-wedge generator. If*

$$\mathcal{L}f \geq G \quad \text{on } S^\circ,$$

then, for every  $T < \infty$ ,

$$\mathbb{E}_z \int_0^T \mathcal{L}f(Z_s) ds \geq \mathbb{E}_z \int_0^T G(Z_s) ds.$$

The analogous statement holds for upper bounds and equalities. Thus open-wedge generator inequalities for bounded localized tests may be used inside time integrals under any initial state, without specifying special values at the boundary or vertex.

*Proof.* The solution has zero occupation time on the whole boundary: Definition 2.1 gives zero occupation time at the vertex, and [\[9, Lemma 3.3\]](#) gives zero occupation time on  $\partial S$ . Hence

$$\int_0^T \mathbf{1}_{\partial S}(Z_s) ds = 0 \quad \text{almost surely}$$

for every initial state. The inequality  $\mathcal{L}f \geq G$  holds whenever  $Z_s \in S^\circ$ , and the set of times at which  $Z_s \in \partial S$  has Lebesgue measure zero. Since both functions are bounded, integration in time and expectation give the claim. Changing the boundary extension of  $\mathcal{L}f$  changes the time integral only on a null set of times.  $\square$

**Proposition 3.6** (scaling invariance along drift rays). *Assume  $\alpha < 2$ , so that the Definition 2.1 Markov family with drift is available. For  $r > 0$ , let*

$$T_r : S \rightarrow S, \quad T_r x = rx.$$

If  $\pi$  is a stationary distribution for drift  $\mu$ , then  $(T_r)_\# \pi$  is a stationary distribution for drift  $\mu/r$ . Equivalently, for every  $c > 0$ , stationary existence for drift  $\mu$  is equivalent to stationary existence for drift  $c\mu$ .

Moreover, if  $\pi$  has interior density  $p_\pi$ , then  $(T_r)_\# \pi$  has density

$$p_r(y) = r^{-2} p_\pi(y/r), \quad y \in S^\circ.$$

Consequently, on every nonzero drift ray, existence and nonexistence are scale-invariant properties. In particular, any conditional analysis along a strict critical boundary ray may normalize the length of the drift vector.

*Proof.* Let  $Z$  solve the submartingale problem with drift  $\mu$  and initial point  $z$ . Define

$$Y(t) := rZ(t/r^2), \quad t \geq 0.$$

For an admissible test function  $g$ , set  $f(x) = g(rx)$ . Then  $f$  is again admissible: it is constant near the vertex whenever  $g$  is, and on  $\partial S_i$ ,

$$D_i f(x) = r D_i g(rx) \geq 0.$$

A direct calculation gives

$$\left(\mu \cdot \nabla + \frac{1}{2}\Delta\right) f(x) = r^2 \left(\frac{\mu}{r} \cdot \nabla + \frac{1}{2}\Delta\right) g(rx).$$

Therefore the submartingale property for  $f(Z(\cdot))$  is exactly the submartingale property for  $g(Y(\cdot))$  with drift  $\mu/r$ . The zero-occupation condition at the vertex is unchanged by deterministic scaling and deterministic time change; the same is true for the zero boundary-occupation property quoted from Lakner–Liu–Reed. Hence  $Y$  solves the Lakner–Liu–Reed submartingale problem with drift  $\mu/r$  and initial point  $rz$ . By uniqueness for  $\alpha < 2$ , this is the canonical solution with drift  $\mu/r$ .

If  $Z(0) \sim \pi$  and  $\pi$  is stationary for drift  $\mu$ , then  $Z(t/r^2) \sim \pi$  for every  $t$ . Thus  $Y(t) = rZ(t/r^2) \sim (T_r)_\# \pi$  for every  $t$ , so  $(T_r)_\# \pi$  is stationary for drift  $\mu/r$ . Replacing  $r$  by  $1/c$  gives the equivalence between  $\mu$  and  $c\mu$ . The density transformation is the usual Jacobian formula for the planar dilation  $T_r$ . □

**Proposition 3.7** (interior adjoint equation and smooth density). *Assume  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the solution to the submartingale problem with drift  $\mu$ . Then*

$$\int_S \mathcal{L}\varphi d\pi = 0$$

for every  $\varphi \in C_c^2(S^\circ)$ . Consequently, the restriction of  $\pi$  to the open wedge  $S^\circ$  is absolutely continuous with respect to Lebesgue measure. Its density  $p_\pi$  is  $C^\infty$  in  $S^\circ$  and satisfies the adjoint stationary equation

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0 \quad \text{in } S^\circ$$

in the classical sense.

*Proof.* If  $\varphi \in C_c^2(S^\circ)$ , then the compact support of  $\varphi$  has positive distance from the vertex and from  $\partial S$ . Hence  $\varphi$  and  $-\varphi$  are both bounded admissible test functions: they are identically zero in a full neighborhood of the vertex and in a neighborhood of each face, so the oblique boundary inequalities are automatic. In particular, no boundary value of the open-wedge generator enters this argument. Applying [proposition 3.2](#) to both signs gives

$$\int_S \mathcal{L}\varphi d\pi \leq 0, \quad - \int_S \mathcal{L}\varphi d\pi \leq 0,$$

and therefore

$$\int_S \mathcal{L}\varphi d\pi = 0.$$

This means that the order-zero distribution

$$T := \pi|_{S^\circ}$$

satisfies  $\mathcal{L}^*T = 0$  in  $\mathcal{D}'(S^\circ)$ . To make the regularity step explicit, define

$$Q := e^{-\mu \cdot x} T, \quad \langle Q, \psi \rangle := \langle T, e^{-\mu \cdot x} \psi \rangle.$$

For  $\psi \in C_c^\infty(S^\circ)$ , put  $\varphi = e^{-\mu \cdot x} \psi$ . A direct calculation gives

$$\mathcal{L}\varphi = \left(\mu \cdot \nabla + \frac{1}{2}\Delta\right) (e^{-\mu \cdot x} \psi) = \frac{1}{2} e^{-\mu \cdot x} (\Delta\psi - |\mu|^2 \psi).$$

Consequently,

$$\langle (\Delta - |\mu|^2)Q, \psi \rangle = 2\langle T, \mathcal{L}\varphi \rangle = 0.$$

Thus  $Q$  is a distributional solution of  $(\Delta - |\mu|^2)Q = 0$ . The operator has principal symbol  $-|\xi|^2$ , nonzero for  $\xi \neq 0$ , and is therefore elliptic. Constant-coefficient elliptic hypoellipticity gives  $\text{sing supp } Q \subset \text{sing supp}((\Delta - |\mu|^2)Q) = \emptyset$ ; see [\[7\]](#). Hence  $Q$  is represented by a  $C^\infty$  function  $q$

on  $S^\circ$ . It follows that

$$T = e^{\mu \cdot x} q(x) dx,$$

so  $\pi|_{S^\circ}$  has the smooth density  $p_\pi = e^{\mu \cdot x} q$ . Returning to  $\mathcal{L}^*T = 0$  gives

$$\mathcal{L}^*p_\pi = \frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0$$

classically in  $S^\circ$ , as claimed.  $\square$

**Corollary 3.8** (strict positivity of the interior stationary density). *Assume  $\alpha < 2$ . Let  $\pi$  be a stationary distribution for the solution to the submartingale problem with drift  $\mu$ , and let  $p_\pi$  be the interior density from [proposition 3.7](#). Then*

$$p_\pi(x) > 0 \quad \text{for every } x \in S^\circ.$$

*Proof.* By [proposition 3.2](#),  $\pi(\{0\}) = 0$  and  $\pi(\partial S) = 0$ . Hence

$$\int_{S^\circ} p_\pi(x) dx = \pi(S^\circ) = 1.$$

In particular,  $p_\pi$  is not identically zero. Because  $\pi$  is a nonnegative measure, its smooth density is nonnegative almost everywhere; continuity therefore gives  $p_\pi \geq 0$  at every point of  $S^\circ$ . Since  $p_\pi$  satisfies

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0 \quad \text{in } S^\circ,$$

the strong maximum principle [[6](#), Chapter 3] applies on every relatively compact connected subdomain of  $S^\circ$ . Suppose that  $p_\pi(x_0) = 0$  for some  $x_0 \in S^\circ$ , and define

$$Z_0 := \{x \in S^\circ : p_\pi(x) = 0\}.$$

The set  $Z_0$  is closed in  $S^\circ$  by continuity. For every  $x \in Z_0$ , choose  $r_x > 0$  with  $\overline{B}_{2r_x}(x) \Subset S^\circ$ . On  $B_{2r_x}(x)$ , the nonnegative solution  $p_\pi$  attains its minimum zero at the interior point  $x$ . The strong maximum principle gives

$$p_\pi \equiv 0 \quad \text{on } B_{2r_x}(x),$$

so  $B_{2r_x}(x) \subset Z_0$ . Thus  $Z_0$  is also relatively open. Because  $S^\circ$  is connected and  $Z_0 \neq \emptyset$ , one would have  $Z_0 = S^\circ$ , hence  $p_\pi \equiv 0$ , contradicting  $\pi(S^\circ) = 1$ . Therefore  $p_\pi > 0$  throughout  $S^\circ$ .  $\square$

**Proposition 3.9** (ground-state transform and analyticity of the interior density). *Assume  $\alpha < 2$ . Let  $\pi$  be a stationary distribution for the solution to the submartingale problem with drift  $\mu$ , and let  $p_\pi$  be the interior density from [proposition 3.7](#). Define*

$$q_\pi(x) := e^{-\mu \cdot x} p_\pi(x), \quad x \in S^\circ.$$

*Then  $q_\pi > 0$  in  $S^\circ$ , and  $q_\pi$  satisfies*

$$\Delta q_\pi = |\mu|^2 q_\pi \quad \text{in } S^\circ.$$

*Consequently both  $q_\pi$  and  $p_\pi$  are real analytic in  $S^\circ$ . In particular, when  $\mu = 0$ , the interior density  $p_\pi$  is a positive harmonic function in the open wedge.*

*Proof.* The positivity of  $q_\pi$  follows from [corollary 3.8](#) and the positivity of the exponential factor. Write

$$p_\pi(x) = e^{\mu \cdot x} q_\pi(x).$$

Then

$$\nabla p_\pi = e^{\mu \cdot x} (\nabla q_\pi + \mu q_\pi),$$

and

$$\Delta p_\pi = e^{\mu \cdot x} (\Delta q_\pi + 2\mu \cdot \nabla q_\pi + |\mu|^2 q_\pi).$$

Substituting these identities into the interior adjoint equation

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0$$

gives

$$\frac{1}{2}\Delta q_\pi - \frac{1}{2}|\mu|^2 q_\pi = 0,$$

which is exactly  $\Delta q_\pi = |\mu|^2 q_\pi$ . Analytic elliptic regularity for operators with analytic coefficients [10] yields real analyticity of  $q_\pi$ , and hence of  $p_\pi = e^{\mu \cdot x} q_\pi$ , in  $S^\circ$ .  $\square$

*Remark 3.10* (interior elliptic reductions impose no boundary condition). The conclusions of [propositions 3.7](#) and [3.9](#) and [corollary 3.8](#) are purely interior statements. They use compactly supported test functions in  $S^\circ$  and the fact that any stationary distribution gives no mass to  $\partial S$ . No Neumann, oblique, or flux boundary condition for the density is asserted or used in the arguments below. This distinction is necessary in the Lakner–Liu–Reed nonsemimartingale regime, where the process is not represented by boundary local times in the way a semimartingale reflected Brownian motion is.

#### 4. AUXILIARY GEOMETRIC AND TRUNCATION LEMMAS

The qualitative classification is taken from [8]. Two elementary ingredients from its proof are used repeatedly below: the geometry of the borderline reflection line and bounded concave truncations of norm-like Lyapunov gauges.

**Lemma 4.1** (borderline reflection geometry). *Assume*

$$0 < \xi < \pi, \quad \alpha = 1.$$

Then

$$\mathcal{L} := \text{cone}\{v_1, v_2\}$$

is a line through the origin,  $\mathcal{L} \cap S = \{0\}$ , and there is a unique unit normal  $n_{\mathcal{L}}$  such that

$$n_{\mathcal{L}} \cdot z > 0 \quad (z \in S \setminus \{0\}), \quad n_{\mathcal{L}} \cdot v_i = 0, \quad i = 1, 2.$$

Moreover, the reflection vectors are opposite generators of the same line: there is  $\rho > 0$  such that

$$v_2 = -\rho v_1.$$

The stationary-existence condition in this case is

$$n_{\mathcal{L}} \cdot \mu < 0.$$

*Proof.* The line property and the identity  $\mathcal{L} \cap S = \{0\}$  are the borderline part of the reflection-cone geometry in [9]; the stationary classification is proved in [8]. Since the convex wedge has opening angle strictly smaller than  $\pi$ , its nonzero points lie in a unique open half-plane bounded by  $\mathcal{L}$ . The inward-pointing unit normal to that half-plane is  $n_{\mathcal{L}}$ , and it is orthogonal to both reflection directions. Because the nonnegative cone generated by  $v_1, v_2$  is the whole line  $\mathcal{L}$ , the two nonzero generators point in opposite directions, which gives  $v_2 = -\rho v_1$  for some  $\rho > 0$ .  $\square$

**Lemma 4.2** (bounded concave truncation profiles). *There exists a sequence  $\{\phi_n\}_{n \geq 1}$  of functions in  $C^\infty([0, \infty))$  with the following properties. For each  $n$ ,*

$$(4.1) \quad \phi_n(0) = 0, \quad 0 \leq \phi_n(s) \leq s, \quad 0 \leq \phi'_n(s) \leq 1, \quad \phi''_n(s) \leq 0,$$

$$(4.2) \quad \phi_n(s) = s \quad (0 \leq s \leq n), \quad \phi_n(s) = \text{constant} \quad (s \geq n + 2),$$

and, for every fixed  $s \geq 0$ ,

$$(4.3) \quad \phi_n(s) \rightarrow s, \quad \phi'_n(s) \rightarrow 1.$$

*Proof.* Choose a function  $\rho \in C^\infty(\mathbb{R})$  such that  $0 \leq \rho \leq 1$ ,  $\rho' \leq 0$ ,  $\rho(u) = 1$  for  $u \leq 0$ , and  $\rho(u) = 0$  for  $u \geq 2$ . Define

$$\phi_n(s) := \int_0^s \rho(r-n) dr, \quad s \geq 0.$$

Then  $\phi_n \in C^\infty([0, \infty))$ ,  $\phi_n' = \rho(\cdot - n)$ , and  $\phi_n'' = \rho'(\cdot - n) \leq 0$ . The plateau identities follow from the two plateau regions of  $\rho$ . Since  $0 \leq \phi_n' \leq 1$  and  $\phi_n(0) = 0$ , one has  $0 \leq \phi_n(s) \leq s$ . Finally, for each fixed  $s$ , once  $n > s$ , the point  $s$  lies in the identity region, so  $\phi_n(s) = s$  and  $\phi_n'(s) = 1$ .  $\square$

**Lemma 4.3** (bounded concave truncations for negative-boundary Lyapunov gauges). *Assume  $\alpha < 2$ . Let  $V : S \rightarrow [0, \infty)$  be continuous, facewise  $C^2$  and locally  $C^2$ -extendable away from the vertex, constant in a neighborhood of the vertex, and norm-like. Suppose*

$$D_i V \leq 0 \quad \text{on } \partial S_i, \quad i = 1, 2,$$

*through open-face limits. Let  $\phi_n$  be one of the bounded concave truncation profiles from lemma 4.2, and set  $V_n = \phi_n(V)$ . Then  $V_n \in C_b^2(S)$ ,  $V_n$  is constant near the vertex, and*

$$D_i V_n \leq 0 \quad \text{on } \partial S_i.$$

*Thus  $-V_n$  is an admissible test function, and*

$$V_n(Z_t) - V_n(Z_0) - \int_0^t \mathcal{L}V_n(Z_s) ds$$

*is a supermartingale. If, in addition, for some compact  $K \subset S$  and constants  $\delta, C > 0$ ,*

$$\mathcal{L}V \leq -\delta V + C \mathbf{1}_K \quad \text{on } S^\circ,$$

*then, on  $S^\circ$ ,*

$$\mathcal{L}V_n \leq -\delta \phi_n'(V)V + C \mathbf{1}_K.$$

*The displayed generator inequalities hold in the open wedge; their pathwise time-integral consequences for the bounded truncations follow from corollary 3.5, and their stationary integral consequences follow from corollary 3.4.*

*Proof.* For fixed  $n$ , the function  $V_n$  is bounded. Since  $V$  is constant near the vertex, so is  $V_n$ . The possible nonconstant region of  $V_n$  is contained in

$$\{V < n + 2\} \subset \{V \leq n + 2\},$$

and this latter set is compact because  $V$  is norm-like. Choose a vertex neighborhood on which  $V$  is constant. Outside that neighborhood, the compact set  $\{V \leq n + 2\}$  is covered by finitely many neighborhoods on which the local  $C^2$ -extensions of  $V$  are available, and the first and second derivatives of  $V$  are bounded on this compact set. The chain rule and the fixed bounded derivatives of  $\phi_n$  therefore give a bounded  $C^2$  extension of  $V_n$  on the nonconstant region; the two constant plateaus complete the extension near the vertex and outside  $\{V < n + 2\}$ . Hence  $V_n \in C_b^2(S)$ . No uniform  $C_b^2$ -bound in  $n$  is asserted or needed. The boundary sign follows from

$$D_i V_n = \phi_n'(V) D_i V \leq 0.$$

Therefore  $-V_n$  is admissible. Applying Definition 2.1 to  $-V_n$  gives that

$$-V_n(Z_t) + V_n(Z_0) + \int_0^t \mathcal{L}V_n(Z_s) ds$$

is a submartingale, equivalently the displayed process with sign reversed is a supermartingale.

Finally, the chain rule in the open wedge gives

$$\mathcal{L}V_n = \phi_n'(V) \mathcal{L}V + \frac{1}{2} \phi_n''(V) |\nabla V|^2 \leq \phi_n'(V) \mathcal{L}V,$$

because  $\phi_n'' \leq 0$ . Combining this with the drift inequality for  $V$  and  $0 \leq \phi_n' \leq 1$  yields

$$\mathcal{L}V_n \leq -\delta \phi_n'(V)V + C\mathbf{1}_K.$$

The asserted integration conventions follow from [corollaries 3.4](#) and [3.5](#).  $\square$

## 5. QUANTITATIVE ESTIMATES AND EXPONENTIAL-MOMENT LOWER BOUNDS

This section develops quantitative estimates that hold for every stationary law in the regimes of [theorem 1.1](#). It begins with the convex geometry of exponential-moment domains, then establishes an explicit inner moment region together with directional, radial, angular-sector, compact-set, large-deviation, and Laplace upper bounds. The normal marginal at  $\alpha = 1$  is identified exactly. The full moment domain and the matching logarithmic lower bounds are obtained in [section 6](#).

Define the exponential-moment domain of a stationary distribution  $\pi$  by

$$(5.1) \quad \mathcal{D}_{\text{exp}}(\pi) := \left\{ \vartheta \in \mathbb{R}^2 : \int_S e^{\vartheta \cdot z} \pi(dz) < \infty \right\}.$$

**Proposition 5.1** (convexity and dual-cone monotonicity of the exponential-moment domain). *Let  $\pi$  be any probability measure on  $S$ , and define  $\mathcal{D}_{\text{exp}}(\pi)$  by (5.1). Then  $\mathcal{D}_{\text{exp}}(\pi)$  is convex. Moreover, it is downward closed with respect to the dual-cone order: if*

$$(5.2) \quad \theta \in \mathcal{D}_{\text{exp}}(\pi), \quad \theta - \eta \in S^\vee,$$

then

$$(5.3) \quad \eta \in \mathcal{D}_{\text{exp}}(\pi).$$

Consequently, for any subset  $D \subset \mathcal{D}_{\text{exp}}(\pi)$ , the dual-cone lower hull

$$(5.4) \quad D - S^\vee := \{ \eta \in \mathbb{R}^2 : \exists \theta \in D \text{ with } \theta - \eta \in S^\vee \}$$

is also contained in  $\mathcal{D}_{\text{exp}}(\pi)$ .

*Proof.* Convexity follows from Hölder's inequality. If  $\theta_1, \theta_2 \in \mathcal{D}_{\text{exp}}(\pi)$  and  $0 < t < 1$ , then

$$\int_S e^{(t\theta_1 + (1-t)\theta_2) \cdot z} \pi(dz) = \int_S (e^{\theta_1 \cdot z})^t (e^{\theta_2 \cdot z})^{1-t} \pi(dz) < \infty.$$

For the order property, assume  $\theta - \eta \in S^\vee$ . Since  $z \in S$ ,

$$(\theta - \eta) \cdot z \geq 0,$$

so  $\eta \cdot z \leq \theta \cdot z$  for all  $z \in S$ . Hence

$$e^{\eta \cdot z} \leq e^{\theta \cdot z},$$

and integrability of  $e^{\theta \cdot z}$  implies integrability of  $e^{\eta \cdot z}$ . The final assertion is the same statement applied pointwise to some  $\theta \in D$ .  $\square$

**Lemma 5.2** (positive-ray extents are unchanged by closure). *Let  $A \subset \mathbb{R}^2$  be nonempty and downward closed with respect to the dual-cone order:*

$$\theta \in A, \quad \theta - \eta \in S^\vee \implies \eta \in A.$$

Let  $c \in S_\circ^\vee$ . Then

$$(5.5) \quad \sup\{\lambda \geq 0 : \lambda c \in A\} = \sup\{\lambda \geq 0 : \lambda c \in \overline{A}\},$$

with equality in the extended interval  $[0, \infty]$ , where the supremum of an empty subset of  $[0, \infty)$  is understood to be zero.

*Proof.* The left-hand side is at most the right-hand side. If  $\lambda = 0$ , the desired comparison is automatic under the stated empty-set convention. Fix therefore  $\lambda > 0$  with  $\lambda c \in \overline{A}$ , and let

$0 \leq q < \lambda$ . Choose  $\theta_n \in A$  with  $\theta_n \rightarrow \lambda c$ . Since  $(\lambda - q)c \in S_\circ^\vee$  and  $S_\circ^\vee$  is open, one has

$$\theta_n - qc \in S^\vee$$

for all sufficiently large  $n$ . Downward closure of  $A$ , applied with  $\theta = \theta_n$  and  $\eta = qc$ , then gives  $qc \in A$ . Thus every  $q < \lambda$  is bounded above by the left-hand ray extent. Letting  $q \uparrow \lambda$ , and then taking the supremum over all  $\lambda c \in \bar{A}$ , proves (5.5). If the right-hand ray extent is  $+\infty$ , then for every  $M > 0$  there exists a finite  $\lambda > M$  with  $\lambda c \in \bar{A}$ . Applying the preceding argument with  $q = M$  gives  $Mc \in A$ . Since this holds for every  $M > 0$ , the left-hand ray extent is also  $+\infty$ .  $\square$

*Remark 5.3.* The directional decay estimates below use only an inner approximation to  $\mathcal{D}_{\text{exp}}(\pi)$ . Once one exponential moment vector is available, every vector below it in the dual-cone order is automatically available as well. Thus the directional tail bound is governed by how far one can move from an available moment vector in the direction  $c$  while remaining in the dual cone.

Recall the dual cone notation  $S^\vee$  and  $S_\circ^\vee$  from the setup, and define the inner exponential-moment region

$$(5.6) \quad \mathcal{E}(\mu) := \left\{ \vartheta \in S_\circ^\vee : \vartheta \cdot v_i \leq 0 \ (i = 1, 2), \ \vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2 < 0 \right\}.$$

Throughout this section we also write

$$(5.7) \quad Q_\mu(\vartheta) := \vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2, \quad \partial_Q \mathcal{E}(\mu) := \overline{\mathcal{E}(\mu)} \cap \{Q_\mu = 0\}.$$

Thus  $\partial_Q \mathcal{E}(\mu)$  denotes the quadratic boundary of the inner moment region, including endpoint points produced by closing the dual and oblique inequalities. Moment estimates below always use vectors in the open set  $\mathcal{E}(\mu)$ ; closed-boundary notation is used only for variational maximization.

**Proposition 5.4** (geometry of the inner exponential-moment region). *Assume  $0 < \xi < \pi$ . For each drift  $\mu$ , the set  $\mathcal{E}(\mu)$  is convex. Moreover,*

$$\mathcal{E}(\mu) \neq \emptyset \iff \text{there exists } a \in \mathfrak{A} \text{ with } a \cdot \mu < 0.$$

More precisely, if  $a \in \mathfrak{A}$  and  $a \cdot \mu < 0$ , then

$$\beta a \in \mathcal{E}(\mu) \quad \text{for all} \quad 0 < \beta < -\frac{2a \cdot \mu}{|a|^2}.$$

Conversely, every  $\vartheta \in \mathcal{E}(\mu)$  belongs to  $\mathfrak{A}$  and satisfies  $\vartheta \cdot \mu < 0$ .

*Proof.* Convexity follows because  $S_\circ^\vee$  is convex, the boundary conditions  $\vartheta \cdot v_i \leq 0$  are half-space constraints, and

$$\vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2 < 0 \iff |\vartheta + \mu|^2 < |\mu|^2,$$

which is an open ball condition.

If  $a \in \mathfrak{A}$  and  $a \cdot \mu < 0$ , then for  $\vartheta = \beta a$ ,

$$\vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2 = \beta a \cdot \mu + \frac{1}{2}\beta^2|a|^2 < 0$$

precisely when  $0 < \beta < -2a \cdot \mu / |a|^2$ . Thus  $\beta a \in \mathcal{E}(\mu)$ . Conversely, if  $\vartheta \in \mathcal{E}(\mu)$ , then  $\vartheta \cdot z > 0$  on  $S \setminus \{0\}$  and  $\vartheta \cdot v_i \leq 0$ , hence  $\vartheta \in \mathfrak{A}$ , and the strict quadratic inequality implies  $\vartheta \cdot \mu < 0$ .  $\square$

**Lemma 5.5** (closure of the inner moment region). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ . Then*

$$\overline{\mathcal{E}(\mu)} = \left\{ \vartheta \in S^\vee : \vartheta \cdot v_i \leq 0 \ (i = 1, 2), \ Q_\mu(\vartheta) \leq 0 \right\}.$$

Consequently  $\overline{\mathcal{E}(\mu)}$  is a nonempty compact convex set,  $\partial_Q \mathcal{E}(\mu)$  is nonempty and compact, and every point of  $\partial_Q \mathcal{E}(\mu)$  is the limit of points of  $\mathcal{E}(\mu)$ .

*Proof.* The inclusion from left to right is immediate from the definition of  $\mathcal{E}(\mu)$ . Conversely, suppose that  $\vartheta \in S^\vee$ ,  $\vartheta \cdot v_i \leq 0$ , and  $Q_\mu(\vartheta) \leq 0$ . Choose  $\vartheta_0 \in \mathcal{E}(\mu)$  and set

$$\vartheta_\delta := (1 - \delta)\vartheta + \delta\vartheta_0, \quad 0 < \delta < 1.$$

For every  $z \in S \setminus \{0\}$ ,

$$\vartheta_\delta \cdot z \geq \delta \vartheta_0 \cdot z > 0,$$

hence  $\vartheta_\delta \in S_\circ^\vee$ . The oblique inequalities are preserved by convexity, and convexity of  $Q_\mu$  gives

$$Q_\mu(\vartheta_\delta) \leq (1 - \delta)Q_\mu(\vartheta) + \delta Q_\mu(\vartheta_0) < 0.$$

Thus  $\vartheta_\delta \in \mathcal{E}(\mu)$ , and  $\vartheta_\delta \rightarrow \vartheta$ . This proves the closure identity and the density assertion.

The closed set above is contained in the ball  $\{\vartheta : |\vartheta + \mu| \leq |\mu|\}$ , so it is compact; convexity follows from the convexity of the closed dual cone, the oblique half-spaces, and the sublevel set  $\{Q_\mu \leq 0\}$ . Finally, if  $\vartheta_0 \in \mathcal{E}(\mu)$ , then

$$t_* := -\frac{2\vartheta_0 \cdot \mu}{|\vartheta_0|^2} > 1, \quad Q_\mu(t_*\vartheta_0) = 0,$$

and  $t\vartheta_0 \in \mathcal{E}(\mu)$  for  $1 < t < t_*$ . Hence  $t_*\vartheta_0 \in \partial_Q \mathcal{E}(\mu)$ , proving that the quadratic boundary is nonempty; compactness follows because it is closed in  $\overline{\mathcal{E}(\mu)}$ .  $\square$

**Proposition 5.6** (inner exponential-moment region). *Assume  $\alpha < 2$  and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Then*

$$\mathcal{E}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi).$$

More precisely, for every  $\vartheta \in \mathcal{E}(\mu)$ ,

$$\int_S e^{\vartheta \cdot z} \pi(dz) < \infty.$$

With

$$(5.8) \quad m(\vartheta) := \min_{u \in S \cap S^1} \vartheta \cdot u > 0,$$

there exists  $C(\vartheta) < \infty$  such that

$$(5.9) \quad \pi(\{|z| \geq r\}) \leq C(\vartheta)e^{-m(\vartheta)r}, \quad r \geq 0.$$

Hence

$$(5.10) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq -m(\vartheta).$$

*Proof.* Fix  $\vartheta \in \mathcal{E}(\mu)$ . Choose  $\eta \in C^\infty([0, \infty))$  such that

$$(5.11) \quad \eta(s) = 0 \quad (0 \leq s \leq \tfrac{1}{2}), \quad \eta(s) = 1 \quad (s \geq 1), \quad 0 \leq \eta' \leq C_\eta.$$

Define

$$(5.12) \quad W_\vartheta(z) := \eta(\vartheta \cdot z)e^{\vartheta \cdot z}.$$

Because  $\vartheta \in S_\circ^\vee$ , the function  $W_\vartheta$  vanishes in a neighborhood of the vertex. On the boundary,

$$(5.13) \quad D_i W_\vartheta(z) = (\eta'(\vartheta \cdot z) + \eta(\vartheta \cdot z))e^{\vartheta \cdot z}(\vartheta \cdot v_i) \leq 0.$$

Set

$$(5.14) \quad \kappa(\vartheta) := \vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2 < 0.$$

On the set  $\{\vartheta \cdot z \geq 1\}$  one has  $W_\vartheta = e^{\vartheta \cdot z}$  and therefore

$$(5.15) \quad \mathcal{L}W_\vartheta = \kappa(\vartheta)W_\vartheta.$$

Let

$$(5.16) \quad K_\vartheta := \{z \in S : \vartheta \cdot z \leq 1\}.$$

The estimate below is used on the open wedge; boundary values are immaterial in the subsequent integration, since stationary laws charge no boundary mass. The function

$$z \mapsto \mathcal{L}W_\vartheta(z) - \kappa(\vartheta)W_\vartheta(z)$$

has a continuous extension to the compact set  $K_\vartheta$ : near the vertex it vanishes because  $W_\vartheta$  is identically zero, and away from the vertex all derivatives extend continuously to the open faces. Define

$$C_\vartheta := \max \left\{ 0, \max_{z \in K_\vartheta} (\mathcal{L}W_\vartheta(z) - \kappa(\vartheta)W_\vartheta(z)) \right\} < \infty,$$

where the continuous facewise extension is used in the maximum. On  $S \setminus K_\vartheta$  the difference is zero by (5.15); on  $K_\vartheta$  it is at most  $C_\vartheta$ . Hence

$$(5.17) \quad \mathcal{L}W_\vartheta(z) \leq C_\vartheta \mathbf{1}_{K_\vartheta}(z) + \kappa(\vartheta)W_\vartheta(z) \quad (z \in S^\circ).$$

Now choose increasing concave truncations  $\chi_n \in C^\infty([0, \infty))$  with

$$(5.18) \quad \chi_n(s) = s \quad (s \leq n), \quad \chi_n(s) = n + 1 \quad (s \geq n + 2), \quad 0 \leq \chi'_n \leq 1, \quad \chi''_n \leq 0,$$

and define

$$(5.19) \quad W_{\vartheta,n} := \chi_n(W_\vartheta).$$

For each fixed  $n$ , the nonconstant region of  $W_{\vartheta,n}$  is contained in the sublevel set  $\{z \in S : W_\vartheta(z) \leq n + 2\}$ , which is compact because  $\vartheta \in S^\circ_\circ$  and the cut-off is fixed. Moreover,  $W_{\vartheta,n}$  is constant in a neighborhood of the vertex. Hence the compact-localization argument used in lemma 4.3 gives  $W_{\vartheta,n} \in C_b^2(S)$ . The boundary derivatives satisfy

$$(5.20) \quad D_i W_{\vartheta,n} = \chi'_n(W_\vartheta) D_i W_\vartheta \leq 0.$$

Hence  $-W_{\vartheta,n}$  is admissible. By proposition 3.2,

$$(5.21) \quad \int_S \mathcal{L}W_{\vartheta,n} d\pi \geq 0.$$

The chain rule and  $\chi''_n \leq 0$  yield

$$(5.22) \quad \mathcal{L}W_{\vartheta,n} = \chi'_n(W_\vartheta) \mathcal{L}W_\vartheta + \frac{1}{2} \chi''_n(W_\vartheta) |\nabla W_\vartheta|^2 \leq \chi'_n(W_\vartheta) \mathcal{L}W_\vartheta.$$

Using the preceding generator estimate,  $0 \leq \chi'_n \leq 1$ , and the zero-boundary-mass conclusion in proposition 3.2,

$$(5.23) \quad \mathcal{L}W_{\vartheta,n} \leq C_\vartheta \mathbf{1}_{K_\vartheta} + \kappa(\vartheta) \chi'_n(W_\vartheta) W_\vartheta \quad \pi\text{-almost everywhere.}$$

Integrating and combining with (5.21) gives

$$(5.24) \quad 0 \leq C_\vartheta \pi(K_\vartheta) + \kappa(\vartheta) \int_S \chi'_n(W_\vartheta) W_\vartheta d\pi.$$

Since  $\kappa(\vartheta) < 0$ ,

$$(5.25) \quad \int_S \chi'_n(W_\vartheta) W_\vartheta d\pi \leq \frac{C_\vartheta \pi(K_\vartheta)}{-\kappa(\vartheta)}.$$

For fixed  $z$ ,  $\chi'_n(W_\vartheta(z)) W_\vartheta(z) \rightarrow W_\vartheta(z)$ ; hence Fatou's lemma yields

$$(5.26) \quad \int_S W_\vartheta(z) \pi(dz) < \infty.$$

Since  $W_\vartheta = e^{\vartheta \cdot z}$  whenever  $\vartheta \cdot z \geq 1$ , and since  $0 \leq \vartheta \cdot z < 1$  on the remaining part of  $S$  contributes at most  $e$  times the finite mass of  $\pi$ , this proves the asserted exponential-moment integrability.

Finally,  $\vartheta \cdot z \geq m(\vartheta)|z|$  on  $S$ , so Markov's inequality gives

$$(5.27) \quad \pi(\{|z| \geq r\}) \leq e^{-m(\vartheta)r} \int_S e^{\vartheta \cdot z} \pi(dz),$$

which is (5.9). Then (5.10) follows.  $\square$

**5.1. Finite-exponential Lyapunov extensions of the moment region.** A single exponential is admissible only when its gradient has the correct sign on both faces. A sum of two exponentials

can have the correct boundary sign asymptotically even when neither exponent satisfies both oblique inequalities. The following compact correction lemma converts that asymptotic sign into a globally admissible Lyapunov gauge.

**Lemma 5.7** (compact correction of asymptotic boundary signs). *Assume  $0 < \xi < \pi$ . Let  $H : S \rightarrow (0, \infty)$  be smooth in the open wedge, facewise  $C^2$ , locally  $C^2$ -extendable away from the vertex, and norm-like. Suppose that there are  $R < \infty$  and  $\delta > 0$  such that*

$$(5.28) \quad \mathcal{L}_\mu H \leq -\delta H \quad \text{on } S^\circ \cap \{|z| \geq R\},$$

and

$$(5.29) \quad D_i H(ru_i) \leq 0, \quad r \geq R, \quad i = 1, 2.$$

*Then there is a norm-like function  $V : S \rightarrow [1, \infty)$ , constant near the vertex, facewise  $C^2$ , and locally  $C^2$ -extendable away from the vertex, such that*

$$(5.30) \quad D_i V \leq 0 \quad \text{on } \partial S_i, \quad i = 1, 2,$$

and, for some compact  $K \subset S$  and constants  $\delta_0, C_0 > 0$ ,

$$(5.31) \quad \mathcal{L}_\mu V \leq -\delta_0 V + C_0 \mathbf{1}_K \quad \text{on } S^\circ.$$

Moreover,  $V - H$  is constant outside a compact set.

*Proof.* Choose  $\chi \in C^\infty([0, \infty))$  that vanishes on a neighborhood of zero and equals one outside a compact interval. For a constant  $c_0 > 0$ , replace  $H$  by

$$\tilde{H}(z) := \chi(|z|^2)H(z) + \{1 - \chi(|z|^2)\}c_0.$$

Because the cutoff is a smooth function of  $|z|^2$ , the modified function is  $C^2$  through the vertex, is constant there, and agrees with  $H$  outside a compact set. This changes the generator and the two boundary derivatives only on a compact set.

We next correct the boundary signs on compact subintervals of the open faces. Because  $\tilde{H}$  is constant near the vertex and agrees with  $H$  for large  $|z|$ , the continuous functions

$$g_i(r) := [D_i \tilde{H}(ru_i)]_+, \quad r > 0,$$

have compact support in  $(0, \infty)$ . Choose numbers  $0 < a_i < b_i < \infty$  with  $\text{supp } g_i \subset [a_i, b_i]$ , and choose  $\rho_i \in C_c^\infty((0, \infty))$  with  $\rho_i = 1$  on  $[a_i, b_i]$ . Extend  $g_i$  by zero to  $\mathbb{R}$  and let  $\varphi_\varepsilon$  be a nonnegative standard mollifier supported in  $(-\varepsilon, \varepsilon)$ , where  $0 < \varepsilon < a_i/2$ . Put

$$\omega_i(\varepsilon) := \sup\{|g_i(r) - g_i(s)| : |r - s| \leq \varepsilon\}.$$

Uniform continuity gives  $\omega_i(\varepsilon) < \infty$ . For every  $r \in \mathbb{R}$  and every  $|y| \leq \varepsilon$ ,  $g_i(r) \leq g_i(r - y) + \omega_i(\varepsilon)$ ; integrating against the mollifier yields

$$g_i(r) \leq (g_i * \varphi_\varepsilon)(r) + \omega_i(\varepsilon).$$

Consequently

$$m_i(r) := \rho_i(r)((g_i * \varphi_\varepsilon)(r) + \omega_i(\varepsilon)) \in C_c^\infty((0, \infty))$$

satisfies  $m_i(r) \geq g_i(r)$  for all  $r > 0$ : on  $[a_i, b_i]$  this follows from  $\rho_i = 1$ , while outside that interval  $g_i = 0$ .

The compact face segment  $F_i := \{ru_i : r \in \text{supp } m_i\}$  is disjoint from the vertex and from the other face. Hence

$$d_i := \text{dist}(F_i, \{0\} \cup \partial S_{3-i}) > 0.$$

Choose  $0 < \delta_i < d_i/3$  and a cutoff  $\chi_i \in C_c^\infty((-\delta_i, \delta_i))$  with  $\chi_i(0) = 1$ . In the tubular coordinates  $z = ru_i + sn_i$  on the  $\delta_i$ -neighborhood of  $F_i$ , define

$$G_i(ru_i + sn_i) := -s\chi_i(s)m_i(r),$$

and extend  $G_i$  by zero outside that tube. Because both cutoffs are compactly supported in the coordinate patch, this extension is a compactly supported  $C^\infty$  function and its support does not meet the other face. At  $s = 0$ ,

$$\partial_r G_i(r, 0) = 0, \quad \partial_s G_i(r, 0) = -m_i(r).$$

Writing  $v_i = \beta_i u_i + n_i$  gives the exact boundary derivative

$$D_i G_i(r u_i) = \beta_i \partial_r G_i(r, 0) + \partial_s G_i(r, 0) = -m_i(r).$$

Because  $G_{3-i}$  vanishes in a neighborhood of face  $i$ ,

$$D_i(\tilde{H} + G_1 + G_2)(r u_i) = D_i \tilde{H}(r u_i) - m_i(r) \leq D_i \tilde{H}(r u_i) - g_i(r) \leq 0.$$

Thus  $\tilde{H} + G_1 + G_2$  has the global boundary signs in (5.30) and differs from  $H$  only on a compact set.

Choose

$$C := \max \left\{ 0, 1 - \inf_{z \in S} (\tilde{H}(z) + G_1(z) + G_2(z)) \right\}, \quad V := \tilde{H} + G_1 + G_2 + C.$$

The infimum is finite because the correction is compactly supported and  $\tilde{H}$  is norm-like. Hence  $V \geq 1$ . Outside a compact set,  $V = H + C$ . Since  $H$  is norm-like, there is  $R_C < \infty$  such that  $H \geq 2C$  on  $S \cap \{|z| \geq R_C\}$ ; consequently  $V = H + C \leq \frac{3}{2}H$  there. With  $\delta_0 := 2\delta/3$ , the exterior estimate (5.28) gives

$$\mathcal{L}_\mu V = \mathcal{L}_\mu H \leq -\delta H \leq -\delta_0 V$$

outside a compact set  $K$ . Enlarge  $K$  so that it contains the supports of all corrections and the set  $\{|z| \leq \max(R, R_C)\}$ . Using the continuous facewise extension, define

$$C_0 := 1 + \max_{z \in K} (\mathcal{L}_\mu V(z) + \delta_0 V(z))_+ < \infty.$$

Then

$$\mathcal{L}_\mu V \leq -\delta_0 V + C_0 \mathbf{1}_K \quad \text{on } S^\circ,$$

which is (5.31). The construction also proves all asserted regularity and compact-support properties.  $\square$

**Definition 5.8** (face-separated subcritical exponential pairs). Let  $\mathfrak{P}_2(\mu)$  be the set of ordered pairs  $(\zeta_1, \zeta_2) \in (\mathbb{R}^2)^2$  such that

$$(5.32) \quad Q_\mu(\zeta_1) < 0, \quad Q_\mu(\zeta_2) < 0,$$

$$(5.33) \quad \zeta_1 \cdot v_1 < 0, \quad \zeta_2 \cdot v_2 < 0,$$

and

$$(5.34) \quad \zeta_1 \cdot u_1 > \zeta_2 \cdot u_1, \quad \zeta_2 \cdot u_2 > \zeta_1 \cdot u_2.$$

Define

$$(5.35) \quad \mathcal{E}_{2 \exp}(\mu) := \left\{ \theta \in S_\circ^\vee : \begin{array}{l} \text{there exist } (\zeta_1, \zeta_2) \in \mathfrak{P}_2(\mu), t \in [0, 1], \eta \in S^\vee, \\ \theta + \eta = t\zeta_1 + (1-t)\zeta_2 \end{array} \right\}.$$

Finally set

$$(5.36) \quad \mathcal{M}_{2 \exp}(\mu) := \text{conv}(\{0\} \cup \mathcal{E}(\mu) \cup \mathcal{E}_{2 \exp}(\mu)) - S^\vee$$

and, for  $c \in S_\circ^\vee$ ,

$$(5.37) \quad \alpha_c^{2 \exp}(\mu) := \sup\{\lambda \geq 0 : \lambda c \in \mathcal{M}_{2 \exp}(\mu)\}.$$

Define also

$$(5.38) \quad \alpha_{\text{rad}}^{2 \exp}(\mu) := \sup_{\theta \in \mathcal{M}_{2 \exp}(\mu)} \min_{u \in S \cap \mathbb{S}^1} \theta \cdot u.$$

**Proposition 5.9** (two-exponential moment region). *Assume  $\alpha < 2$ , and let  $\pi$  be stationary for the drift  $\mu$ . Then*

$$(5.39) \quad \mathcal{E}_{2\text{exp}}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi), \quad \mathcal{M}_{2\text{exp}}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi).$$

Consequently, for every  $c \in S_{\circ}^{\vee}$ ,

$$(5.40) \quad \alpha_c^{\text{in}}(\mu) \leq \alpha_c^{2\text{exp}}(\mu) \leq \alpha_c^{\pi},$$

and

$$(5.41) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^{2\text{exp}}(\mu) \leq \alpha_{\text{rad}}^{\pi}.$$

*Proof.* Fix  $\theta \in \mathcal{E}_{2\text{exp}}(\mu)$ , and choose  $(\zeta_1, \zeta_2), t, \eta$  as in (5.35). Put

$$H(z) := e^{\zeta_1 \cdot z} + e^{\zeta_2 \cdot z}.$$

For  $z \in S$ , weighted arithmetic–geometric mean and  $\eta \cdot z \geq 0$  give

$$H(z) \geq e^{(t\zeta_1 + (1-t)\zeta_2) \cdot z} = e^{(\theta + \eta) \cdot z} \geq e^{\theta \cdot z}.$$

Because  $\theta \in S_{\circ}^{\vee}$ , the last function is norm-like; hence so is  $H$ .

Let

$$\delta := \min\{-Q_{\mu}(\zeta_1), -Q_{\mu}(\zeta_2)\} > 0.$$

Then

$$\mathcal{L}_{\mu}H = Q_{\mu}(\zeta_1)e^{\zeta_1 \cdot z} + Q_{\mu}(\zeta_2)e^{\zeta_2 \cdot z} \leq -\delta H.$$

On face one,

$$D_1H(ru_1) = (\zeta_1 \cdot v_1)e^{r\zeta_1 \cdot u_1} + (\zeta_2 \cdot v_1)e^{r\zeta_2 \cdot u_1}.$$

Put

$$\Delta_1 := (\zeta_1 - \zeta_2) \cdot u_1 > 0, \quad A_1 := -\zeta_1 \cdot v_1 > 0.$$

After factoring out  $e^{r\zeta_1 \cdot u_1}$ ,

$$e^{-r\zeta_1 \cdot u_1} D_1H(ru_1) = -A_1 + (\zeta_2 \cdot v_1)e^{-r\Delta_1}.$$

If  $(\zeta_2 \cdot v_1)_+ = 0$ , this is negative for every  $r \geq 0$ . Otherwise, for

$$r \geq R_1 := \frac{1}{\Delta_1} \left[ \log \left( \frac{2(\zeta_2 \cdot v_1)_+}{A_1} \right) \right]_+, \quad (\log x)_+ := \max\{\log x, 0\},$$

the second term is at most  $A_1/2$ , and hence  $D_1H(ru_1) \leq -(A_1/2)e^{r\zeta_1 \cdot u_1} < 0$ . On face two, define

$$\Delta_2 := (\zeta_2 - \zeta_1) \cdot u_2 > 0, \quad A_2 := -\zeta_2 \cdot v_2 > 0.$$

Factoring the dominant exponential gives

$$e^{-r\zeta_2 \cdot u_2} D_2H(ru_2) = -A_2 + (\zeta_1 \cdot v_2)e^{-r\Delta_2}.$$

If  $(\zeta_1 \cdot v_2)_+ = 0$ , the right-hand side is negative for all  $r \geq 0$ . Otherwise set

$$R_2 := \frac{1}{\Delta_2} \left[ \log \left( \frac{2(\zeta_1 \cdot v_2)_+}{A_2} \right) \right]_+.$$

For  $r \geq R_2$ ,

$$(\zeta_1 \cdot v_2)e^{-r\Delta_2} \leq (\zeta_1 \cdot v_2)_+ e^{-r\Delta_2} \leq \frac{A_2}{2},$$

and therefore

$$D_2H(ru_2) \leq -\frac{A_2}{2} e^{r\zeta_2 \cdot u_2} < 0.$$

Thus both boundary signs hold for  $r \geq R := \max\{R_1, R_2\}$ . Applying lemma 5.7 yields a norm-like  $V \geq 1$ , constant near the vertex, with negative boundary derivatives and the drift bound (5.31).

Let  $V_n = \phi_n(V)$  be the bounded concave truncations from [lemma 4.3](#). Since  $-V_n$  is admissible, stationarity gives

$$0 \leq \int_S \mathcal{L}_\mu V_n d\pi \leq -\delta_0 \int_S \phi'_n(V) V d\pi + C_0.$$

Fatou's lemma and  $\phi'_n(s) \rightarrow 1$  imply  $\int_S V d\pi < \infty$ . Outside a compact set,  $V - H$  is constant; hence  $\int_S H d\pi < \infty$ . Since  $e^{\theta \cdot z} \leq H(z)$ , the required exponential moment is finite. This proves the first inclusion in [\(5.39\)](#).

The full moment domain is convex and downward closed by [proposition 5.1](#). It contains the origin,  $\mathcal{E}(\mu)$ , and  $\mathcal{E}_{2\text{exp}}(\mu)$ , so it contains the convex dual-cone lower hull  $\mathcal{M}_{2\text{exp}}(\mu)$ . Taking ray extents gives [\(5.40\)](#); the first inequality follows because  $\mathcal{E}(\mu) - S^\vee \subset \mathcal{M}_{2\text{exp}}(\mu)$ . Taking the supremum of the minimum angular support gives [\(5.41\)](#).  $\square$

**Definition 5.10** (finite face-dominating exponential families). For an integer  $N \geq 1$ , let  $\mathfrak{P}_N^{\text{fin}}(\mu)$  be the collection of families

$$((a_k, \zeta_k))_{k=1}^N, \quad a_k > 0, \quad \zeta_k \in \mathbb{R}^2,$$

such that

$$(5.42) \quad Q_\mu(\zeta_k) < 0, \quad k = 1, \dots, N,$$

and the following dominant-face condition holds. For  $i = 1, 2$ , put

$$m_i := \max_{1 \leq k \leq N} \zeta_k \cdot u_i, \quad I_i := \{k : \zeta_k \cdot u_i = m_i\}.$$

Then require

$$(5.43) \quad \sum_{k \in I_i} a_k (\zeta_k \cdot v_i) < 0, \quad i = 1, 2.$$

Define  $\mathcal{E}_{\text{fin}}(\mu)$  to be the set of  $\theta \in S^\vee$  for which there exist  $N \geq 1$ , a family in  $\mathfrak{P}_N^{\text{fin}}(\mu)$ , weights  $t_k \geq 0$  with  $\sum_{k=1}^N t_k = 1$ , and  $\eta \in S^\vee$  such that

$$(5.44) \quad \theta + \eta = \sum_{k=1}^N t_k \zeta_k.$$

Set

$$(5.45) \quad \mathcal{M}_{\text{fin}}(\mu) := \text{conv}(\{0\} \cup \mathcal{E}(\mu) \cup \mathcal{E}_{\text{fin}}(\mu)) - S^\vee,$$

and define, for  $c \in S^\vee$ ,

$$(5.46) \quad \alpha_c^{\text{fin}}(\mu) := \sup\{\lambda \geq 0 : \lambda c \in \mathcal{M}_{\text{fin}}(\mu)\},$$

$$(5.47) \quad \alpha_{\text{rad}}^{\text{fin}}(\mu) := \sup_{\theta \in \mathcal{M}_{\text{fin}}(\mu)} \min_{u \in S \cap \mathbb{S}^1} \theta \cdot u.$$

**Lemma 5.11** (two-face coefficient selection and stability). *The following two elementary coefficient facts will be used repeatedly.*

(i) Let  $r_1, r_2 \in \mathbb{R}^2$  satisfy

$$(5.48) \quad r_1 \cdot u_1 \geq r_2 \cdot u_1, \quad r_2 \cdot u_2 \geq r_1 \cdot u_2,$$

and

$$(5.49) \quad r_1 \cdot v_1 < 0, \quad r_2 \cdot v_2 < 0.$$

*If both inequalities in [\(5.48\)](#) are equalities, assume in addition that  $r_1 = r_2$ ; then the common vector is required to satisfy both inequalities in [\(5.49\)](#). There is a positive-coefficient exponential family, using  $r_1, r_2$  or only their common value, whose leading oblique sum is strictly negative on both faces. More explicitly:*

- if both inequalities in (5.48) are strict, any positive coefficients work;
- if equality holds only on face one, take  $a_2 = 1$  and any

$$(5.50) \quad a_1 > \frac{(r_2 \cdot v_1)_+}{-r_1 \cdot v_1};$$

- if equality holds only on face two, take  $a_1 = 1$  and any

$$(5.51) \quad a_2 > \frac{(r_1 \cdot v_2)_+}{-r_2 \cdot v_2};$$

- if both equalities hold, use the single exponent  $r_1 = r_2$ .

(ii) Let  $\{(a_k, \zeta_k) : 1 \leq k \leq N\}$  be a finite exponential family. Write

$$m_i = \max_k \zeta_k \cdot u_i, \quad b_i = \sum_{k: \zeta_k \cdot u_i = m_i} a_k \zeta_k \cdot v_i < 0, \quad i = 1, 2.$$

Let  $\eta_1, \dots, \eta_M$  be additional exponents satisfying  $\eta_j \cdot u_i \leq m_i$  for both faces. Define

$$T_i := \{j : \eta_j \cdot u_i = m_i\}, \quad P_i := \sum_{j \in T_i} (\eta_j \cdot v_i)_+.$$

If every added exponent is given the same coefficient  $0 < \varepsilon < \varepsilon_*$ , where

$$(5.52) \quad \varepsilon_* := \min_{i=1,2} \frac{-b_i}{1 + P_i} > 0,$$

then the enlarged family has the same face maxima and strictly negative leading oblique sums on both faces.

*Proof.* For part (i), if both tangential inequalities are strict, the leading set on face one is  $\{r_1\}$  and the leading set on face two is  $\{r_2\}$ ; (5.49) gives the result. If only the face-one rates tie, the face-one leading sum with the choices in (5.50) is

$$a_1 r_1 \cdot v_1 + r_2 \cdot v_1 \leq -a_1 (-r_1 \cdot v_1) + (r_2 \cdot v_1)_+ < 0.$$

On face two,  $r_2$  is the unique leader, so its leading sum is  $r_2 \cdot v_2 < 0$ .

If equality holds only on face two, choose  $a_1 = 1$  and

$$a_2 > \frac{(r_1 \cdot v_2)_+}{-r_2 \cdot v_2}.$$

The face-two leading sum then satisfies

$$r_1 \cdot v_2 + a_2 r_2 \cdot v_2 \leq (r_1 \cdot v_2)_+ - a_2 (-r_2 \cdot v_2) < 0,$$

while  $r_1$  is the unique leader on face one and contributes  $r_1 \cdot v_1 < 0$ . Finally, if both tangential equalities hold, then

$$(r_1 - r_2) \cdot u_1 = (r_1 - r_2) \cdot u_2 = 0.$$

Since  $u_1, u_2$  are linearly independent,  $r_1 = r_2$ . The single common exponent has negative oblique coefficient on each face by the additional hypothesis.

For part (ii), an exponent with  $\eta_j \cdot u_i < m_i$  is not in the enlarged leading set on face  $i$ . Hence the new leading sum there is

$$b_i + \varepsilon \sum_{j \in T_i} \eta_j \cdot v_i \leq b_i + \varepsilon P_i.$$

By (5.52),  $\varepsilon P_i < -b_i$  when  $P_i > 0$ , while the inequality is immediate when  $P_i = 0$ . Thus both new leading sums remain strictly negative.  $\square$

**Proposition 5.12** (finite-exponential moment region). *Assume  $\alpha < 2$ , and let  $\pi$  be stationary for the drift  $\mu$ . Then*

$$(5.53) \quad \mathcal{E}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi), \quad \mathcal{M}_{\text{fin}}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi).$$

Moreover,

$$(5.54) \quad \mathcal{M}_{2\text{exp}}(\mu) \subset \mathcal{M}_{\text{fin}}(\mu),$$

and consequently, for every  $c \in S_\circ^\vee$ ,

$$(5.55) \quad \alpha_c^{\text{in}}(\mu) \leq \alpha_c^{2\text{exp}}(\mu) \leq \alpha_c^{\text{fin}}(\mu) \leq \alpha_c^\pi,$$

while

$$(5.56) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^{2\text{exp}}(\mu) \leq \alpha_{\text{rad}}^{\text{fin}}(\mu) \leq \alpha_{\text{rad}}^\pi.$$

*Proof.* Fix  $\theta \in \mathcal{E}_{\text{fin}}(\mu)$ , and choose data as in [definition 5.10](#). Put

$$H(z) := \sum_{k=1}^N a_k e^{\zeta_k \cdot z}.$$

Let  $I_t := \{k : t_k > 0\}$ . Weighted arithmetic–geometric mean gives

$$H(z) \geq \sum_{k \in I_t} t_k \frac{a_k}{t_k} e^{\zeta_k \cdot z} \geq C_t \exp \left\{ \sum_{k=1}^N t_k \zeta_k \cdot z \right\}, \quad C_t := \prod_{k \in I_t} \left( \frac{a_k}{t_k} \right)^{t_k} > 0.$$

By [\(5.44\)](#) and  $\eta \cdot z \geq 0$  on  $S$ ,

$$H(z) \geq C_t e^{(\theta+\eta) \cdot z} \geq C_t e^{\theta \cdot z}.$$

Since  $\theta \in S_\circ^\vee$ , the function  $H$  is norm-like.

With

$$\delta := \min_{1 \leq k \leq N} \{-Q_\mu(\zeta_k)\} > 0,$$

one has

$$\mathcal{L}_\mu H = \sum_{k=1}^N a_k Q_\mu(\zeta_k) e^{\zeta_k \cdot z} \leq -\delta H.$$

On face  $i$ , factor out the maximal tangential exponential rate:

$$e^{-m_i r} D_i H(r u_i) = \sum_{k \in I_i} a_k (\zeta_k \cdot v_i) + \sum_{k \notin I_i} a_k (\zeta_k \cdot v_i) e^{-r(m_i - \zeta_k \cdot u_i)}.$$

Write

$$b_i := \sum_{k \in I_i} a_k (\zeta_k \cdot v_i) < 0.$$

If  $I_i = \{1, \dots, N\}$ , the second sum is absent and the displayed quantity equals  $b_i < 0$ . Otherwise define

$$\Delta_i := \min_{k \notin I_i} (m_i - \zeta_k \cdot u_i) > 0, \quad P_i := \sum_{k \notin I_i} a_k |\zeta_k \cdot v_i|.$$

Then

$$\left| \sum_{k \notin I_i} a_k (\zeta_k \cdot v_i) e^{-r(m_i - \zeta_k \cdot u_i)} \right| \leq P_i e^{-r \Delta_i}.$$

If  $P_i = 0$ , the boundary expression is negative for all  $r \geq 0$ . If  $P_i > 0$ , then for

$$r \geq R_i := \frac{1}{\Delta_i} \left[ \log \left( \frac{2P_i}{-b_i} \right) \right]_+$$

one has  $P_i e^{-r \Delta_i} \leq -b_i/2$ , and therefore

$$e^{-m_i r} D_i H(r u_i) \leq \frac{b_i}{2} < 0.$$

Taking  $R = \max\{R_1, R_2\}$  proves the two boundary signs for every  $r \geq R$ . The compact correction lemma and the concave-truncation argument used in the proof of [proposition 5.9](#) therefore yield  $\int_S H d\pi < \infty$ . Since  $e^{\theta \cdot z} \leq C_t^{-1} H(z)$ , this proves  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$ .

Convexity and dual-cone downward closure of the full moment domain give the second inclusion in (5.53). Every pair in  $\mathfrak{P}_2(\mu)$ , with coefficients  $a_1 = a_2 = 1$ , satisfies the dominant-face condition: the strict tangential inequalities identify a single dominant exponent on each face, and its own-face coefficient is negative. Thus  $\mathcal{E}_{2\text{exp}}(\mu) \subset \mathcal{E}_{\text{fin}}(\mu)$ , proving (5.54). The projected chain follows by taking ray extents. For the radial chain, if  $\theta \in \mathcal{M}_{\text{fin}}(\mu)$  and  $m(\theta) := \min_{u \in S \cap \mathbb{S}^1} \theta \cdot u > 0$ , then  $e^{m(\theta)|z|} \leq e^{\theta \cdot z}$  on  $S$ ; taking suprema gives the last inequality in (5.56), and the remaining inequalities follow from the set inclusions.  $\square$

**Proposition 5.13** (norm-like finite-exponential moment criterion). *Assume  $\alpha < 2$ , and let  $\pi$  be stationary for the drift  $\mu$ . Let*

$$((a_k, \zeta_k))_{k=1}^N, \quad a_k > 0,$$

be a finite family satisfying

$$Q_\mu(\zeta_k) < 0, \quad k = 1, \dots, N.$$

For  $i = 1, 2$ , put

$$m_i := \max_k \zeta_k \cdot u_i, \quad I_i := \{k : \zeta_k \cdot u_i = m_i\},$$

and suppose that

$$\sum_{k \in I_i} a_k \zeta_k \cdot v_i < 0, \quad i = 1, 2.$$

Assume in addition that the exponential sum is coercive on the wedge, in the equivalent form

$$(5.57) \quad \gamma := \min_{u \in S \cap \mathbb{S}^1} \max_{1 \leq k \leq N} \zeta_k \cdot u > 0.$$

If  $\theta \in \mathbb{R}^2$ ,  $t_k \geq 0$ ,  $\sum_k t_k = 1$ , and  $\eta \in S^\vee$  satisfy

$$(5.58) \quad \theta + \eta = \sum_{k=1}^N t_k \zeta_k,$$

then

$$\theta \in \mathcal{D}_{\text{exp}}(\pi).$$

*Proof.* Set

$$H(z) := \sum_{k=1}^N a_k e^{\zeta_k \cdot z}.$$

Condition (5.57) implies

$$H(ru) \geq \min_k a_k e^{\gamma r}, \quad r \geq 0, \quad u \in S \cap \mathbb{S}^1,$$

so  $H$  is norm-like. The strict kernel inequalities give

$$\mathcal{L}_\mu H \leq -\delta H, \quad \delta := \min_k \{-Q_\mu(\zeta_k)\} > 0.$$

The boundary sign is obtained directly from the leading sums. On face  $i$ ,

$$e^{-m_i r} D_i H(ru_i) = \sum_{k \in I_i} a_k \zeta_k \cdot v_i + \sum_{k \notin I_i} a_k \zeta_k \cdot v_i e^{-r(m_i - \zeta_k \cdot u_i)}.$$

Every exponent gap in the second sum is strictly positive, so that sum converges to zero as  $r \rightarrow \infty$ . The first sum is strictly negative by hypothesis. Hence there is  $R_i < \infty$  such that  $D_i H(ru_i) < 0$  for  $r \geq R_i$ . Taking  $R = \max(R_1, R_2)$ , apply lemma 5.7. We obtain a norm-like function  $V : S \rightarrow [1, \infty)$ , constant near the vertex, with

$$D_i V \leq 0 \quad \text{on } \partial S_i, \quad i = 1, 2,$$

and constants  $\delta_0, C_0 > 0$  and a compact set  $K \subset S$  such that

$$(5.59) \quad \mathcal{L}_\mu V \leq -\delta_0 V + C_0 \mathbf{1}_K \quad \text{on } S^\circ.$$

Moreover, there are a compact set  $K_0 \subset S$  and a constant  $c_0 \in \mathbb{R}$  for which

$$(5.60) \quad V(z) = H(z) + c_0, \quad z \in S \setminus K_0.$$

Let  $\phi_n$  be the concave truncations of lemma 4.2 and put  $V_n := \phi_n(V)$ . By lemma 4.3,  $V_n \in C_b^2(S)$ , it is constant near the vertex, and  $-V_n$  is admissible. The chain rule and (5.59) give

$$\mathcal{L}_\mu V_n = \phi_n'(V) \mathcal{L}_\mu V + \frac{1}{2} \phi_n''(V) |\nabla V|^2 \leq -\delta_0 \phi_n'(V) V + C_0 \mathbf{1}_K.$$

Stationarity applied to the admissible test  $-V_n$  yields

$$0 \leq \int_S \mathcal{L}_\mu V_n d\pi \leq -\delta_0 \int_S \phi_n'(V) V d\pi + C_0 \pi(K),$$

and therefore

$$(5.61) \quad \int_S \phi_n'(V) V d\pi \leq \frac{C_0}{\delta_0}.$$

For each  $z \in S$ ,  $\phi_n'(V(z))V(z) \rightarrow V(z)$ . Fatou's lemma applied to (5.61) gives

$$\int_S V d\pi \leq \frac{C_0}{\delta_0} < \infty.$$

Finally, (5.60) implies

$$\int_S H d\pi = \int_{K_0} H d\pi + \int_{S \setminus K_0} (V - c_0) d\pi < \infty,$$

because  $H$  is bounded on the compact set  $K_0$  and  $\int_S V d\pi < \infty$ . Let  $I_t = \{k : t_k > 0\}$ . Weighted arithmetic–geometric mean gives

$$H(z) \geq \prod_{k \in I_t} \left( \frac{a_k}{t_k} \right)^{t_k} \exp \left\{ \sum_k t_k \zeta_k \cdot z \right\}.$$

By (5.58) and  $\eta \cdot z \geq 0$  on  $S$ , the right-hand side dominates a positive constant times  $e^{\theta \cdot z}$ . Hence the  $\theta$ -moment is finite.  $\square$

**Lemma 5.14** (interior decomposition of a compact-core lower hull). *Let  $\mathbb{C} \subset \mathbb{R}^2$  be compact, convex, and have nonempty interior, and set*

$$\mathcal{K}_{\mathbb{C}} := \mathbb{C} - S^\vee.$$

*Every  $\theta \in \text{int } \mathcal{K}_{\mathbb{C}}$  admits a representation*

$$(5.62) \quad \theta = \zeta - \eta, \quad \zeta \in \text{int } \mathbb{C}, \quad \eta \in S_\circ^\vee.$$

*Proof.* Choose  $q \in S_\circ^\vee$ . Since  $\theta \in \text{int } \mathcal{K}_{\mathbb{C}}$ , there is  $r_\theta > 0$  such that  $B_{r_\theta}(\theta) \subset \mathcal{K}_{\mathbb{C}}$ . Choose

$$0 < \varepsilon < \frac{r_\theta}{1 + |q|}.$$

Then  $\theta + \varepsilon q \in \mathcal{K}_{\mathbb{C}}$ , so

$$\theta + \varepsilon q = \zeta_0 - \eta_0$$

for some  $\zeta_0 \in \mathbb{C}$  and  $\eta_0 \in S^\vee$ . Put  $\bar{\eta} := \eta_0 + \varepsilon q \in S_\circ^\vee$ . Since the latter cone is open, choose  $r_\eta > 0$  with  $B_{r_\eta}(\bar{\eta}) \subset S_\circ^\vee$ .

Fix  $\zeta_* \in \text{int } \mathbb{C}$ , set  $d := \zeta_* - \theta - \bar{\eta}$ , and choose

$$0 < t < \min \left\{ 1, \frac{r_\eta}{1 + |d|} \right\}.$$

Define

$$\zeta_t = (1 - t)\zeta_0 + t\zeta_*.$$

Then  $\zeta_t \in \text{int } \mathbb{C}$ , and

$$\zeta_t - \theta = \bar{\eta} + td.$$

The choice of  $t$  gives  $|td| < r_\eta$ , hence  $\zeta_t - \theta \in B_{r_\eta}(\bar{\eta}) \subset S^\vee$ . Taking  $\zeta = \zeta_t$  and  $\eta = \zeta_t - \theta$  proves the claim.  $\square$

**Proposition 5.15** (convexity, drift scaling, and five-exponential reduction). *Let  $\mathcal{E}_{\text{fin}}^{(\leq 5)}(\mu)$  denote the set obtained from [definition 5.10](#) by restricting the witnessing family to  $1 \leq N \leq 5$ . Then:*

- (i)  $\mathcal{E}_{\text{fin}}(\mu)$  is convex.
- (ii) For every  $\rho > 0$ ,

$$\mathcal{E}_{\text{fin}}(\rho\mu) = \rho\mathcal{E}_{\text{fin}}(\mu), \quad \mathcal{M}_{\text{fin}}(\rho\mu) = \rho\mathcal{M}_{\text{fin}}(\mu).$$

Consequently,

$$\alpha_c^{\text{fin}}(\rho\mu) = \rho\alpha_c^{\text{fin}}(\mu), \quad \alpha_{\text{rad}}^{\text{fin}}(\rho\mu) = \rho\alpha_{\text{rad}}^{\text{fin}}(\mu).$$

- (iii) Arbitrarily long finite families are unnecessary:

$$(5.63) \quad \mathcal{E}_{\text{fin}}^{(\leq 5)}(\mu) = \mathcal{E}_{\text{fin}}(\mu).$$

Thus the building blocks of  $\mathcal{M}_{\text{fin}}(\mu)$  are described by a finite-dimensional feasibility problem with at most five exponent vectors.

*Proof.* For convexity, let  $\theta^{(j)} \in \mathcal{E}_{\text{fin}}(\mu)$ ,  $j = 1, 2$ , and choose witnessing families, target weights, and cone shifts. Concatenate the two exponential families. On face  $i$ , if one family has the larger maximal tangential exponent, its strictly negative leading boundary coefficient remains the leading coefficient of the concatenated family. If the two maxima agree, the new leading coefficient is the sum of two strictly negative numbers. Hence the concatenated family is again face-dominating. For  $s \in (0, 1)$ , multiply the target weights in the first family by  $s$ , those in the second by  $1 - s$ , and use the cone shift  $s\eta^{(1)} + (1 - s)\eta^{(2)}$ . This witnesses  $s\theta^{(1)} + (1 - s)\theta^{(2)}$ . The endpoint cases are immediate.

For scaling, replace every exponent  $\zeta_k$  by  $\rho\zeta_k$ , every target and cone shift by its  $\rho$ -multiple, and note that

$$Q_{\rho\mu}(\rho\zeta_k) = \rho^2 Q_\mu(\zeta_k).$$

The maximizing index sets on the two faces are unchanged, while each leading boundary coefficient is multiplied by  $\rho$ . This proves the first scaling identity; the corresponding identity for  $\mathcal{M}_{\text{fin}}$  also uses  $\rho S^\vee = S^\vee$ . The scalar-rate formulas follow directly.

It remains to prove (5.63). Fix a witness  $((a_k, \zeta_k))_{k=1}^N$  for  $\theta \in \mathcal{E}_{\text{fin}}(\mu)$ , and write

$$\bar{\zeta} := \theta + \eta = \sum_{k=1}^N t_k \zeta_k.$$

By Carathéodory's theorem in  $\mathbb{R}^2$ , there is a set  $J_t \subset \{1, \dots, N\}$  with  $|J_t| \leq 3$  such that  $\bar{\zeta}$  is a convex combination of  $\{\zeta_k : k \in J_t\}$ .

Let  $m_i$  and  $I_i$  be the two face maxima and maximizing index sets of the original family. For each  $k$ , define

$$q_k := (\mathbf{1}_{\{k \in I_1\}} \zeta_k \cdot v_1, \mathbf{1}_{\{k \in I_2\}} \zeta_k \cdot v_2) \in \mathbb{R}^2.$$

The dominant-face assumptions say that

$$q := \sum_{k=1}^N a_k q_k \in (-\infty, 0)^2.$$

The conic form of Carathéodory's theorem in dimension two gives a set  $J_b$  with  $|J_b| \leq 2$  and nonnegative coefficients  $b_k$ ,  $k \in J_b$ , such that

$$q = \sum_{k \in J_b} b_k q_k.$$

After discarding indices with zero coefficient, every retained coefficient is strictly positive. Put  $J = J_t \cup J_b$ , so  $|J| \leq 5$ . Since both coordinates of  $q$  are negative,  $J_b \cap I_i \neq \emptyset$  for  $i = 1, 2$ ; hence

the maximal tangential exponent of the restricted family on face  $i$  is still  $m_i$ . Give the retained exponents the positive coefficients

$$\widehat{a}_k = \begin{cases} b_k + \varepsilon, & k \in J_b, \\ \varepsilon, & k \in J \setminus J_b. \end{cases}$$

Put  $w := \sum_{k \in J} q_k$ , and choose

$$0 < \varepsilon < \varepsilon_J := \min_{i=1,2} \frac{-q_i}{2(1 + |w_i|)}.$$

Then, for  $i = 1, 2$ ,

$$q_i + \varepsilon w_i \leq q_i + \varepsilon |w_i| < q_i + \frac{-q_i}{2} = \frac{q_i}{2} < 0.$$

Thus the two leading boundary sums, whose vector is

$$q + \varepsilon \sum_{k \in J} q_k,$$

remain strictly negative. All retained exponents still satisfy  $Q_\mu(\zeta_k) < 0$ . Using the Carathéodory weights on  $J_t$ , and zero target weights on the other retained indices, preserves the identity  $\bar{\zeta} = \theta + \eta$ . The restricted family therefore witnesses  $\theta$  and has at most five terms.  $\square$

**Proposition 5.16** (locally uniform bounds on compact subsets). *Assume  $\alpha < 2$  and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Let  $\Theta \subset \mathcal{E}(\mu)$  be nonempty and compact. Then there exists a constant  $C_{\Theta, \eta} < \infty$  such that*

$$(5.64) \quad \sup_{\vartheta \in \Theta} \int_S \eta(\vartheta \cdot z) e^{\vartheta \cdot z} \pi(dz) \leq C_{\Theta, \eta},$$

where  $\eta \in C^\infty([0, \infty))$  is any fixed cut-off satisfying

$$(5.65) \quad \eta(s) = 0 \quad (0 \leq s \leq 1/2), \quad \eta(s) = 1 \quad (s \geq 1),$$

and  $0 \leq \eta' \leq C_\eta$ . Consequently, for each nonempty compact  $\Theta \subset \mathcal{E}(\mu)$ ,

$$(5.66) \quad \sup_{\vartheta \in \Theta} \int_{\{\vartheta \cdot z \geq 1\}} e^{\vartheta \cdot z} \pi(dz) < \infty.$$

Since  $\Theta$  is a nonempty compact subset of  $S_\circ^\vee$ ,

$$(5.67) \quad m_\Theta := \inf_{\vartheta \in \Theta} \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u > 0.$$

Moreover, there exists  $C'_{\Theta, \eta} < \infty$  such that

$$(5.68) \quad \pi(\{|z| \geq r\}) \leq C'_{\Theta, \eta} e^{-m_\Theta r}, \quad r \geq 0.$$

*Proof.* For  $\vartheta \in \Theta$ , let

$$W_\vartheta(z) = \eta(\vartheta \cdot z) e^{\vartheta \cdot z}.$$

Since  $\Theta \subset \mathcal{E}(\mu)$  is compact and  $\mathcal{E}(\mu)$  is defined by the strict quadratic inequality

$$\vartheta \cdot \mu + \frac{1}{2} |\vartheta|^2 < 0,$$

there exists  $\delta_\Theta > 0$  such that

$$(5.69) \quad \vartheta \cdot \mu + \frac{1}{2} |\vartheta|^2 \leq -\delta_\Theta, \quad \vartheta \in \Theta.$$

Since  $\Theta$  is a compact subset of  $S_\circ^\vee$ , the quantity

$$m_\Theta := \inf_{\vartheta \in \Theta} \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u$$

is strictly positive. The set  $\Theta$  is bounded. Hence all coefficients appearing when  $\mathcal{L}$  acts on  $W_\vartheta$  are bounded uniformly over  $\vartheta \in \Theta$ , except for the negative exponential term itself. On the region

$\vartheta \cdot z \geq 1$  we have

$$(5.70) \quad \mathcal{L}W_\vartheta = \left( \vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2 \right) W_\vartheta \leq -\delta_\Theta W_\vartheta.$$

On the complementary region  $\vartheta \cdot z \leq 1$ , the uniform lower bound  $m_\Theta > 0$  implies  $|z| \leq 1/m_\Theta$ . Put

$$K_\Theta := S \cap \overline{B}_{1/m_\Theta}(0).$$

Thus  $\{z \in S : \vartheta \cdot z \leq 1\} \subset K_\Theta$  for every  $\vartheta \in \Theta$ . To take the compact bound on an actual compact parameter set, extend the open-wedge expression

$$\mathcal{L}W_\vartheta + \delta_\Theta W_\vartheta$$

continuously to the two open faces and assign the value 0 at the vertex. The extension is compatible with the fixed cut-off uniformly in  $\vartheta$ : if

$$M_\Theta := \sup_{\vartheta \in \Theta} |\vartheta|,$$

then  $M_\Theta > 0$ , and  $|z| \leq (2M_\Theta)^{-1}$  implies  $\vartheta \cdot z \leq 1/2$  for every  $\vartheta \in \Theta$ , hence  $W_\vartheta(z) = 0$ . On the compact set  $\Theta \times K_\Theta$  the extended positive part is bounded: the exponential factor, the first two derivatives of  $\eta$ , and the parameters  $\vartheta \in \Theta$  are uniformly bounded. Define

$$C_\Theta^0 := 1 + \sup_{\vartheta \in \Theta, z \in K_\Theta} (\mathcal{L}W_\vartheta(z) + \delta_\Theta W_\vartheta(z))_+ < \infty,$$

where the expression inside the supremum denotes this continuous extension. Combining this compact bound with the estimate on  $\{\vartheta \cdot z \geq 1\}$  gives the global open-wedge estimate

$$(5.71) \quad \mathcal{L}W_\vartheta \leq C_\Theta^0 \mathbf{1}_{\{\vartheta \cdot z \leq 1\}} - \delta_\Theta W_\vartheta, \quad \vartheta \in \Theta,$$

with boundary values understood through the stationary zero-boundary-mass convention.

Let  $\chi_n$  be the same increasing concave truncations used in [proposition 5.6](#) and set

$$W_{\vartheta,n} := \chi_n(W_\vartheta).$$

For each fixed pair  $(\vartheta, n)$ , the compact-localization argument above gives  $W_{\vartheta,n} \in C_b^2(S)$  and the function is constant near the vertex. Because  $\vartheta \cdot v_i \leq 0$ , one has  $D_i W_\vartheta \leq 0$ ; hence  $-W_{\vartheta,n}$  is admissible. The stationary inequality gives

$$\int_S \mathcal{L}W_{\vartheta,n} d\pi \geq 0.$$

Since  $\chi_n'' \leq 0$ ,

$$\mathcal{L}W_{\vartheta,n} \leq \chi_n'(W_\vartheta) \mathcal{L}W_\vartheta \leq C_\Theta^0 \mathbf{1}_{\{\vartheta \cdot z \leq 1\}} - \delta_\Theta \chi_n'(W_\vartheta) W_\vartheta.$$

The first term has integral at most  $C_\Theta^0$ , because  $\pi$  is a probability measure. Therefore

$$\int_S \chi_n'(W_\vartheta) W_\vartheta d\pi \leq \frac{C_\Theta^0}{\delta_\Theta}, \quad \vartheta \in \Theta.$$

Fatou's lemma gives [\(5.64\)](#), uniformly over  $\vartheta \in \Theta$ . Since  $\eta = 1$  on  $[1, \infty)$ , [\(5.66\)](#) follows.

The preceding display gives  $m_\Theta > 0$ . Hence, for every  $\vartheta \in \Theta$  and  $z \in S$ ,

$$\vartheta \cdot z \geq m_\Theta |z|.$$

Fix any  $\vartheta_0 \in \Theta$ . Since  $\eta = 1$  on  $[1, \infty)$ , [\(5.64\)](#) gives

$$\int_{\{\vartheta_0 \cdot z \geq 1\}} e^{\vartheta_0 \cdot z} \pi(dz) \leq C_{\Theta, \eta}.$$

On the complementary region  $\{\vartheta_0 \cdot z < 1\}$  one has  $e^{\vartheta_0 \cdot z} \leq e$ . Therefore

$$\int_S e^{\vartheta_0 \cdot z} \pi(dz) \leq e + C_{\Theta, \eta}.$$

Markov's inequality yields

$$\pi(\{|z| \geq r\}) \leq (e + C_{\Theta, \eta})e^{-m_{\Theta} r}, \quad r \geq 0.$$

This proves (5.68), with  $C'_{\Theta, \eta} = e + C_{\Theta, \eta}$ .  $\square$

*Remark 5.17* (why this uniformity matters). The estimates above do not identify the full exponential-moment domain, but they show that the inner region  $\mathcal{E}(\mu)$  is not merely a pointwise collection of admissible exponential directions. On compact subsets the Lyapunov argument gives uniform control. This is the form needed for later transform or kernel-equation arguments, where one usually needs integrability on regions rather than at isolated transform points.

**Proposition 5.18** (local analyticity of the Laplace transform on the inner region). *Assume  $\alpha < 2$  and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Suppose  $\mathcal{E}(\mu) \neq \emptyset$ . Define*

$$M_{\pi}(\zeta) := \int_S e^{\zeta \cdot z} \pi(dz), \quad \zeta \in \mathbb{C}^2,$$

whenever the integral is absolutely convergent. Then for every nonempty compact set  $\Theta \subset \mathcal{E}(\mu)$  there exists an open neighborhood  $\mathcal{O}_{\Theta} \subset \mathbb{R}^2$  of  $\Theta$  such that  $M_{\pi}$  is finite and real analytic on  $\mathcal{O}_{\Theta}$ . More precisely,  $M_{\pi}$  extends to a holomorphic function on the tube

$$\mathcal{T}_{\Theta} := \{\zeta \in \mathbb{C}^2 : \operatorname{Re} \zeta \in \mathcal{O}_{\Theta}\}.$$

In particular, all polynomially weighted exponential moments

$$\int_S |z|^k e^{\theta \cdot z} \pi(dz)$$

are finite locally uniformly for  $\theta \in \mathcal{E}(\mu)$  and  $k \geq 0$ .

*Proof.* It is enough to prove the assertion locally uniformly on a nonempty compact set  $\Theta \subset \mathcal{E}(\mu)$ . Choose  $a_0 \in \mathcal{E}(\mu)$ . Since  $a_0 \in S_{\circ}^{\vee}$ , there exists  $m_0 > 0$  such that

$$a_0 \cdot z \geq m_0 |z|, \quad z \in S.$$

The set  $\Theta$  is compact and the quadratic function

$$q(\theta) := \theta \cdot \mu + \frac{1}{2} |\theta|^2$$

is strictly negative on  $\Theta$ . Define the positive margin

$$m_{\Theta}^q := -\max_{\theta \in \Theta} q(\theta) > 0$$

and the finite constant

$$M_{\Theta}^q := \max_{\theta \in \Theta} |a_0 \cdot (\mu + \theta)| + \frac{1}{2} |a_0|^2 > 0.$$

Choose

$$(5.72) \quad 0 < \delta < \min \left\{ 1, \frac{m_{\Theta}^q}{2M_{\Theta}^q} \right\}.$$

For every  $\theta \in \Theta$ ,

$$\begin{aligned} q(\theta + \delta a_0) &= q(\theta) + \delta a_0 \cdot (\mu + \theta) + \frac{1}{2} \delta^2 |a_0|^2 \\ &\leq -m_{\Theta}^q + \delta M_{\Theta}^q < -\frac{1}{2} m_{\Theta}^q < 0. \end{aligned}$$

Moreover,  $a_0 \in S_{\circ}^{\vee}$  and  $a_0 \cdot v_i \leq 0$ , so  $\theta + \delta a_0 \in S_{\circ}^{\vee}$  and  $(\theta + \delta a_0) \cdot v_i \leq 0$ . Therefore

$$\Theta_{\delta} := \{\theta + \delta a_0 : \theta \in \Theta\} \subset \mathcal{E}(\mu).$$

By the locally uniform compact-subset estimate in [proposition 5.16](#),

$$\sup_{\theta \in \Theta} \int_{\{(\theta + \delta a_0) \cdot z \geq 1\}} e^{(\theta + \delta a_0) \cdot z} \pi(dz) < \infty.$$

The remaining region  $\{(\theta + \delta a_0) \cdot z < 1\}$  is uniformly bounded in  $z$ . Indeed, since  $\Theta_\delta$  is a compact subset of  $S_\circ^\vee$ , the number

$$m_\delta := \inf\{(\theta + \delta a_0) \cdot u : \theta \in \Theta, u \in S \cap \mathbb{S}^1\}$$

is strictly positive. Hence  $(\theta + \delta a_0) \cdot z < 1$  implies  $|z| < m_\delta^{-1}$ . Thus

$$\sup_{\theta \in \Theta} \int_S e^{(\theta + \delta a_0) \cdot z} \pi(dz) < \infty.$$

Now take

$$\mathcal{O}_\Theta := \{\eta \in \mathbb{R}^2 : \text{dist}(\eta, \Theta) < \frac{1}{2} \delta m_0\}.$$

If  $\eta \in \mathcal{O}_\Theta$ , choose  $\theta \in \Theta$  with  $|\eta - \theta| < \delta m_0/2$ . Then for  $z \in S$ ,

$$(\eta - \theta) \cdot z \leq |\eta - \theta| |z| \leq \frac{\delta}{2} a_0 \cdot z.$$

Therefore

$$e^{\eta \cdot z} \leq e^{(\theta + \delta a_0) \cdot z}.$$

This gives absolute integrability of  $e^{\eta \cdot z}$  for each  $\eta \in \mathcal{O}_\Theta$ . The finite-cover argument below gives the compact-tube domination needed for local uniformity and differentiation under the integral sign.

For complex  $\zeta$  with  $\text{Re } \zeta \in \mathcal{O}_\Theta$ , the preceding pointwise domination gives absolute convergence of  $M_\pi(\zeta)$ . The compact-uniform domination needed for differentiating under the integral sign is supplied by the finite-cover argument below.

Fix a compact set  $K \Subset \mathcal{O}_\Theta$  and put

$$d_K := \max_{\eta \in K} \text{dist}(\eta, \Theta) < \frac{\delta m_0}{2}.$$

Let  $\rho_K := \delta m_0/2 - d_K > 0$ . For each  $\eta \in K$  choose  $\theta_\eta \in \Theta$  with

$$|\eta - \theta_\eta| \leq d_K.$$

If  $\eta' \in K$  and  $|\eta' - \eta| < \rho_K/2$ , then

$$|\eta' - \theta_\eta| \leq \frac{\delta m_0}{2} - \frac{\rho_K}{2}.$$

A finite number of such balls covers  $K$ ; denote their centers by  $\eta_1, \dots, \eta_N$  and set  $\theta_\ell := \theta_{\eta_\ell}$ . If  $\eta \in K$  belongs to the  $\ell$ th ball, then, for all  $z \in S$ ,

$$(\eta - \theta_\ell) \cdot z \leq \left( \frac{\delta}{2} - \varepsilon_K \right) a_0 \cdot z, \quad \varepsilon_K := \frac{\rho_K}{2m_0} > 0.$$

Because  $a_0 \cdot z \geq m_0 |z|$ , for every multi-index  $\gamma$  there is a constant  $C_{K,\gamma} < \infty$  such that

$$|z|^{|\gamma|} \leq C_{K,\gamma} \exp(\varepsilon_K a_0 \cdot z), \quad z \in S.$$

Consequently, for every  $\zeta$  with  $\text{Re } \zeta \in K$ ,

$$|z|^{|\gamma|} |e^{\zeta \cdot z}| \leq C_{K,\gamma} \sum_{\ell=1}^N \exp((\theta_\ell + \frac{\delta}{2} a_0) \cdot z), \quad z \in S.$$

For each  $\ell$ , the vector  $\theta_\ell + \frac{\delta}{2} a_0$  belongs to  $\mathcal{E}(\mu)$  by convexity, since both  $\theta_\ell$  and  $\theta_\ell + \delta a_0$  lie in  $\mathcal{E}(\mu)$ . Hence the right-hand side is  $\pi$ -integrable by [proposition 5.6](#). This single integrable majorant on the compact tube over  $K$  justifies differentiation under the integral sign for all derivatives. Thus  $M_\pi$  is holomorphic on  $\mathcal{T}_\Theta$ , and the polynomially weighted moment statement follows from the same finite-cover majorization.  $\square$

**Corollary 5.19** (a Lyapunov ray inside the moment region). *Let  $a \in S_\circ^\vee$  satisfy*

$$a \cdot v_i \leq 0, \quad i = 1, 2, \quad a \cdot \mu < 0,$$

and let  $\pi$  be any stationary distribution for the drift  $\mu$ . Set

$$(5.73) \quad \beta_* := -\frac{2a \cdot \mu}{|a|^2} > 0.$$

Then

$$(5.74) \quad \{\beta a : 0 < \beta < \beta_*\} \subset \mathcal{E}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi).$$

Consequently, for every  $0 < \beta < \beta_*$ ,

$$(5.75) \quad \int_S e^{\beta a \cdot z} \pi(dz) < \infty.$$

Moreover,

$$(5.76) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \log \pi(\{a \cdot z \geq x\}) \leq -\beta_*,$$

and with

$$(5.77) \quad m_a := \min_{u \in S \cap S^1} a \cdot u > 0,$$

one has

$$(5.78) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq -\beta_* m_a.$$

*Proof.* For  $0 < \beta < \beta_*$ ,

$$(\beta a) \cdot \mu + \frac{1}{2} |\beta a|^2 = \beta a \cdot \mu + \frac{1}{2} \beta^2 |a|^2 = \frac{1}{2} \beta |a|^2 (\beta - \beta_*) < 0,$$

so  $\beta a \in \mathcal{E}(\mu)$ . This proves (5.74); (5.75) follows from [proposition 5.6](#). For  $x \geq 0$ , Markov's inequality gives

$$\pi\{a \cdot z \geq x\} \leq e^{-\beta x} \int_S e^{\beta a \cdot z} \pi(dz),$$

and hence

$$\limsup_{x \rightarrow \infty} \frac{1}{x} \log \pi\{a \cdot z \geq x\} \leq -\beta.$$

Letting  $\beta \uparrow \beta_*$  proves (5.76). Since

$$a \cdot z \geq m_a |z| \quad (z \in S),$$

one also has

$$\pi\{|z| \geq r\} \leq \pi\{a \cdot z \geq m_a r\} \leq e^{-\beta m_a r} \int_S e^{\beta a \cdot z} \pi(dz).$$

Thus

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi\{|z| \geq r\} \leq -\beta m_a,$$

and  $\beta \uparrow \beta_*$  proves (5.78). □

**Corollary 5.20** (explicit borderline tail bound when  $\alpha = 1$ ). *Assume*

$$0 < \xi < \pi, \quad \alpha = 1, \quad n_{\mathcal{L}} \cdot \mu < 0,$$

where  $n_{\mathcal{L}}$  is the unit normal from [lemma 4.1](#). Let  $\pi$  be any stationary distribution for the drift  $\mu$ . Then, for every

$$0 < \beta < -2n_{\mathcal{L}} \cdot \mu,$$

one has

$$\int_S e^{\beta n_{\mathcal{L}} \cdot z} \pi(dz) < \infty.$$

Consequently,

$$\limsup_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z \in S : n_{\mathcal{L}} \cdot z \geq x\} \leq 2n_{\mathcal{L}} \cdot \mu < 0.$$

If

$$m_{\mathcal{L}} := \min_{u \in S \cap \mathbb{S}^1} n_{\mathcal{L}} \cdot u > 0,$$

then

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq 2m_{\mathcal{L}} n_{\mathcal{L}} \cdot \mu < 0.$$

*Proof.* By lemma 4.1, the vector  $n_{\mathcal{L}}$  belongs to  $S_{\circ}^{\vee}$  and satisfies  $n_{\mathcal{L}} \cdot v_i = 0$  for  $i = 1, 2$ . Since  $|n_{\mathcal{L}}| = 1$  and  $n_{\mathcal{L}} \cdot \mu < 0$ , corollary 5.19 applied with  $a = n_{\mathcal{L}}$  gives

$$\{\beta n_{\mathcal{L}} : 0 < \beta < -2n_{\mathcal{L}} \cdot \mu\} \subset \mathcal{D}_{\text{exp}}(\pi),$$

which proves the exponential-moment claim. Markov's inequality gives, for each such  $\beta$ ,

$$\pi\{n_{\mathcal{L}} \cdot z \geq x\} \leq e^{-\beta x} \int_S e^{\beta n_{\mathcal{L}} \cdot z} \pi(dz).$$

Letting  $\beta \uparrow -2n_{\mathcal{L}} \cdot \mu$  yields the normal-coordinate logarithmic bound. Finally,  $n_{\mathcal{L}} \cdot z \geq m_{\mathcal{L}}|z|$  on  $S$ , and therefore

$$\pi\{|z| \geq r\} \leq \pi\{n_{\mathcal{L}} \cdot z \geq m_{\mathcal{L}}r\}.$$

The radial bound follows.  $\square$

**Lemma 5.21** (finite-measure solutions of the one-dimensional adjoint equation). *Let  $m \in \mathbb{R}$ , and let  $\nu$  be a finite nonnegative Borel measure on  $(0, \infty)$ . Suppose that*

$$\int_{(0, \infty)} \left( \frac{1}{2} \varphi''(y) + m \varphi'(y) \right) \nu(dy) = 0, \quad \varphi \in C_c^2((0, \infty)).$$

*Then  $\nu$  is absolutely continuous with a smooth density  $q$ . If  $m \neq 0$ , then*

$$q(y) = A + B e^{2my}$$

*for constants  $A, B \in \mathbb{R}$ . If  $m = 0$ , then*

$$q(y) = A + By.$$

*Proof.* The displayed identity says, in the sense of distributions on  $(0, \infty)$ , that

$$\frac{1}{2} \nu'' - m \nu' = 0.$$

Set

$$T := \frac{1}{2} \nu' - m \nu.$$

Then  $T' = 0$ , hence  $T$  is a constant distribution: for some  $C_0 \in \mathbb{R}$ ,

$$T = C_0$$

where the right-hand side denotes the distribution induced by the constant function  $C_0$ . Equivalently,

$$\nu' - 2m\nu = 2C_0.$$

If  $m \neq 0$ , multiply by the smooth integrating factor  $e^{-2my}$ . In distributions,

$$(e^{-2my} \nu)' = 2C_0 e^{-2my}.$$

The right-hand side is a smooth function, so  $e^{-2my} \nu$  differs from an absolutely continuous distribution by a constant distribution. Hence it is itself represented by a smooth function of the form

$$C_1 - \frac{C_0}{m} e^{-2my},$$

and  $\nu$  has density

$$q(y) = C_1 e^{2my} - \frac{C_0}{m} = A + B e^{2my}.$$

If  $m = 0$ , then  $\nu' = 2C_0$ . Thus  $\nu$  differs from the absolutely continuous distribution with density  $2C_0y$  by a constant distribution, and is represented by the smooth affine density

$$q(y) = C_1 + 2C_0y = A + By.$$

This proves the claim and excludes any singular component in  $\nu$ .  $\square$

**Proposition 5.22** (exact normal marginal in the borderline case). *Assume*

$$0 < \xi < \pi, \quad \alpha = 1,$$

and let  $n_{\mathcal{L}}$  be as in [lemma 4.1](#). Set

$$m := n_{\mathcal{L}} \cdot \mu.$$

If  $m \geq 0$ , then no stationary distribution exists. If  $m < 0$  and  $\pi$  is any stationary distribution for the drift  $\mu$ , then the marginal law

$$\nu := \pi \circ (z \mapsto n_{\mathcal{L}} \cdot z)^{-1}$$

of the normal coordinate satisfies

$$(5.79) \quad \nu(dy) = (-2m)e^{2my} \mathbf{1}_{(0,\infty)}(y) dy.$$

In particular,

$$(5.80) \quad \pi\{z \in S : n_{\mathcal{L}} \cdot z \geq x\} = e^{2mx}, \quad x \geq 0.$$

*Proof.* Suppose first that  $\pi$  is a stationary distribution for the drift  $\mu$ , and let

$$\nu := \pi \circ (z \mapsto n_{\mathcal{L}} \cdot z)^{-1}.$$

Let  $\varphi \in C_c^2((0, \infty))$ . Since the support of  $\varphi$  is compact in  $(0, \infty)$ , there is  $\varepsilon_\varphi > 0$  such that  $\varphi = 0$  on  $(0, \varepsilon_\varphi)$ . Extend  $\varphi$  to  $[0, \infty)$  by setting  $\varphi(0) = 0$ ; the extended function is  $C^2$  at the endpoint. Define

$$f(z) := \varphi(n_{\mathcal{L}} \cdot z).$$

The continuous function  $u \mapsto n_{\mathcal{L}} \cdot u$  has a positive minimum on  $S \cap \mathbb{S}^1$ , so  $n_{\mathcal{L}} \cdot z \geq m_{\mathcal{L}}|z|$  on  $S$  for some  $m_{\mathcal{L}} > 0$ . Since  $\text{supp } \varphi \Subset (0, \infty)$ , the support of  $f$  is compact in  $S \setminus \{0\}$ . In particular  $f \in C_b^2(S)$  and is constant in a neighborhood of the vertex. Moreover, by [lemma 4.1](#),

$$n_{\mathcal{L}} \cdot v_i = 0, \quad i = 1, 2.$$

Therefore

$$D_i f(z) = \varphi'(n_{\mathcal{L}} \cdot z) n_{\mathcal{L}} \cdot v_i = 0,$$

so both  $f$  and  $-f$  are admissible. Applying the stationary inequality to both signs gives equality:

$$\int_S \mathcal{L}f(z) \pi(dz) = 0.$$

Now

$$\mathcal{L}f(z) = \frac{1}{2}\varphi''(n_{\mathcal{L}} \cdot z) + (n_{\mathcal{L}} \cdot \mu)\varphi'(n_{\mathcal{L}} \cdot z) = \frac{1}{2}\varphi''(n_{\mathcal{L}} \cdot z) + m\varphi'(n_{\mathcal{L}} \cdot z).$$

Thus the marginal measure  $\nu$  satisfies

$$(5.81) \quad \int_{(0,\infty)} \left( \frac{1}{2}\varphi''(y) + m\varphi'(y) \right) \nu(dy) = 0, \quad \varphi \in C_c^2((0, \infty)).$$

By [lemma 5.21](#),  $\nu$  has a smooth density  $q$  on  $(0, \infty)$ . Moreover

$$q(y) = A + Be^{2my}$$

if  $m \neq 0$ , and  $q(y) = A + By$  if  $m = 0$ . Impose finiteness and nonnegativity of the probability measure  $\nu$ . If  $m > 0$ , then continuity and nonnegativity of  $A + Be^{2my}$  rule out  $B < 0$ , while  $B > 0$  gives a nonintegrable growing exponential tail. Hence  $B = 0$ , and the remaining constant density is finite only if  $A = 0$ , contradicting total mass one. If  $m = 0$ , nonnegativity of  $A + By$  rules out  $B < 0$ , while  $B > 0$  is nonintegrable; hence  $B = 0$ , and finiteness again forces  $A = 0$ , a

contradiction. If  $m < 0$ , finiteness and nonnegativity force the constant term to vanish:  $A > 0$  is nonintegrable and  $A < 0$  makes the density eventually negative, so  $A = 0$ . Nonnegativity and positive total mass then force  $B > 0$ . Because  $\pi(\{0\}) = 0$  and  $n_{\mathcal{L}} \cdot z > 0$  on  $S \setminus \{0\}$ , the marginal  $\nu$  has total mass one. Therefore

$$1 = \int_0^\infty B e^{2my} dy = \frac{B}{-2m},$$

so  $B = -2m$ . This proves (5.79), and direct integration gives

$$\nu([x, \infty)) = \int_x^\infty (-2m) e^{2my} dy = e^{2mx}, \quad x \geq 0,$$

which is (5.80). The cases  $m > 0$  and  $m = 0$  treated above prove nonexistence whenever  $m \geq 0$ .  $\square$

**Corollary 5.23** (optimal normal exponential moments when  $\alpha = 1$ ). *Assume*

$$0 < \xi < \pi, \quad \alpha = 1,$$

and let  $n_{\mathcal{L}}$  be as in lemma 4.1. Let  $\pi$  be a stationary distribution for the drift  $\mu$  and set

$$m := n_{\mathcal{L}} \cdot \mu < 0.$$

Then the normal-coordinate moment generating function is explicit:

$$(5.82) \quad \int_S e^{\beta n_{\mathcal{L}} \cdot z} \pi(dz) = \frac{-2m}{-2m - \beta}, \quad \beta < -2m.$$

For  $\beta \geq -2m$ ,

$$(5.83) \quad \int_S e^{\beta n_{\mathcal{L}} \cdot z} \pi(dz) = \infty.$$

Consequently,

$$(5.84) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z \in S : n_{\mathcal{L}} \cdot z \geq x\} = 2m.$$

Moreover, since  $n_{\mathcal{L}}$  is a unit vector and

$$m_{\mathcal{L}} := \min_{u \in S \cap \mathbb{S}^1} n_{\mathcal{L}} \cdot u > 0,$$

one has the radial logarithmic bracket

$$(5.85) \quad 2m \leq \liminf_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq 2mm_{\mathcal{L}}.$$

*Proof.* By proposition 5.22, the law of  $Y := n_{\mathcal{L}} \cdot Z$  under  $\pi$  is

$$(-2m)e^{2my} \mathbf{1}_{(0, \infty)}(y) dy.$$

Therefore, for  $\beta < -2m$ ,

$$\int_S e^{\beta n_{\mathcal{L}} \cdot z} \pi(dz) = \int_0^\infty e^{\beta y} (-2m) e^{2my} dy = \frac{-2m}{-2m - \beta},$$

whereas the same integral diverges for  $\beta \geq -2m$ . The exact tail formula

$$\pi\{n_{\mathcal{L}} \cdot z \geq x\} = e^{2mx}$$

gives (5.84). Finally,  $n_{\mathcal{L}} \cdot z \leq |z|$  implies

$$\{n_{\mathcal{L}} \cdot z \geq r\} \subseteq \{|z| \geq r\},$$

while  $n_{\mathcal{L}} \cdot z \geq m_{\mathcal{L}}|z|$  implies

$$\{|z| \geq r\} \subseteq \{n_{\mathcal{L}} \cdot z \geq m_{\mathcal{L}}r\}.$$

Using the exact normal tail on both inclusions gives (5.85).  $\square$

**Proposition 5.24** (directional upper logarithmic rates). *Assume  $\alpha < 2$ , let  $\pi$  be a stationary distribution for the drift  $\mu$ , and let  $c \in S^\vee \cap \mathbb{S}^1$ . Define*

$$(5.86) \quad \Pi_c(x) := \pi(\{z \in S : c \cdot z \geq x\}),$$

and

$$(5.87) \quad \alpha_c^{\text{in}}(\mu) := \sup\left\{\ell \geq 0 : \exists \vartheta \in \mathcal{E}(\mu) \text{ with } \vartheta - \ell c \in S^\vee\right\},$$

with the convention that this supremum is 0 when the set on the right-hand side is empty. Then

$$(5.88) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \log \Pi_c(x) \leq -\alpha_c^{\text{in}}(\mu).$$

*Proof.* If the admissible set in (5.87) is empty, then  $\alpha_c^{\text{in}}(\mu) = 0$ , and the asserted upper bound follows from  $\Pi_c(x) \leq 1$ . Otherwise, let  $\vartheta \in \mathcal{E}(\mu)$  and  $\ell \geq 0$  satisfy  $\vartheta - \ell c \in S^\vee$ . Then

$$(5.89) \quad \vartheta \cdot z \geq \ell c \cdot z \quad (z \in S).$$

Thus on  $\{c \cdot z \geq x\}$  one has  $\vartheta \cdot z \geq \ell x$ , so by Markov's inequality and [proposition 5.6](#),

$$(5.90) \quad \Pi_c(x) \leq e^{-\ell x} \int_S e^{\vartheta \cdot z} \pi(dz).$$

Hence

$$(5.91) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \log \Pi_c(x) \leq -\ell.$$

Taking the supremum over all admissible pairs  $(\ell, \vartheta)$  yields (5.88).  $\square$

**Proposition 5.25** (variational representation of the inner directional rate). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ , and let  $c \in S^\vee \cap \mathbb{S}^1$ . For  $\vartheta \in \mathcal{E}(\mu)$  define*

$$(5.92) \quad \rho_c(\vartheta) := \min_{u \in S \cap \mathbb{S}^1} \frac{\vartheta \cdot u}{c \cdot u}.$$

Then  $0 < \rho_c(\vartheta) < \infty$ , and the inner directional rate from [proposition 5.24](#) satisfies

$$(5.93) \quad \alpha_c^{\text{in}}(\mu) = \sup_{\vartheta \in \mathcal{E}(\mu)} \rho_c(\vartheta).$$

In particular,  $\alpha_c^{\text{in}}(\mu) > 0$  for every  $c \in S^\vee \cap \mathbb{S}^1$  whenever  $\mathcal{E}(\mu) \neq \emptyset$ .

*Proof.* Because  $c \in S^\vee$ , one has  $c \cdot u > 0$  for every  $u \in S \cap \mathbb{S}^1$ . Because  $\vartheta \in S^\vee$ , the numerator is also strictly positive on the same compact set. Hence  $\rho_c(\vartheta)$  is a finite positive number.

For fixed  $\vartheta \in \mathcal{E}(\mu)$  and  $\ell \geq 0$ , the condition

$$\vartheta - \ell c \in S^\vee$$

is equivalent to

$$(\vartheta - \ell c) \cdot u \geq 0 \quad \text{for all } u \in S \cap \mathbb{S}^1.$$

Since  $c \cdot u > 0$ , this is equivalent to

$$\ell \leq \inf_{u \in S \cap \mathbb{S}^1} \frac{\vartheta \cdot u}{c \cdot u} = \rho_c(\vartheta).$$

Thus, for each fixed  $\vartheta \in \mathcal{E}(\mu)$ , the largest admissible  $\ell$  in the definition of  $\alpha_c^{\text{in}}(\mu)$  is precisely  $\rho_c(\vartheta)$ . Taking the supremum over  $\vartheta \in \mathcal{E}(\mu)$  gives (5.93).  $\square$

**Proposition 5.26** (boundary attainment of the inner decay variational problem). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$  and let  $c \in S^\vee \cap \mathbb{S}^1$ . Then*

$$(5.94) \quad \alpha_c^{\text{in}}(\mu) = \max\left\{\rho_c(\vartheta) : \vartheta \in \partial_Q \mathcal{E}(\mu)\right\}.$$

For the radial objective, define

$$\alpha_{\text{rad}}^{\text{in}}(\mu) := \sup_{\vartheta \in \mathcal{E}(\mu)} \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u,$$

then

$$(5.95) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) = \max \left\{ \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u : \vartheta \in \partial_Q \mathcal{E}(\mu) \right\}.$$

*Proof.* By lemma 5.5,  $\overline{\mathcal{E}(\mu)}$  is compact and  $\partial_Q \mathcal{E}(\mu)$  is nonempty and compact.

Let  $\vartheta \in \mathcal{E}(\mu)$ . Then  $Q_\mu(\vartheta) < 0$ , so necessarily  $\vartheta \cdot \mu < 0$ . Set

$$t_* := -\frac{2\vartheta \cdot \mu}{|\vartheta|^2}.$$

Since  $Q_\mu(\vartheta) < 0$ , we have  $t_* > 1$ , and a direct calculation gives

$$Q_\mu(t_*\vartheta) = 0.$$

Moreover, for every  $1 < t < t_*$ , the vector  $t\vartheta$  still satisfies the dual and oblique inequalities defining  $\mathcal{E}(\mu)$  and satisfies  $Q_\mu(t\vartheta) < 0$ . Thus  $t\vartheta \in \mathcal{E}(\mu)$  for  $1 < t < t_*$ , while

$$t_*\vartheta \in \partial_Q \mathcal{E}(\mu).$$

The formula defining  $\rho_c$  extends continuously from  $S_\circ^\vee$  to  $S^\vee$ , because the denominators  $c \cdot u$  are bounded away from zero on  $S \cap \mathbb{S}^1$ . The extended function is positively homogeneous of degree one:

$$\rho_c(t\vartheta) = t\rho_c(\vartheta), \quad t > 0.$$

For every  $\vartheta \in \mathcal{E}(\mu)$ , the radially pushed point  $t_*\vartheta$  belongs to  $\partial_Q \mathcal{E}(\mu)$  and satisfies  $\rho_c(t_*\vartheta) > \rho_c(\vartheta)$ . Hence

$$\sup_{\vartheta \in \mathcal{E}(\mu)} \rho_c(\vartheta) \leq \sup_{\eta \in \partial_Q \mathcal{E}(\mu)} \rho_c(\eta).$$

For the reverse inequality, let  $\eta \in \partial_Q \mathcal{E}(\mu)$ . Since  $\eta \in \overline{\mathcal{E}(\mu)}$ , choose  $\eta_n \in \mathcal{E}(\mu)$  with  $\eta_n \rightarrow \eta$ . Continuity of the extended function  $\rho_c$  gives

$$\rho_c(\eta) = \lim_{n \rightarrow \infty} \rho_c(\eta_n) \leq \sup_{\vartheta \in \mathcal{E}(\mu)} \rho_c(\vartheta).$$

Taking the supremum over  $\eta \in \partial_Q \mathcal{E}(\mu)$  gives equality of the two suprema. Since  $\partial_Q \mathcal{E}(\mu)$  is compact and  $\rho_c$  is continuous on it, the supremum is attained. This proves (5.94).

For the radial objective, define

$$m_{\text{rad}}(\vartheta) := \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u.$$

This map is continuous and positively homogeneous. For each  $\vartheta \in \mathcal{E}(\mu)$ , the same radial factor  $t_* > 1$  constructed above satisfies

$$m_{\text{rad}}(t_*\vartheta) = t_* m_{\text{rad}}(\vartheta) > m_{\text{rad}}(\vartheta), \quad t_*\vartheta \in \partial_Q \mathcal{E}(\mu).$$

Hence

$$\sup_{\vartheta \in \mathcal{E}(\mu)} m_{\text{rad}}(\vartheta) \leq \sup_{\eta \in \partial_Q \mathcal{E}(\mu)} m_{\text{rad}}(\eta).$$

Conversely, for  $\eta \in \partial_Q \mathcal{E}(\mu)$  choose  $\eta_n \in \mathcal{E}(\mu)$  with  $\eta_n \rightarrow \eta$ . Continuity gives

$$m_{\text{rad}}(\eta) = \lim_{n \rightarrow \infty} m_{\text{rad}}(\eta_n) \leq \sup_{\vartheta \in \mathcal{E}(\mu)} m_{\text{rad}}(\vartheta).$$

The two suprema are therefore equal, and compactness of  $\partial_Q \mathcal{E}(\mu)$  makes the boundary supremum a maximum. This proves (5.95).  $\square$

*Remark 5.27.* Proposition 5.26 is the analogue, for the inner exponential-moment region available in the present submartingale setting, of the usual principle in two-dimensional reflected models that logarithmic directional decay rates are read from the boundary of a convergence domain.

The result is an inner-bound statement: it identifies the relevant closed quadratic boundary of the exponential-moment region obtained by the Lyapunov method. The boundary points are variational support points; exact asymptotics may require additional information on the closed transform domain.

**Proposition 5.28** (endpoint formula in a two-dimensional wedge). *Assume  $0 < \xi < \pi$ . Let*

$$\omega_1 := u_1 = (1, 0), \quad \omega_2 := u_2 = (\cos \xi, \sin \xi)$$

*be the two generating unit vectors of the wedge. If  $c \in S_\circ^\vee$  and  $\vartheta \in S^\vee$ , define*

$$\rho_c(\vartheta) := \min_{u \in S \cap \mathbb{S}^1} \frac{\vartheta \cdot u}{c \cdot u}.$$

*Then*

$$\rho_c(\vartheta) = \min \left\{ \frac{\vartheta \cdot \omega_1}{c \cdot \omega_1}, \frac{\vartheta \cdot \omega_2}{c \cdot \omega_2} \right\}.$$

*Consequently, if  $\mathcal{E}(\mu) \neq \emptyset$ , then*

$$\alpha_c^{\text{in}}(\mu) = \sup_{\vartheta \in \mathcal{E}(\mu)} \min \left\{ \frac{\vartheta \cdot \omega_1}{c \cdot \omega_1}, \frac{\vartheta \cdot \omega_2}{c \cdot \omega_2} \right\}.$$

*If  $\mathcal{E}(\mu) = \emptyset$ , then  $\alpha_c^{\text{in}}(\mu) = 0$  by the convention in [proposition 5.24](#).*

*Moreover,*

$$\min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u = \min\{\vartheta \cdot \omega_1, \vartheta \cdot \omega_2\}.$$

*Proof.* Every  $u \in S \cap \mathbb{S}^1$  can be written as  $u(\phi) = (\cos \phi, \sin \phi)$  for  $0 \leq \phi \leq \xi$ . Write

$$R(\phi) := \frac{\vartheta \cdot u(\phi)}{c \cdot u(\phi)}.$$

Since  $c \in S_\circ^\vee$ , the denominator is strictly positive on  $[0, \xi]$ . A direct differentiation gives

$$R'(\phi) = \frac{\vartheta_2 c_1 - \vartheta_1 c_2}{(c \cdot u(\phi))^2},$$

which has constant sign. Thus  $R$  is monotone on the whole angular interval, and its minimum is attained at one of the two endpoints  $\phi = 0$  or  $\phi = \xi$ . This proves the formula for  $\rho_c(\vartheta)$ . If  $\mathcal{E}(\mu) \neq \emptyset$ , the formula for  $\alpha_c^{\text{in}}(\mu)$  follows from [proposition 5.25](#); if  $\mathcal{E}(\mu) = \emptyset$ , the convention in [proposition 5.24](#) gives  $\alpha_c^{\text{in}}(\mu) = 0$ .

For the last claim, the assertion is trivial if  $\vartheta = 0$ . Assume  $\vartheta \neq 0$ . The map  $\phi \mapsto \vartheta \cdot u(\phi)$  has derivative  $\vartheta \cdot u'(\phi)$ . At an interior critical point,  $u(\phi)$  is parallel or antiparallel to  $\vartheta$ . The antiparallel alternative is impossible for nonzero  $\vartheta \in S^\vee$  and  $u(\phi) \in S \cap \mathbb{S}^1$ , because it would give  $\vartheta \cdot u(\phi) < 0$ . Hence an interior critical point can only occur when  $u(\phi) = \vartheta/|\vartheta|$ , where the function attains its maximum  $|\vartheta|$ , not a local minimum. Therefore its minimum on the compact interval  $[0, \xi]$  is also attained at an endpoint.  $\square$

**Proposition 5.29** (one-dimensional parametrization of the inner variational boundary). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ . For  $\omega \in \mathbb{R}/2\pi\mathbb{Z}$ , set*

$$e(\omega) := (\cos \omega, \sin \omega), \quad \vartheta_\mu(\omega) := -\mu + |\mu|e(\omega).$$

*Define*

$$\mathcal{I}_\mu := \left\{ \omega \in \mathbb{R}/2\pi\mathbb{Z} : \vartheta_\mu(\omega) \cdot u_j \geq 0 \ (j = 1, 2), \quad \vartheta_\mu(\omega) \cdot v_i \leq 0 \ (i = 1, 2) \right\}.$$

*The set  $\mathcal{I}_\mu$  is nonempty and compact, and*

$$\partial_Q \mathcal{E}(\mu) = \{\vartheta_\mu(\omega) : \omega \in \mathcal{I}_\mu\}.$$

Consequently, for each  $c \in S^\vee \cap \mathbb{S}^1$ ,

$$\alpha_c^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min \left\{ \frac{\vartheta_\mu(\omega) \cdot u_1}{c \cdot u_1}, \frac{\vartheta_\mu(\omega) \cdot u_2}{c \cdot u_2} \right\}.$$

Likewise,

$$\alpha_{\text{rad}}^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min \{ \vartheta_\mu(\omega) \cdot u_1, \vartheta_\mu(\omega) \cdot u_2 \}.$$

*Proof.* The equation  $Q_\mu(\vartheta) = 0$  is equivalent to

$$|\vartheta + \mu|^2 = |\mu|^2.$$

Since  $\mathcal{E}(\mu) \neq \emptyset$ , necessarily  $\mu \neq 0$ , and the quadratic boundary is exactly the circle

$$\vartheta = -\mu + |\mu|e(\omega), \quad \omega \in \mathbb{R}/2\pi\mathbb{Z}.$$

The closure  $\overline{\mathcal{E}(\mu)}$  is contained in the set determined by the closed constraints

$$\vartheta \in S^\vee, \quad \vartheta \cdot v_i \leq 0, \quad Q_\mu(\vartheta) \leq 0.$$

The set  $\mathcal{I}_\mu$  is closed in  $\mathbb{R}/2\pi\mathbb{Z}$ , hence compact. It is nonempty: choose  $\vartheta_0 \in \mathcal{E}(\mu)$ , and let  $t_* > 1$  be the positive number for which  $Q_\mu(t_*\vartheta_0) = 0$ . Then  $t_*\vartheta_0$  satisfies the closed dual and oblique constraints and therefore has the form  $\vartheta_\mu(\omega_*)$  for some  $\omega_* \in \mathcal{I}_\mu$ .

We next identify the boundary part of the closure. If  $\vartheta \in \overline{\mathcal{E}(\mu)}$  and  $Q_\mu(\vartheta) = 0$ , then  $\vartheta$  satisfies the closed dual and oblique constraints above. Since the quadratic boundary is the circle  $\vartheta = -\mu + |\mu|e(\omega)$ , this point equals  $\vartheta_\mu(\omega)$  for some  $\omega \in \mathcal{I}_\mu$ . Conversely, let  $\vartheta = \vartheta_\mu(\omega)$  with  $\omega \in \mathcal{I}_\mu$ . Fix  $\vartheta_0 \in \mathcal{E}(\mu)$ . For every  $0 < \delta < 1$ ,

$$\vartheta_\delta := (1 - \delta)\vartheta + \delta\vartheta_0$$

belongs to  $S^\vee$ , because  $\vartheta \in S^\vee$  and  $\vartheta_0 \in S^\vee$ . The oblique inequalities remain nonpositive by linearity. Finally, since  $Q_\mu(\vartheta) = 0$  and  $Q_\mu(\vartheta_0) < 0$ , strict convexity of  $Q_\mu$  gives

$$Q_\mu(\vartheta_\delta) \leq (1 - \delta)Q_\mu(\vartheta) + \delta Q_\mu(\vartheta_0) - \frac{1}{2}\delta(1 - \delta)|\vartheta - \vartheta_0|^2 < 0.$$

Hence  $\vartheta_\delta \in \mathcal{E}(\mu)$  and  $\vartheta_\delta \rightarrow \vartheta$ . This proves the boundary identity.

Combining [proposition 5.26](#) with [proposition 5.28](#) and the identity  $\partial_Q \mathcal{E}(\mu) = \{\vartheta_\mu(\omega) : \omega \in \mathcal{I}_\mu\}$  gives

$$\alpha_c^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min \left\{ \frac{\vartheta_\mu(\omega) \cdot u_1}{c \cdot u_1}, \frac{\vartheta_\mu(\omega) \cdot u_2}{c \cdot u_2} \right\}.$$

For the radial objective, [proposition 5.26](#) gives

$$\alpha_{\text{rad}}^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min_{u \in S \cap \mathbb{S}^1} \vartheta_\mu(\omega) \cdot u.$$

The endpoint identity in [proposition 5.28](#) converts the inner minimum to  $\min\{\vartheta_\mu(\omega) \cdot u_1, \vartheta_\mu(\omega) \cdot u_2\}$ , proving the stated radial formula.  $\square$

**Proposition 5.30** (variational form of angular-sector exponents). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ . Let  $C \subset S \cap \mathbb{S}^1$  be nonempty and compact. Define*

$$I_C^{\text{in}}(\mu) := \sup_{\vartheta \in \mathcal{E}(\mu)} \inf_{u \in C} \vartheta \cdot u.$$

Then

$$(5.96) \quad I_C^{\text{in}}(\mu) = \max \left\{ \inf_{u \in C} \vartheta \cdot u : \vartheta \in \partial_Q \mathcal{E}(\mu) \right\}.$$

Equivalently, with the one-dimensional parametrization

$$\vartheta_\mu(\omega) = -\mu + |\mu|(\cos \omega, \sin \omega), \quad \omega \in \mathcal{I}_\mu,$$

one has

$$(5.97) \quad I_C^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \inf_{u \in C} \vartheta_\mu(\omega) \cdot u.$$

If, in addition,  $C$  is an angular interval

$$C = \{(\cos \phi, \sin \phi) : \phi_- \leq \phi \leq \phi_+\} \quad \text{with} \quad 0 \leq \phi_- \leq \phi_+ \leq \xi,$$

then

$$(5.98) \quad \inf_{u \in C} \vartheta \cdot u = \min\{\vartheta \cdot u(\phi_-), \vartheta \cdot u(\phi_+)\},$$

where  $u(\phi) = (\cos \phi, \sin \phi)$ . Consequently,

$$(5.99) \quad I_C^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min\{\vartheta_\mu(\omega) \cdot u(\phi_-), \vartheta_\mu(\omega) \cdot u(\phi_+)\}.$$

*Proof.* The map

$$\vartheta \mapsto m_C(\vartheta) := \inf_{u \in C} \vartheta \cdot u$$

is continuous and positively homogeneous of degree one. Continuity follows because  $C$  is compact and the support function of a compact set is continuous. Fix  $\vartheta \in \mathcal{E}(\mu)$ . Since  $Q_\mu(\vartheta) < 0$ , one has  $\vartheta \cdot \mu < 0$ , and with

$$t_* = -\frac{2\vartheta \cdot \mu}{|\vartheta|^2} > 1,$$

one has  $t\vartheta \in \mathcal{E}(\mu)$  for  $1 < t < t_*$ , and  $t_*\vartheta \in \partial_Q \mathcal{E}(\mu)$ . Since

$$m_C(t\vartheta) = t m_C(\vartheta),$$

every value obtained in  $\mathcal{E}(\mu)$  is dominated by a value on the quadratic boundary. Hence

$$\sup_{\vartheta \in \mathcal{E}(\mu)} m_C(\vartheta) \leq \sup_{\eta \in \partial_Q \mathcal{E}(\mu)} m_C(\eta).$$

Conversely, if  $\eta \in \partial_Q \mathcal{E}(\mu)$ , choose  $\eta_n \in \mathcal{E}(\mu)$  with  $\eta_n \rightarrow \eta$ . Continuity gives

$$m_C(\eta) = \lim_{n \rightarrow \infty} m_C(\eta_n) \leq \sup_{\vartheta \in \mathcal{E}(\mu)} m_C(\vartheta).$$

Thus the boundary and open-region suprema are equal. The compactness of  $\partial_Q \mathcal{E}(\mu)$  gives the maximum in (5.96). Formula (5.97) follows from [proposition 5.29](#).

It remains only to prove the endpoint formula for an angular interval. For fixed  $\vartheta \in S^\vee$ , set

$$g(\phi) := \vartheta \cdot u(\phi), \quad \phi_- \leq \phi \leq \phi_+.$$

If  $\vartheta = 0$ , the assertion is trivial. Otherwise, whenever  $g'$  vanishes at an interior point,  $u(\phi)$  is parallel or antiparallel to  $\vartheta$ . The antiparallel alternative is incompatible with  $\vartheta \in S^\vee \setminus \{0\}$  and  $u(\phi) \in S \cap \mathbb{S}^1$ , since it would make  $\vartheta \cdot u(\phi) < 0$ . Thus an interior critical point can only occur at  $u(\phi) = \vartheta/|\vartheta|$ , where  $g$  attains its maximum  $|\vartheta|$ , not a strict interior minimum. Hence the minimum of  $g$  on the compact interval  $[\phi_-, \phi_+]$  is attained at one of the two endpoints. This proves (5.98), and (5.99) follows by inserting the parametrization (5.97).  $\square$

**Proposition 5.31** (continuity of inner directional and angular-sector rates). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ . Then the map*

$$c \mapsto \alpha_c^{\text{in}}(\mu)$$

*is continuous on  $S_\circ^\vee \cap \mathbb{S}^1$ . More precisely, it is locally uniformly continuous on that open arc.*

*Moreover, for angular intervals*

$$C_{\phi_-, \phi_+} := \{(\cos \phi, \sin \phi) : \phi_- \leq \phi \leq \phi_+\}, \quad 0 \leq \phi_- \leq \phi_+ \leq \xi,$$

*the inner angular-sector exponent*

$$I_{C_{\phi_-, \phi_+}}^{\text{in}}(\mu)$$

is continuous as a function of  $(\phi_-, \phi_+)$  on the closed triangle

$$\{(\phi_-, \phi_+) : 0 \leq \phi_- \leq \phi_+ \leq \xi\}.$$

*Proof.* By [proposition 5.29](#),

$$\alpha_c^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min \left\{ \frac{\vartheta_\mu(\omega) \cdot u_1}{c \cdot u_1}, \frac{\vartheta_\mu(\omega) \cdot u_2}{c \cdot u_2} \right\}.$$

The set  $\mathcal{I}_\mu$  is compact, because it is a closed subset of the circle. Fix  $c_0 \in S_\circ^\vee \cap \mathbb{S}^1$ . Since  $c_0 \cdot u_j > 0$  for  $j = 1, 2$ , there is a neighborhood  $U$  of  $c_0$  in  $S_\circ^\vee \cap \mathbb{S}^1$  and a number  $\eta > 0$  such that

$$c \cdot u_j \geq \eta, \quad c \in U, \quad j = 1, 2.$$

Put

$$M_\mu := \max_{\omega \in \mathcal{I}_\mu} |\vartheta_\mu(\omega)| < \infty.$$

For  $c, c' \in U$ ,  $\omega \in \mathcal{I}_\mu$ , and  $j \in \{1, 2\}$ ,

$$\left| \frac{\vartheta_\mu(\omega) \cdot u_j}{c \cdot u_j} - \frac{\vartheta_\mu(\omega) \cdot u_j}{c' \cdot u_j} \right| = \frac{|\vartheta_\mu(\omega) \cdot u_j| |(c - c') \cdot u_j|}{(c \cdot u_j)(c' \cdot u_j)} \leq \frac{M_\mu}{\eta^2} |c - c'|.$$

Since  $|(\min_j x_j) - (\min_j y_j)| \leq \max_j |x_j - y_j|$ , the function

$$F(c, \omega) := \min \left\{ \frac{\vartheta_\mu(\omega) \cdot u_1}{c \cdot u_1}, \frac{\vartheta_\mu(\omega) \cdot u_2}{c \cdot u_2} \right\}$$

satisfies

$$|F(c, \omega) - F(c', \omega)| \leq \frac{M_\mu}{\eta^2} |c - c'|$$

uniformly in  $\omega$ . Therefore

$$\left| \max_{\omega \in \mathcal{I}_\mu} F(c, \omega) - \max_{\omega \in \mathcal{I}_\mu} F(c', \omega) \right| \leq \frac{M_\mu}{\eta^2} |c - c'|.$$

This proves continuity at  $c_0$ . On every compact subarc of  $S_\circ^\vee \cap \mathbb{S}^1$ , the denominators admit one common positive lower bound  $\eta$ , so the same estimate proves uniform continuity there.

For the angular-sector exponent, [proposition 5.30](#) gives

$$I_{C_{\phi_-, \phi_+}}^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \min \{ \vartheta_\mu(\omega) \cdot u(\phi_-), \vartheta_\mu(\omega) \cdot u(\phi_+) \},$$

where  $u(\phi) = (\cos \phi, \sin \phi)$ . The function

$$(\omega, \phi_-, \phi_+) \mapsto \min \{ \vartheta_\mu(\omega) \cdot u(\phi_-), \vartheta_\mu(\omega) \cdot u(\phi_+) \}$$

is continuous on the compact set

$$\mathcal{I}_\mu \times \{(\phi_-, \phi_+) : 0 \leq \phi_- \leq \phi_+ \leq \xi\}.$$

Taking the maximum over the fixed compact set  $\mathcal{I}_\mu$  preserves continuity. This proves the second claim.  $\square$

**Corollary 5.32** (explicit separator-direction estimate). *Assume*

$$0 < \xi < \pi, \quad 1 \leq \alpha < 2, \quad b \cdot \mu > 0.$$

Let  $\pi$  be any stationary distribution for this drift, and let  $c \in S_\circ^\vee \cap \mathbb{S}^1$ . Then, with

$$\beta_b^* := \frac{2b \cdot \mu}{|b|^2}, \quad \rho_b^{\text{end}}(c) := \min \left\{ \frac{(-b) \cdot \omega_1}{c \cdot \omega_1}, \frac{(-b) \cdot \omega_2}{c \cdot \omega_2} \right\},$$

one has

$$\limsup_{x \rightarrow \infty} \frac{1}{x} \log \pi \{z \in S : c \cdot z \geq x\} \leq -\beta_b^* \rho_b^{\text{end}}(c).$$

In particular, the separator ray gives a completely explicit lower bound on the directional logarithmic decay rate in every dual-cone direction  $c$ .

*Proof.* By the separator property,  $-b \in S_\circ^\vee$  and  $(-b) \cdot v_i \leq 0$ . For every  $0 < \beta < \beta_b^*$ ,  $\vartheta = -\beta b$  belongs to  $\mathcal{E}(\mu)$ . Applying [proposition 5.28](#) to  $\vartheta = -\beta b$  gives

$$\rho_c(-\beta b) = \beta \rho_b^{\text{end}}(c).$$

Then [proposition 5.24](#) yields

$$\limsup_{x \rightarrow \infty} \frac{1}{x} \log \pi\{c \cdot z \geq x\} \leq -\beta \rho_b^{\text{end}}(c).$$

Letting  $\beta \uparrow \beta_b^*$  proves the claim.  $\square$

**Corollary 5.33** (radial inner decay exponent). *Assume  $\alpha < 2$  and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Define*

$$(5.100) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) := \sup_{\vartheta \in \mathcal{E}(\mu)} \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u,$$

with the convention that the supremum is zero if  $\mathcal{E}(\mu) = \emptyset$ . Then

$$(5.101) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq -\alpha_{\text{rad}}^{\text{in}}(\mu).$$

*Proof.* If  $\mathcal{E}(\mu) = \emptyset$ , then  $\alpha_{\text{rad}}^{\text{in}}(\mu) = 0$  and the asserted bound follows from  $\pi(\{|z| \geq r\}) \leq 1$ . Otherwise fix any  $\vartheta \in \mathcal{E}(\mu)$ . By [proposition 5.6](#),

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(\{|z| \geq r\}) \leq -m(\vartheta), \quad m(\vartheta) := \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u.$$

Taking the supremum over  $\vartheta \in \mathcal{E}(\mu)$  gives [\(5.101\)](#).  $\square$

**Proposition 5.34** (angular-sector radial upper bounds). *Assume  $\alpha < 2$  and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Let  $C \subset S \cap \mathbb{S}^1$  be a nonempty compact set of directions and define*

$$I_C^{\text{in}}(\mu) := \sup_{\vartheta \in \mathcal{E}(\mu)} \inf_{u \in C} \vartheta \cdot u,$$

with the convention that the supremum is zero if  $\mathcal{E}(\mu) = \emptyset$ . Then

$$(5.102) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi\{z \in S : |z| \geq r, z/|z| \in C\} \leq -I_C^{\text{in}}(\mu).$$

In particular, taking  $C = S \cap \mathbb{S}^1$  recovers the radial inner decay bound in [corollary 5.33](#).

*Proof.* If  $\mathcal{E}(\mu) = \emptyset$ , then  $I_C^{\text{in}}(\mu) = 0$  and the asserted bound follows from  $\pi\{z \in S : |z| \geq r, z/|z| \in C\} \leq 1$ . Assume  $\mathcal{E}(\mu) \neq \emptyset$ . Fix  $\vartheta \in \mathcal{E}(\mu)$  and set

$$m_C(\vartheta) := \inf_{u \in C} \vartheta \cdot u.$$

Since  $C \subset S \cap \mathbb{S}^1$  is compact and  $\vartheta \in S_\circ^\vee$ , one has  $m_C(\vartheta) > 0$ . On the event

$$\{|z| \geq r, z/|z| \in C\},$$

we have

$$\vartheta \cdot z = |z| \vartheta \cdot (z/|z|) \geq r m_C(\vartheta).$$

Therefore Markov's inequality and [proposition 5.6](#) give

$$\pi\{|z| \geq r, z/|z| \in C\} \leq \exp\{-r m_C(\vartheta)\} \int_S e^{\vartheta \cdot z} \pi(dz).$$

Taking logarithms, dividing by  $r$ , and sending  $r \rightarrow \infty$  yields

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi\{|z| \geq r, z/|z| \in C\} \leq -m_C(\vartheta).$$

Finally take the supremum over  $\vartheta \in \mathcal{E}(\mu)$ . If  $C = S \cap \mathbb{S}^1$ , then  $m_C(\vartheta) = \min_{u \in S \cap \mathbb{S}^1} \vartheta \cdot u$ , giving [corollary 5.33](#).  $\square$

**Proposition 5.35** (compact-set exponential upper bounds). *Assume  $\alpha < 2$  and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Let  $G \subset S \setminus \{0\}$  be a nonempty compact set. Define*

$$J_G^{\text{in}}(\mu) := \sup_{\vartheta \in \mathcal{E}(\mu)} \inf_{z \in G} \vartheta \cdot z,$$

with the convention that the supremum is zero if  $\mathcal{E}(\mu) = \emptyset$ . Then

$$(5.103) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(rG) \leq -J_G^{\text{in}}(\mu),$$

where

$$rG := \{rz : z \in G\}.$$

If  $\mathcal{E}(\mu) \neq \emptyset$ , then the variational value is attained on the quadratic boundary of the inner exponential-moment region:

$$(5.104) \quad J_G^{\text{in}}(\mu) = \max \left\{ \inf_{z \in G} \vartheta \cdot z : \vartheta \in \partial_Q \mathcal{E}(\mu) \right\}.$$

Equivalently, using the circular parametrization

$$\vartheta_\mu(\omega) = -\mu + |\mu|(\cos \omega, \sin \omega), \quad \omega \in \mathcal{I}_\mu,$$

one has

$$(5.105) \quad J_G^{\text{in}}(\mu) = \max_{\omega \in \mathcal{I}_\mu} \inf_{z \in G} \vartheta_\mu(\omega) \cdot z.$$

*Proof.* If  $\mathcal{E}(\mu) = \emptyset$ , then  $J_G^{\text{in}}(\mu) = 0$  and the asserted bound follows from  $\pi(rG) \leq 1$ . Assume  $\mathcal{E}(\mu) \neq \emptyset$ . Fix  $\vartheta \in \mathcal{E}(\mu)$  and set

$$m_G(\vartheta) := \inf_{z \in G} \vartheta \cdot z.$$

Since  $G$  is compact and does not contain the origin, and since  $\vartheta \in S_o^\vee$ , one has  $m_G(\vartheta) > 0$ . On the event  $rG$ ,

$$\vartheta \cdot Z \geq r m_G(\vartheta).$$

Therefore Markov's inequality and the exponential moment inclusion  $\mathcal{E}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi)$  give

$$\pi(rG) \leq \exp\{-r m_G(\vartheta)\} \int_S e^{\vartheta \cdot z} \pi(dz).$$

Taking logarithms, dividing by  $r$ , and sending  $r \rightarrow \infty$  yields

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(rG) \leq -m_G(\vartheta).$$

Taking the supremum over  $\vartheta \in \mathcal{E}(\mu)$  proves (5.103).

It remains to prove the boundary-attainment formula. The map

$$\vartheta \mapsto m_G(\vartheta) = \inf_{z \in G} \vartheta \cdot z$$

is continuous and positively homogeneous of degree one. If  $\vartheta \in \mathcal{E}(\mu)$ , then the radial-pushing argument used in [propositions 5.26](#) and [5.30](#) applies: with

$$t_* = -\frac{2\vartheta \cdot \mu}{|\vartheta|^2} > 1,$$

one has  $t\vartheta \in \mathcal{E}(\mu)$  for  $1 < t < t_*$  and  $t_*\vartheta \in \partial_Q \mathcal{E}(\mu)$ . Since  $m_G(t\vartheta) = t m_G(\vartheta)$ , every value obtained from  $\mathcal{E}(\mu)$  is dominated by a quadratic-boundary value. Conversely, if  $\eta \in \partial_Q \mathcal{E}(\mu)$ , choose  $\eta_n \in \mathcal{E}(\mu)$  with  $\eta_n \rightarrow \eta$ ; then continuity gives

$$m_G(\eta) = \lim_{n \rightarrow \infty} m_G(\eta_n) \leq \sup_{\vartheta \in \mathcal{E}(\mu)} m_G(\vartheta).$$

Hence the boundary and open-region suprema are equal. The set  $\partial_Q \mathcal{E}(\mu)$  is compact, so the maximum in (5.104) is attained. Formula (5.105) follows from the circular parametrization of that boundary.

□

**Proposition 5.36** (support-function representation of the inner rate). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ . Define*

$$(5.106) \quad I_\mu^{\text{in}}(z) := \sup_{\vartheta \in \mathcal{E}(\mu)} \vartheta \cdot z = \sup_{\vartheta \in \overline{\mathcal{E}(\mu)}} \vartheta \cdot z, \quad z \in S.$$

Then  $I_\mu^{\text{in}}$  is finite, convex, continuous, and positively homogeneous of degree one on  $S$ . Moreover,

$$(5.107) \quad I_\mu^{\text{in}}(z) > 0 \quad (z \in S \setminus \{0\}).$$

Finally, with the circular parametrization from [proposition 5.29](#),

$$(5.108) \quad I_\mu^{\text{in}}(z) = \max_{\omega \in \mathcal{I}_\mu} \vartheta_\mu(\omega) \cdot z, \quad z \in S.$$

*Proof.* The set  $\mathcal{E}(\mu)$  is bounded because

$$\vartheta \cdot \mu + \frac{1}{2}|\vartheta|^2 < 0 \iff |\vartheta + \mu|^2 < |\mu|^2.$$

For fixed  $z$ , the map  $\vartheta \mapsto \vartheta \cdot z$  is continuous, and  $\mathcal{E}(\mu)$  is dense in  $\overline{\mathcal{E}(\mu)}$ . This proves the equality of the two suprema in (5.106). Hence the support function  $I_\mu^{\text{in}}$  is finite. For  $z, y \in S$  and  $t \in [0, 1]$ ,

$$\begin{aligned} I_\mu^{\text{in}}(tz + (1-t)y) &= \sup_{\vartheta \in \overline{\mathcal{E}(\mu)}} \{t\vartheta \cdot z + (1-t)\vartheta \cdot y\} \\ &\leq tI_\mu^{\text{in}}(z) + (1-t)I_\mu^{\text{in}}(y), \end{aligned}$$

so the function is convex. For every  $r \geq 0$ ,

$$I_\mu^{\text{in}}(rz) = \sup_{\vartheta \in \overline{\mathcal{E}(\mu)}} r\vartheta \cdot z = rI_\mu^{\text{in}}(z),$$

so it is positively homogeneous of degree one. If

$$M_E := \sup_{\vartheta \in \mathcal{E}(\mu)} |\vartheta| < \infty,$$

then

$$|I_\mu^{\text{in}}(z) - I_\mu^{\text{in}}(y)| \leq M_E |z - y|, \quad z, y \in S,$$

so  $I_\mu^{\text{in}}$  is continuous.

Because  $\mathcal{E}(\mu) \neq \emptyset$ , choose  $\vartheta_0 \in \mathcal{E}(\mu)$ . Since  $\vartheta_0 \in S_\circ^\vee$ , one has  $\vartheta_0 \cdot z > 0$  for every  $z \in S \setminus \{0\}$ , and therefore  $I_\mu^{\text{in}}(z) \geq \vartheta_0 \cdot z > 0$ .

It remains to prove the boundary representation. Fix  $z \in S$ . If  $z = 0$ , both sides of (5.108) are zero. Suppose  $z \neq 0$ . For every  $\vartheta \in \mathcal{E}(\mu)$ , the radial pushing argument used earlier gives

$$t_* := -\frac{2\vartheta \cdot \mu}{|\vartheta|^2} > 1, \quad Q_\mu(t_*\vartheta) = 0, \quad t_*\vartheta \in \partial_Q \mathcal{E}(\mu).$$

Since  $\vartheta \in S_\circ^\vee$  and  $z \in S \setminus \{0\}$ , we have  $\vartheta \cdot z > 0$ , hence

$$t_*\vartheta \cdot z > \vartheta \cdot z.$$

Thus

$$\sup_{\vartheta \in \mathcal{E}(\mu)} \vartheta \cdot z \leq \sup\{\vartheta \cdot z : \vartheta \in \partial_Q \mathcal{E}(\mu)\}.$$

The reverse inequality follows pointwise on the boundary set: if  $\eta \in \partial_Q \mathcal{E}(\mu)$ , then choose  $\eta_n \in \mathcal{E}(\mu)$  with  $\eta_n \rightarrow \eta$ , and continuity gives  $\eta \cdot z = \lim_n \eta_n \cdot z \leq \sup_{\mathcal{E}(\mu)} \vartheta \cdot z$ . The compact boundary set therefore attains the same supremum. By [proposition 5.29](#), this quadratic boundary is exactly

$$\{\vartheta_\mu(\omega) : \omega \in \mathcal{I}_\mu\}.$$

This proves (5.108). □

**Proposition 5.37** (dual variational representation of directional inner rates). *Assume  $0 < \xi < \pi$  and  $\mathcal{E}(\mu) \neq \emptyset$ , and let  $c \in S_\circ^\vee \cap \mathbb{S}^1$ . Then*

$$(5.109) \quad \alpha_c^{\text{in}}(\mu) = \inf\{I_\mu^{\text{in}}(z) : z \in S, c \cdot z \geq 1\} = \inf\{I_\mu^{\text{in}}(z) : z \in S, c \cdot z = 1\}.$$

Consequently, the directional upper rate obtained from the inner exponential-moment region can be read either from the dual-cone formula in [proposition 5.24](#) or from the inner support function  $I_\mu^{\text{in}}$  on the cross-section  $\{z \in S : c \cdot z = 1\}$ .

*Proof.* Set

$$C_c := \{z \in S : c \cdot z = 1\}.$$

Because  $c \in S_\circ^\vee$ , this set is nonempty, compact, and convex. If  $z \in S$  and  $s := c \cdot z \geq 1$ , then  $y := z/s \in C_c$  and positive homogeneity gives  $I_\mu^{\text{in}}(z) = sI_\mu^{\text{in}}(y) \geq I_\mu^{\text{in}}(y)$ . Therefore the infimum over  $\{z \in S : c \cdot z \geq 1\}$  equals the infimum over  $C_c$ . Hence it is enough to prove

$$(5.110) \quad \alpha_c^{\text{in}}(\mu) = \inf_{z \in C_c} I_\mu^{\text{in}}(z).$$

Taking the closure of  $\mathcal{E}(\mu)$  does not change the support function, so

$$I_\mu^{\text{in}}(z) = \sup_{\vartheta \in \overline{\mathcal{E}(\mu)}} \vartheta \cdot z.$$

The set  $\overline{\mathcal{E}(\mu)}$  is nonempty, compact, and convex, and  $C_c$  is nonempty, compact, and convex. Therefore Sion's minimax theorem applies to the bilinear map  $(\vartheta, z) \mapsto \vartheta \cdot z$ , giving

$$(5.111) \quad \inf_{z \in C_c} I_\mu^{\text{in}}(z) = \inf_{z \in C_c} \sup_{\vartheta \in \overline{\mathcal{E}(\mu)}} \vartheta \cdot z = \sup_{\vartheta \in \overline{\mathcal{E}(\mu)}} \inf_{z \in C_c} \vartheta \cdot z.$$

For fixed  $\vartheta \in \overline{\mathcal{E}(\mu)}$ , the quantity  $\inf_{z \in C_c} \vartheta \cdot z$  is the largest  $\ell$  such that

$$\vartheta \cdot z \geq \ell c \cdot z \quad (z \in S),$$

which is equivalent to

$$\vartheta - \ell c \in S^\vee.$$

Equivalently,

$$(5.112) \quad \inf_{z \in C_c} \vartheta \cdot z = \rho_c(\vartheta) := \sup\{\ell \geq 0 : \vartheta - \ell c \in S^\vee\}.$$

Because  $0 < \xi < \pi$  and  $c \in S_\circ^\vee$ , the continuous function  $u \mapsto c \cdot u$  is bounded below by a positive constant on the compact angular interval  $S \cap \mathbb{S}^1$ . Therefore

$$(5.113) \quad \rho_c(\vartheta) = \min_{u \in S \cap \mathbb{S}^1} \frac{\vartheta \cdot u}{c \cdot u},$$

and the right-hand side is continuous in  $\vartheta$  as the minimum of a continuous function over a compact set. Since  $\mathcal{E}(\mu)$  is dense in its closure,  $\sup_{\overline{\mathcal{E}(\mu)}} \rho_c = \sup_{\mathcal{E}(\mu)} \rho_c$ . Hence the right side of [\(5.111\)](#) is exactly  $\alpha_c^{\text{in}}(\mu)$ . This proves [\(5.110\)](#), and therefore [\(5.109\)](#).  $\square$

**Proposition 5.38** (finite-cover compact-set upper bound). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Let  $G \subset S \setminus \{0\}$  be a nonempty compact set. Let  $I_\mu^{\text{in}}$  denote the inner support function from [proposition 5.36](#), with the convention that  $I_\mu^{\text{in}} \equiv 0$  if  $\mathcal{E}(\mu) = \emptyset$ . Then*

$$(5.114) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(rG) \leq - \inf_{z \in G} I_\mu^{\text{in}}(z).$$

*In particular, whenever  $\mathcal{E}(\mu) \neq \emptyset$ , the right-hand side is strictly negative.*

*Proof.* If  $\mathcal{E}(\mu) = \emptyset$ , the estimate gives only the trivial upper bound 0, so assume  $\mathcal{E}(\mu) \neq \emptyset$ . Set

$$\gamma_G := \inf_{z \in G} I_\mu^{\text{in}}(z).$$

Choose  $\vartheta_0 \in \mathcal{E}(\mu)$ . Since  $\vartheta_0 \in S_0^\vee$  and  $G \subset S \setminus \{0\}$  is compact,  $\inf_{z \in G} \vartheta_0 \cdot z > 0$ . Hence  $\gamma_G \geq \inf_{z \in G} \vartheta_0 \cdot z > 0$ .

Fix  $\eta > 0$ . For every  $z \in G$ , by the definition of the supremum there exists  $\vartheta_z \in \mathcal{E}(\mu)$  such that

$$\vartheta_z \cdot z \geq \gamma_G - \eta.$$

By continuity, there is a relatively open neighborhood  $U_z$  of  $z$  in  $G$  such that

$$\vartheta_z \cdot y \geq \gamma_G - 2\eta \quad (y \in U_z).$$

Choose a finite subcover  $G \subset \bigcup_{j=1}^N U_{z_j}$  and write  $\vartheta_j := \vartheta_{z_j}$ . Then

$$rG \subset \bigcup_{j=1}^N rU_{z_j}.$$

For  $Z \in rU_{z_j}$ , one has

$$\vartheta_j \cdot Z \geq r(\gamma_G - 2\eta).$$

Thus Markov's inequality and  $\vartheta_j \in \mathcal{E}(\mu) \subset \mathcal{D}_{\exp}(\pi)$  give

$$\pi(rU_{z_j}) \leq \exp\{-r(\gamma_G - 2\eta)\} \int_S e^{\vartheta_j \cdot z} \pi(dz).$$

Summing over  $j$  yields

$$\pi(rG) \leq \exp\{-r(\gamma_G - 2\eta)\} \sum_{j=1}^N \int_S e^{\vartheta_j \cdot z} \pi(dz).$$

Taking logarithms, dividing by  $r$ , and sending  $r \rightarrow \infty$  gives

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(rG) \leq -\gamma_G + 2\eta.$$

Letting  $\eta \downarrow 0$  proves (5.114). □

**Proposition 5.39** (large-deviation upper bound from the inner support function). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , let  $\pi$  be a stationary distribution for the drift  $\mu$ , and suppose that  $\mathcal{E}(\mu) \neq \emptyset$ . For  $r > 0$  define the scaled measures*

$$\Pi_r(F) := \pi(rF), \quad F \subset S.$$

Then for every closed set  $F \subset S$ ,

$$(5.115) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \Pi_r(F) \leq - \inf_{z \in F} I_\mu^{\text{in}}(z),$$

with the standard conventions  $\inf \emptyset = +\infty$  and  $\log 0 = -\infty$ , where  $I_\mu^{\text{in}}$  is the inner support function from proposition 5.36. In particular, this is the closed-set large-deviation upper bound with good rate function  $I_\mu^{\text{in}}$ ; no matching lower bound is asserted.

*Proof.* By proposition 5.36,  $I_\mu^{\text{in}}$  is continuous, positively homogeneous, and strictly positive on  $S \setminus \{0\}$ . Its level sets are compact: indeed, by continuity on the compact set  $S \cap \mathbb{S}^1$ ,

$$m_I := \min_{u \in S \cap \mathbb{S}^1} I_\mu^{\text{in}}(u) > 0,$$

and therefore

$$I_\mu^{\text{in}}(z) \geq m_I |z|, \quad z \in S.$$

If  $F = \emptyset$ , then  $\Pi_r(F) = 0$  for all  $r$ , and the asserted inequality holds under the stated conventions. Let  $F \subset S$  be nonempty and closed and set

$$I_F := \inf_{z \in F} I_\mu^{\text{in}}(z).$$

If  $I_F = 0$ , the claim is trivial because  $\Pi_r(F) \leq 1$ . Assume  $I_F > 0$ . Then  $0 \notin F$ . Fix  $z_0 \in F$  and choose  $R > |z_0|$ ; increasing  $R$  later if necessary preserves this inequality. For such an  $R$ ,

$$F_R := F \cap \overline{B_R}$$

is a nonempty compact subset of  $S \setminus \{0\}$ .

For such  $R$ ,

$$\Pi_r(F) \leq \pi(rF_R) + \pi\{|Z| \geq rR\}.$$

By [proposition 5.38](#),

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi(rF_R) \leq - \inf_{z \in F_R} I_\mu^{\text{in}}(z) \leq -I_F.$$

On the other hand, [corollary 5.33](#) gives

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \pi\{|Z| \geq rR\} \leq -R\alpha_{\text{rad}}^{\text{in}}(\mu),$$

where  $\alpha_{\text{rad}}^{\text{in}}(\mu) > 0$  because  $\mathcal{E}(\mu) \neq \emptyset$ . Since the preceding estimates hold for every  $R > |z_0|$ , choose  $R$  also large enough that  $R\alpha_{\text{rad}}^{\text{in}}(\mu) > I_F$ . The elementary logarithmic bound

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log(A_r + B_r) \leq \max \left\{ \limsup_{r \rightarrow \infty} \frac{1}{r} \log A_r, \limsup_{r \rightarrow \infty} \frac{1}{r} \log B_r \right\}$$

for nonnegative sequences, with the convention  $\log 0 = -\infty$ , gives

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \Pi_r(F) \leq \max\{-I_F, -R\alpha_{\text{rad}}^{\text{in}}(\mu)\} = -I_F.$$

This proves [\(5.115\)](#). □

**Proposition 5.40** (Laplace upper bound for bounded upper-semicontinuous test functions). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , let  $\pi$  be a stationary distribution for the drift  $\mu$ , and suppose that  $\mathcal{E}(\mu) \neq \emptyset$ . Let  $I_\mu^{\text{in}}$  be the inner support function from [proposition 5.36](#). For  $r > 0$  let*

$$\Pi_r(F) := \pi(rF), \quad F \subset S.$$

If  $g : S \rightarrow \mathbb{R}$  is bounded and upper semicontinuous, then

$$(5.116) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_S e^{rg(z)} \Pi_r(dz) \leq \sup_{z \in S} \{g(z) - I_\mu^{\text{in}}(z)\}.$$

Equivalently,

$$(5.117) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_S e^{rg(z/r)} \pi(dz) \leq \sup_{z \in S} \{g(z) - I_\mu^{\text{in}}(z)\}.$$

*Proof.* Write

$$\mathcal{B}_g := \sup_{z \in S} \{g(z) - I_\mu^{\text{in}}(z)\}, \quad M_g := \sup_{z \in S} g(z) < \infty.$$

The function  $I_\mu^{\text{in}}$  is continuous, positively homogeneous, and strictly positive on  $S \setminus \{0\}$  by [proposition 5.36](#). Hence, with

$$m_I := \min_{u \in S \cap \mathbb{S}^1} I_\mu^{\text{in}}(u) > 0,$$

one has

$$(5.118) \quad I_\mu^{\text{in}}(z) \geq m_I |z|, \quad z \in S.$$

In particular, applying the weak upper bound of [proposition 5.39](#) to the closed exterior set  $\{z \in S : |z| \geq R\}$  gives

$$(5.119) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \Pi_r\{z \in S : |z| \geq R\} \leq -m_I R, \quad R > 0.$$

Fix  $\eta > 0$ . Choose  $R$  sufficiently large that

$$(5.120) \quad M_g - m_I R \leq \mathcal{B}_g + \eta.$$

Then

$$\int_{\{z \in S : |z| \geq R\}} e^{rg(z)} \Pi_r(dz) \leq e^{rM_g} \Pi_r\{z \in S : |z| \geq R\},$$

and (5.119)–(5.120) give

$$(5.121) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_{\{z \in S : |z| \geq R\}} e^{rg(z)} \Pi_r(dz) \leq \mathcal{B}_g + \eta.$$

It remains to control the integral over the compact set  $K_R := S \cap \overline{B_R}$ . Let  $m_g := \inf_S g$ . Since  $g$  is bounded,  $m_g > -\infty$ . If  $m_g = M_g$ , then  $g \equiv M_g$  on  $S$  and

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_{K_R} e^{rg} d\Pi_r \leq M_g = \mathcal{B}_g,$$

because  $I_\mu^{\text{in}}(0) = 0$ . Hence assume  $m_g < M_g$  and choose a finite mesh

$$m_g = a_0 < a_1 < \cdots < a_N = M_g$$

with mesh size at most  $\eta$ . For  $1 \leq k \leq N$ , put

$$F_k := K_R \cap \{z : g(z) \geq a_{k-1}\}.$$

Since  $g$  is upper semicontinuous, each  $F_k$  is closed, hence compact. Empty sets are omitted from the finite sum below. On the set  $K_R \cap \{a_{k-1} \leq g < a_k\}$  one has  $e^{rg} \leq e^{ra_k}$ ; points with  $g = M_g = a_N$ , if any, are included in the  $k = N$  term. Therefore

$$(5.122) \quad \int_{K_R} e^{rg} d\Pi_r \leq \sum_{k=1}^N e^{ra_k} \Pi_r(F_k).$$

By [proposition 5.39](#),

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \Pi_r(F_k) \leq - \inf_{z \in F_k} I_\mu^{\text{in}}(z).$$

For  $z \in F_k$ ,  $a_k \leq a_{k-1} + \eta \leq g(z) + \eta$ , and hence

$$a_k - I_\mu^{\text{in}}(z) \leq g(z) - I_\mu^{\text{in}}(z) + \eta \leq \mathcal{B}_g + \eta.$$

Thus

$$(5.123) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_{K_R} e^{rg} d\Pi_r \leq \mathcal{B}_g + \eta.$$

Combining (5.121) and (5.123) with the elementary fact that the logarithmic limsup of a finite sum of nonnegative terms is bounded by the maximum of their logarithmic limsups, and then letting  $\eta \downarrow 0$ , proves (5.116). The equivalent form (5.117) follows from the definition of  $\Pi_r$ .  $\square$

**Proposition 5.41** (Laplace upper bound for subcritical linear-growth test functions). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , let  $\pi$  be a stationary distribution for the drift  $\mu$ , and suppose that  $\mathcal{E}(\mu) \neq \emptyset$ . Let  $I_\mu^{\text{in}}$  be the inner support function from [proposition 5.36](#), and set*

$$m_I := \min_{u \in S \cap \mathbb{S}^1} I_\mu^{\text{in}}(u) > 0.$$

*Let  $g : S \rightarrow \mathbb{R}$  be upper semicontinuous and suppose that there are constants  $C_g < \infty$  and  $\kappa < m_I$  such that*

$$(5.124) \quad g(z) \leq C_g + \kappa|z|, \quad z \in S.$$

*For  $r > 0$  define  $\Pi_r(F) := \pi(rF)$ . Then*

$$(5.125) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_S e^{rg(z)} \Pi_r(dz) \leq \sup_{z \in S} \{g(z) - I_\mu^{\text{in}}(z)\}.$$

Equivalently,

$$(5.126) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_S e^{rg(z/r)} \pi(dz) \leq \sup_{z \in S} \{g(z) - I_\mu^{\text{in}}(z)\}.$$

*Proof.* Set

$$\mathcal{B}_g := \sup_{z \in S} \{g(z) - I_\mu^{\text{in}}(z)\}.$$

Because  $I_\mu^{\text{in}}(z) \geq m_I |z|$  and  $\kappa < m_I$ , the quantity  $\mathcal{B}_g$  is finite and  $g(z) - I_\mu^{\text{in}}(z) \rightarrow -\infty$  as  $|z| \rightarrow \infty$ .

We first control the contribution from large  $|z|$ . Choose  $\delta > 0$  such that  $\kappa + 3\delta < m_I$ . For each  $u \in S \cap \mathbb{S}^1$ , by the definition of  $I_\mu^{\text{in}}(u)$  there is  $\vartheta_u \in \mathcal{E}(\mu)$  with

$$\vartheta_u \cdot u > \kappa + 3\delta.$$

For each such  $u$ , continuity gives a relatively open neighborhood of  $u$  in  $S \cap \mathbb{S}^1$  on which the same strict inequality holds with  $\kappa + 2\delta$  in place of  $\kappa + 3\delta$ . By compactness of  $S \cap \mathbb{S}^1$ , choose finitely many  $\vartheta_1, \dots, \vartheta_N \in \mathcal{E}(\mu)$  such that for every  $u \in S \cap \mathbb{S}^1$  there exists  $j$  with

$$(5.127) \quad \vartheta_j \cdot u \geq \kappa + 2\delta.$$

Then, if  $Z \in S$  and  $|Z| > rR$ , (5.127) gives some  $j$  such that

$$e^{\kappa|Z|} \leq e^{-2\delta|Z|} e^{\vartheta_j \cdot Z} \leq e^{-2\delta rR} e^{\vartheta_j \cdot Z}.$$

Using the assumed linear-growth bound,

$$(5.128) \quad \begin{aligned} \int_{\{z \in S : |z| \geq R\}} e^{rg(z)} \Pi_r(dz) &= \int_{\{|Z| \geq rR\}} e^{rg(Z/r)} \pi(dZ) \\ &\leq e^{rC_g} \int_{\{|Z| \geq rR\}} e^{\kappa|Z|} \pi(dZ) \\ &\leq e^{r(C_g - 2\delta R)} \sum_{j=1}^N \int_S e^{\vartheta_j \cdot Z} \pi(dZ). \end{aligned}$$

Each integral in the sum is finite, since  $\vartheta_j \in \mathcal{E}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi)$ . Hence

$$(5.129) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_{\{z \in S : |z| \geq R\}} e^{rg(z)} \Pi_r(dz) \leq C_g - 2\delta R.$$

This tends to  $-\infty$  as  $R \rightarrow \infty$ .

Now fix  $R < \infty$  and write  $K_R := S \cap \overline{B_R}$ . Since  $g$  is upper semicontinuous, it is bounded above on  $K_R$ ; write  $M_R := \sup_{K_R} g$ . Fix  $\eta > 0$ . If  $M_R \leq \mathcal{B}_g + \eta$ , then

$$\int_{K_R} e^{rg} d\Pi_r \leq e^{r(\mathcal{B}_g + \eta)},$$

and the compact estimate follows directly. We may therefore assume  $M_R > \mathcal{B}_g + \eta$ . Choose  $a_0 < \mathcal{B}_g + \eta$  with  $a_0 < M_R$ . Then the contribution of  $K_R \cap \{g < a_0\}$  is bounded above by  $e^{r(\mathcal{B}_g + \eta)}$  for every  $r > 0$ . On  $K_R \cap \{g \geq a_0\}$  choose a finite mesh

$$a_0 < a_1 < \dots < a_N = M_R$$

with mesh size at most  $\eta$ , and define

$$F_k := K_R \cap \{z : g(z) \geq a_{k-1}\}, \quad 1 \leq k \leq N.$$

Each  $F_k$  is compact. Put

$$E_0 := K_R \cap \{g < a_0\},$$

and, for  $1 \leq k < N$ ,

$$E_k := K_R \cap \{a_{k-1} \leq g < a_k\}, \quad E_N := K_R \cap \{a_{N-1} \leq g \leq a_N\}.$$

These sets cover  $K_R$ ; moreover  $E_k \subset F_k$  and  $e^{rg(z)} \leq e^{ra_k}$  on  $E_k$ . Since  $e^{rg(z)} \leq e^{r(\mathcal{B}_g + \eta)}$  on  $E_0$ ,

$$\begin{aligned} \int_{K_R} e^{rg} d\Pi_r &\leq \int_{E_0} e^{rg} d\Pi_r + \sum_{k=1}^N \int_{E_k} e^{rg} d\Pi_r \\ &\leq e^{r(\mathcal{B}_g + \eta)} + \sum_{k=1}^N e^{ra_k} \Pi_r(F_k). \end{aligned}$$

Thus

$$(5.130) \quad \int_{K_R} e^{rg} d\Pi_r \leq e^{r(\mathcal{B}_g + \eta)} + \sum_{k=1}^N e^{ra_k} \Pi_r(F_k).$$

By [proposition 5.39](#),

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \Pi_r(F_k) \leq - \inf_{z \in F_k} I_\mu^{\text{in}}(z).$$

For  $z \in F_k$ , one has  $a_k \leq a_{k-1} + \eta \leq g(z) + \eta$ , and therefore

$$a_k - I_\mu^{\text{in}}(z) \leq g(z) - I_\mu^{\text{in}}(z) + \eta \leq \mathcal{B}_g + \eta.$$

Thus

$$(5.131) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log \int_{K_R} e^{rg} d\Pi_r \leq \mathcal{B}_g + \eta.$$

Combining [\(5.129\)](#) and [\(5.131\)](#), choose  $R$  large enough that

$$C_g - 2\delta R \leq \mathcal{B}_g + \eta.$$

For this fixed  $R$ , both the exterior and compact contributions have logarithmic limsup at most  $\mathcal{B}_g + \eta$ . Since the logarithmic limsup of a sum of two nonnegative terms is bounded by the maximum of their logarithmic limsups, the whole integral has logarithmic limsup at most  $\mathcal{B}_g + \eta$ . Letting  $\eta \downarrow 0$  proves [\(5.125\)](#). Formula [\(5.126\)](#) follows from the definition of  $\Pi_r$ .  $\square$

*Remark 5.42* (large-deviation interpretation). [Propositions 5.35](#) and [5.38](#) are finite-dimensional Chernoff bounds generated by the inner exponential-moment region. The large-deviation upper bound in [proposition 5.39](#) is the corresponding support-function upper bound obtained from the exponential moments proved in this paper. The directional, radial, and angular-sector bounds are special geometric projections of the same mechanism.

*Remark 5.43* (angular sectors and boundary influence). [Proposition 5.30](#) is the angular-sector analogue of the directional endpoint formula. It shows that the logarithmic radial decay in a fixed angular sector is again read from the same quadratic boundary of the inner exponential-moment region. For interval sectors, only the two angular endpoints enter the variational expression. The exact sector rates obtained in [theorem 6.88](#) show how this inner variational quantity is completed by the matching pathwise lower bound.

*Remark 5.44* (geometric meaning). The quantity  $\rho_c(\vartheta)$  is the largest amount of the direction  $c$  that can be subtracted from the exponential moment vector  $\vartheta$  while staying inside the dual cone  $S^\vee$ . Thus [proposition 5.25](#) is a purely convex-geometric version of the usual principle that directional tail exponents are read from the boundary of an exponential-moment domain. The full moment domain identified in [theorem 6.88](#) turns this geometric construction into the exact logarithmic rate.

## 6. THE CLOSED EXPONENTIAL-MOMENT DOMAIN AND LOGARITHMIC TAILS

This section identifies the closed exponential-moment domain and derives the exact logarithmic tails stated in the introduction. We begin with one-dimensional abscissa identities and support-function representations. Killed-wedge estimates, one-face path estimates, and broken paths then give the outer frontier, while finite sums of exponentials provide matching interior moments.

**6.1. Full scalar abscissae and exact logarithmic tail rates.** We first establish the one-dimensional relation between exponential-moment abscissae and logarithmic tails. This relation identifies the rate encoded by a scalar moment domain; the existence of a full logarithmic limit will follow from the matching pathwise lower bounds proved later in the section.

**Lemma 6.1** (one-dimensional tail-abscissa identity). *Let  $m$  be a finite nonnegative Borel measure on  $[0, \infty)$ , and set*

$$\bar{m}(x) := m([x, \infty)), \quad x \geq 0.$$

Define the extended abscissa

$$\alpha_m := \sup \left\{ \lambda \geq 0 : \int_{[0, \infty)} e^{\lambda y} m(dy) < \infty \right\} \in [0, \infty],$$

with the convention  $\sup \emptyset = 0$ . If  $\bar{m}(x) = 0$  for all sufficiently large  $x$ , the right-hand side below is interpreted as  $+\infty$ . Then

$$(6.1) \quad \alpha_m = \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \bar{m}(x).$$

Consequently, if the logarithmic tail limit exists, then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \bar{m}(x) = -\alpha_m.$$

*Proof.* If  $m = 0$ , or if  $\bar{m}$  vanishes eventually, then the exponential integral is finite for every  $\lambda \geq 0$ , and both sides of (6.1) are  $+\infty$ . Assume henceforth that  $\bar{m}(x) > 0$  along arbitrarily large  $x$ . The layer-cake identity gives, for  $\lambda > 0$ ,

$$(6.2) \quad \int_{[0, \infty)} e^{\lambda y} m(dy) = m([0, \infty)) + \lambda \int_0^\infty e^{\lambda x} \bar{m}(x) dx.$$

Put

$$r := \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \bar{m}(x) \in [0, \infty].$$

If  $0 \leq \lambda < r$ , choose  $\delta > 0$  with  $\lambda + 2\delta < r$ . For all sufficiently large  $x$ ,

$$\bar{m}(x) \leq e^{-(\lambda+\delta)x}.$$

The integral in (6.2) is therefore finite, and so  $\lambda \leq \alpha_m$ . Hence  $\alpha_m \geq r$ , with the extended interpretation that  $\alpha_m = +\infty$  when this holds for every finite  $\lambda$ .

If  $r = +\infty$ , this proves the identity. It remains to consider  $r < +\infty$ . Let  $\lambda > r$ . Choose  $\delta > 0$  with  $\lambda - 2\delta > r$ . There is a sequence  $x_n \rightarrow \infty$  such that

$$\bar{m}(x_n) \geq e^{-(\lambda-\delta)x_n}.$$

For all large  $n$ , monotonicity of  $\bar{m}$  gives

$$\int_0^\infty e^{\lambda x} \bar{m}(x) dx \geq \int_{x_n-1}^{x_n} e^{\lambda x} \bar{m}(x) dx \geq e^{\lambda(x_n-1)} \bar{m}(x_n) \geq e^{-\lambda} e^{\delta x_n}.$$

The right-hand side is unbounded along  $n$ , so the tail integral in (6.2) is infinite. Thus  $\lambda > \alpha_m$ . Since this holds for every  $\lambda > r$ ,  $\alpha_m \leq r$ , and the identity follows.  $\square$

For the full-domain geometric statements below, we work in the convex wedge setting  $0 < \xi < \pi$ . This is the setting of the quantitative theorem in the introduction and of the classified existence regimes. In this setting  $S = \text{cone}\{u_1, u_2\}$ , the dual cone has nonempty interior, and  $S = (S^\vee)^\vee$ .

For a stationary distribution  $\pi$ , define the full projected moment abscissa in a dual direction  $c \in S_\circ^\vee$  by

$$(6.3) \quad \alpha_c^\pi := \sup\{\lambda \geq 0 : \lambda c \in \mathcal{D}_{\text{exp}}(\pi)\} \in [0, \infty].$$

Equivalently, if  $Y_c = c \cdot Z$ , then  $\alpha_c^\pi$  is the abscissa of convergence of  $\mathbb{E}_\pi e^{\lambda Y_c}$ .

**Proposition 6.2** (projected tail abscissa formula). *Let  $\pi$  be a stationary distribution, let  $c \in S_\circ^\vee$ , and set*

$$\Pi_c(x) := \pi\{z \in S : c \cdot z \geq x\}, \quad x \geq 0.$$

*Then, with the convention that the liminf is  $+\infty$  if  $\Pi_c(x) = 0$  for all sufficiently large  $x$ ,*

$$(6.4) \quad \alpha_c^\pi = \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \Pi_c(x).$$

*Consequently, if the logarithmic projection-tail limit exists, then*

$$(6.5) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \Pi_c(x) = -\alpha_c^\pi.$$

*Proof.* Apply lemma 6.1 to the push-forward measure  $\pi \circ (z \mapsto c \cdot z)^{-1}$  on  $[0, \infty)$ . Since  $c \in S_\circ^\vee$ , the push-forward is supported on  $[0, \infty)$ , and its exponential abscissa is exactly (6.3).  $\square$

**Corollary 6.3** (inner rates are lower bounds for the exact projection rates). *Assume  $\alpha < 2$ , let  $\pi$  be a stationary distribution for the drift  $\mu$ , and let  $c \in S_\circ^\vee \cap \mathbb{S}^1$ . Then*

$$(6.6) \quad \alpha_c^{\text{in}}(\mu) \leq \alpha_c^\pi.$$

*Thus  $\alpha_c^{\text{in}}(\mu)$  is the part of the true projection-tail rate forced by the elementary inner transform domain. In the classified regimes the later full-domain theorem computes  $\alpha_c^\pi$  and proves the exact logarithmic limit.*

*Proof.* If  $\alpha_c^{\text{in}}(\mu) = 0$ , the inequality is immediate. Otherwise choose any  $\lambda$  with  $0 \leq \lambda < \alpha_c^{\text{in}}(\mu)$ . By the definition of  $\alpha_c^{\text{in}}(\mu)$ , there exists  $\vartheta \in \mathcal{E}(\mu)$  such that  $\vartheta - \lambda c \in S^\vee$ . Since  $\mathcal{E}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi)$  by proposition 5.6, the dual-cone monotonicity of proposition 5.1 gives  $\lambda c \in \mathcal{D}_{\text{exp}}(\pi)$ . Hence  $\lambda \leq \alpha_c^\pi$ . Letting  $\lambda \uparrow \alpha_c^{\text{in}}(\mu)$  proves (6.6). The final assertion follows from proposition 6.2.  $\square$

The same scalar principle applies to radial and angular-sector tails. For a nonempty Borel angular set  $C \subset S \cap \mathbb{S}^1$ , define

$$(6.7) \quad \alpha_C^\pi := \sup \left\{ \lambda \geq 0 : \int_{\{z \neq 0: z/|z| \in C\}} e^{\lambda|z|} \pi(dz) < \infty \right\} \in [0, \infty],$$

where the integral over the empty set is interpreted as zero. The radial case is  $C = S \cap \mathbb{S}^1$ , and we write  $\alpha_{\text{rad}}^\pi$  for that value.

**Proposition 6.4** (radial and angular-sector tail abscissae). *Let  $\pi$  be a stationary distribution and let  $C \subset S \cap \mathbb{S}^1$  be a nonempty Borel set. Put*

$$\Pi_C(r) := \pi\{z \in S : |z| \geq r, z/|z| \in C\}, \quad r > 0.$$

*Then*

$$(6.8) \quad \alpha_C^\pi = \liminf_{r \rightarrow \infty} -\frac{1}{r} \log \Pi_C(r),$$

*with the convention that the right-hand side is  $+\infty$  if  $\Pi_C(r) = 0$  for all sufficiently large  $r$ . Consequently, if the sector radial logarithmic tail limit exists, then*

$$(6.9) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log \Pi_C(r) = -\alpha_C^\pi.$$

*For  $C = S \cap \mathbb{S}^1$ , this gives the corresponding exact scalar abscissa formula for the full radial tail.*

*Proof.* Apply lemma 6.1 to the finite measure on  $[0, \infty)$  obtained by pushing forward the restricted measure

$$\mathbf{1}_{\{z \neq 0: z/|z| \in C\}} \pi(dz)$$

under  $z \mapsto |z|$ . Its tail is exactly  $\Pi_C(r)$ , and its exponential abscissa is  $\alpha_C^\pi$ .  $\square$

**Corollary 6.5** (inner sector rates are lower bounds for full sector abscissae). *Assume  $\alpha < 2$ , let  $\pi$  be a stationary distribution for the drift  $\mu$ , and let  $C \subset S \cap \mathbb{S}^1$  be nonempty and compact. Then*

$$(6.10) \quad I_C^{\text{in}}(\mu) \leq \alpha_C^\pi.$$

*In particular,*

$$(6.11) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^\pi.$$

*Proof.* If  $I_C^{\text{in}}(\mu) = 0$ , there is nothing to prove. Let  $0 \leq \lambda < I_C^{\text{in}}(\mu)$ . By definition of  $I_C^{\text{in}}(\mu)$ , there exists  $\vartheta \in \mathcal{E}(\mu)$  such that

$$\lambda < \inf_{u \in C} \vartheta \cdot u.$$

Thus, on the sector  $z/|z| \in C$ , one has  $\lambda|z| \leq \vartheta \cdot z$ . Since  $\vartheta \in \mathcal{E}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi)$ , the restricted sector moment

$$\int_{\{z \neq 0: z/|z| \in C\}} e^{\lambda|z|} \pi(dz)$$

is finite. Hence  $\lambda \leq \alpha_C^\pi$ . Letting  $\lambda \uparrow I_C^{\text{in}}(\mu)$  proves (6.10). The radial assertion is the case  $C = S \cap \mathbb{S}^1$ .  $\square$

**Theorem 6.6** (quantitative scaling along drift rays). *Assume  $\alpha < 2$ . Let  $\pi$  be stationary for drift  $\mu$ , let  $r > 0$ , and put*

$$\pi_r := (T_r)_\# \pi, \quad T_r z = rz.$$

*Then  $\pi_r$  is stationary for drift  $\mu/r$ , and*

$$(6.12) \quad \mathcal{D}_{\text{exp}}(\pi_r) = r^{-1} \mathcal{D}_{\text{exp}}(\pi), \quad D_{\pi_r} = r^{-1} D_\pi.$$

*For every  $c \in S_\circ^\vee$  and every compact angular set  $C \subset S \cap \mathbb{S}^1$ ,*

$$(6.13) \quad \alpha_c^{\pi_r} = r^{-1} \alpha_c^\pi, \quad \alpha_{\text{rad}}^{\pi_r} = r^{-1} \alpha_{\text{rad}}^\pi, \quad \alpha_C^{\pi_r} = r^{-1} \alpha_C^\pi.$$

*The corresponding tails satisfy*

$$(6.14) \quad \pi_r\{z : c \cdot z \geq x\} = \pi\{z : c \cdot z \geq x/r\},$$

*and*

$$\pi_r\{z : |z| \geq x, z/|z| \in C\} = \pi\{z : |z| \geq x/r, z/|z| \in C\}.$$

*Consequently the existence of any projected, radial, or sector logarithmic tail limit is invariant along a nonzero drift ray, and its rate is homogeneous of degree one in the drift magnitude. More precisely, for  $a > 0$  the map*

$$\pi \longmapsto (T_{1/a})_\# \pi$$

*is a bijection from stationary laws for  $\mu$  to stationary laws for  $a\mu$ , and all scalar rates are multiplied by  $a$ .*

*Proof.* The stationarity assertion and the bijection are [proposition 3.6](#). If  $Y = rX$  with  $X \sim \pi$ , then

$$\int_S e^{\theta \cdot y} \pi_r(dy) = \int_S e^{(r\theta) \cdot x} \pi(dx).$$

This proves (6.12). For projected moments,

$$\lambda c \in \mathcal{D}_{\text{exp}}(\pi_r) \iff r\lambda c \in \mathcal{D}_{\text{exp}}(\pi),$$

so

$$\alpha_c^{\pi_r} = \sup\{\lambda \geq 0 : r\lambda c \in \mathcal{D}_{\text{exp}}(\pi)\} = r^{-1} \alpha_c^\pi.$$

For radial moments,

$$\int_S e^{\lambda|y|} \pi_r(dy) = \int_S e^{(r\lambda)|x|} \pi(dx),$$

and, because  $y/|y| = x/|x|$  under  $y = rx$ ,

$$\int_{\{y \neq 0: y/|y| \in C\}} e^{\lambda|y|} \pi_r(dy) = \int_{\{x \neq 0: x/|x| \in C\}} e^{(r\lambda)|x|} \pi(dx).$$

Taking the two convergence abscissae gives the remaining identities in (6.13). Finally,

$$\begin{aligned} \{y : c \cdot y \geq x\} &= T_r\{x' : c \cdot x' \geq x/r\}, \\ \{y : |y| \geq x, y/|y| \in C\} &= T_r\{x' : |x'| \geq x/r, x'/|x'| \in C\}, \end{aligned}$$

which proves the two tail identities. Setting  $r = 1/a$  proves the final statement.  $\square$

**Corollary 6.7** (drift-ray normalization of scalar tail rates). *In each stationary-existence region of the phase diagram, the logarithmic-tail problem for all nonzero drifts reduces to one representative on every drift ray. Equivalently, it is enough to treat drifts on any fixed transversal section, such as  $|\mu| = 1$ , and then recover arbitrary magnitudes by [theorem 6.6](#).*

*Proof.* Zero drift is not in either stationary-existence region considered here. Every admissible drift therefore has a unique positive multiple on the chosen transversal section, and [theorem 6.6](#) transfers stationary laws and all scalar logarithmic rates bijectively along that ray.  $\square$

**Corollary 6.8** (exact scalar exponential envelopes). *Let  $\pi$  be a stationary distribution.*

(i) *Let  $c \in S_\circ^\vee$  and assume  $\alpha_c^\pi < \infty$ . Then, for every  $\eta > 0$ ,*

$$(6.15) \quad \Pi_c(x) \leq \exp\{-(\alpha_c^\pi - \eta)x\}$$

*for all sufficiently large  $x$ , and*

$$(6.16) \quad \Pi_c(x_n) \geq \exp\{-(\alpha_c^\pi + \eta)x_n\}$$

*for some sequence  $x_n \rightarrow \infty$ . Moreover, the exact logarithmic projection-tail limit exists if and only if the lower envelope (6.16) can be strengthened to hold for all sufficiently large  $x$ , for every  $\eta > 0$ . In that case*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \Pi_c(x) = -\alpha_c^\pi.$$

(ii) *Let  $C \subset S \cap \mathbb{S}^1$  be nonempty and compact and assume  $\alpha_C^\pi < \infty$ . Then, for every  $\eta > 0$ ,*

$$(6.17) \quad \Pi_C(r) \leq \exp\{-(\alpha_C^\pi - \eta)r\}$$

*for all sufficiently large  $r$ , and*

$$(6.18) \quad \Pi_C(r_n) \geq \exp\{-(\alpha_C^\pi + \eta)r_n\}$$

*for some sequence  $r_n \rightarrow \infty$ . The exact logarithmic sector-tail limit exists if and only if the lower envelope (6.18) can be strengthened to hold for all sufficiently large  $r$ , for every  $\eta > 0$ . In that case*

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log \Pi_C(r) = -\alpha_C^\pi.$$

*Proof.* For the projection statement, [proposition 6.2](#) gives

$$\alpha_c^\pi = \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \Pi_c(x).$$

The eventual upper envelope (6.15) is exactly the assertion that the preceding liminf is at least  $\alpha_c^\pi$ , with margin  $\eta$ . The subsequential lower envelope (6.16) is exactly the assertion that the same liminf is at most  $\alpha_c^\pi$ . The logarithmic limit exists precisely when the corresponding limsup is no larger than  $\alpha_c^\pi$ , which is equivalent to the eventual lower envelope for every  $\eta > 0$ . This proves (i). The sector statement follows in the same way from [proposition 6.4](#).  $\square$

*Remark 6.9* (scalar envelope consequence). [Corollary 6.8](#) isolates the information supplied by a scalar abscissa before its value is computed geometrically. Every scalar tail with finite abscissa

has an eventual exponential upper envelope with exponent arbitrarily close to that abscissa and a matching lower envelope along arbitrarily large scales. A scalar abscissa alone does not exclude exponential-scale oscillation, and therefore does not imply a full logarithmic limit. In the classified wedge regimes, the finite-exponential construction and the broken-path lower bounds compute the abscissa and exclude such oscillation.

**Definition 6.10** (full transform support functions). For a stationary distribution  $\pi$ , define the full transform-domain support function

$$(6.19) \quad I_\pi^{\text{full}}(z) := \sup\{\theta \cdot z : \theta \in \mathcal{D}_{\text{exp}}(\pi)\} \in [0, \infty], \quad z \in S.$$

The same value is obtained if  $\mathcal{D}_{\text{exp}}(\pi)$  is replaced by its closure, because, for each fixed  $z$ , the map  $\theta \mapsto \theta \cdot z$  is continuous.

If  $C \subset S \cap \mathbb{S}^1$  is a nonempty compact angular set, let

$$\pi_C(dz) := \mathbf{1}_{\{z \neq 0: z/|z| \in C\}} \pi(dz)$$

be the restricted sector measure, and define

$$\mathcal{D}_{\text{exp}}(\pi; C) := \left\{ \theta \in \mathbb{R}^2 : \int_{\{z \neq 0: z/|z| \in C\}} e^{\theta \cdot z} \pi(dz) < \infty \right\}.$$

The restricted sector support function is

$$(6.20) \quad I_{\pi, C}^{\text{full}}(u) := \sup\{\theta \cdot u : \theta \in \mathcal{D}_{\text{exp}}(\pi; C)\}, \quad u \in C.$$

If  $\pi_C = 0$ , then  $\mathcal{D}_{\text{exp}}(\pi; C) = \mathbb{R}^2$  and  $I_{\pi, C}^{\text{full}} \equiv +\infty$  on  $C$ .

**Proposition 6.11** (full-domain variational formula for projected abscissae). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ . Let  $\pi$  be a stationary distribution and let  $c \in S_\circ^\vee$ . For  $\theta \in \mathbb{R}^2$ , set*

$$\rho_c(\theta) := \sup\{\lambda \geq 0 : \theta - \lambda c \in S^\vee\},$$

*with the convention that the supremum is zero if the set is empty. If  $\theta \in S^\vee$ , then*

$$(6.21) \quad \rho_c(\theta) = \min_{u \in S \cap \mathbb{S}^1} \frac{\theta \cdot u}{c \cdot u} = \min \left\{ \frac{\theta \cdot u_1}{c \cdot u_1}, \frac{\theta \cdot u_2}{c \cdot u_2} \right\}.$$

*Moreover,*

$$(6.22) \quad \alpha_c^\pi = \sup_{\theta \in \mathcal{D}_{\text{exp}}(\pi)} \rho_c(\theta).$$

*Consequently, the exact logarithmic projection rate, whenever the projection-tail limit exists, is obtained from the closed transform domain by the same variational formula as the inner rate, with  $\mathcal{E}(\mu)$  replaced by  $\mathcal{D}_{\text{exp}}(\pi)$ .*

*Proof.* If  $\lambda c \in \mathcal{D}_{\text{exp}}(\pi)$ , then choosing  $\theta = \lambda c$  gives  $\rho_c(\theta) \geq \lambda$ . Hence the right-hand side of (6.22) is at least  $\alpha_c^\pi$ . Conversely, suppose  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$  and  $0 \leq \lambda < \rho_c(\theta)$ . Then  $\theta - \lambda c \in S^\vee$ . By the dual-cone monotonicity of [proposition 5.1](#),  $\lambda c \in \mathcal{D}_{\text{exp}}(\pi)$ . Thus  $\lambda \leq \alpha_c^\pi$ . Taking the supremum over such  $\lambda$  and then over  $\theta$  gives the reverse inequality.

For  $\theta \in S^\vee$ , the first equality in (6.21) is just the definition of the largest  $\lambda$  for which  $\theta - \lambda c \in S^\vee$ . The endpoint formula follows because, writing  $u(\varphi) = (\cos \varphi, \sin \varphi)$ , the derivative of  $(\theta \cdot u(\varphi))/(c \cdot u(\varphi))$  has numerator  $\theta_2 c_1 - \theta_1 c_2$  and denominator  $(c \cdot u(\varphi))^2$ , so it has constant sign on  $[0, \xi]$ . Hence the minimum is attained at one of the two boundary directions. The final assertion follows from [proposition 6.2](#).  $\square$

**Proposition 6.12** (full radial and sector support-function formulas). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ . Let  $\pi$  be a stationary distribution. Then the full radial abscissa satisfies*

$$(6.23) \quad \alpha_{\text{rad}}^\pi = \inf_{u \in S \cap \mathbb{S}^1} I_\pi^{\text{full}}(u).$$

More generally, for every nonempty compact angular set  $C \subset S \cap \mathbb{S}^1$ , the sector radial abscissa satisfies

$$(6.24) \quad \alpha_C^\pi = \inf_{u \in C} I_{\pi, C}^{\text{full}}(u).$$

Both identities are understood in the extended sense, allowing the value  $+\infty$ .

*Proof.* We first prove the radial identity. If  $0 \leq \lambda < \alpha_{\text{rad}}^\pi$ , then  $\int_S e^{\lambda|z|} \pi(dz) < \infty$ . For every  $u \in S \cap \mathbb{S}^1$ , the inequality  $\lambda u \cdot z \leq \lambda|z|$  gives  $\lambda u \in \mathcal{D}_{\text{exp}}(\pi)$ . Hence  $I_\pi^{\text{full}}(u) \geq \lambda$  for all  $u$ , and  $\lambda \leq \inf_{S \cap \mathbb{S}^1} I_\pi^{\text{full}}$ . Letting  $\lambda \uparrow \alpha_{\text{rad}}^\pi$  gives one inequality.

Conversely, let  $0 \leq \lambda < \inf_{u \in S \cap \mathbb{S}^1} I_\pi^{\text{full}}(u)$ . For each  $u \in S \cap \mathbb{S}^1$ , choose  $\theta_u \in \mathcal{D}_{\text{exp}}(\pi)$  with  $\theta_u \cdot u > \lambda$ . By continuity, this strict inequality holds in a relative neighborhood of  $u$ . Compactness of  $S \cap \mathbb{S}^1$  gives finitely many  $\theta_1, \dots, \theta_N \in \mathcal{D}_{\text{exp}}(\pi)$  such that for every  $v \in S \cap \mathbb{S}^1$  at least one  $j$  satisfies  $\theta_j \cdot v \geq \lambda$ . Hence, for every nonzero  $z \in S$ , with  $v = z/|z|$ ,

$$e^{\lambda|z|} \leq \sum_{j=1}^N e^{\theta_j \cdot z}.$$

The right-hand side is  $\pi$ -integrable, so  $\lambda \leq \alpha_{\text{rad}}^\pi$ . This proves (6.23).

For the sector identity, suppose first that  $0 \leq \lambda < \alpha_C^\pi$ . Then

$$\int e^{\lambda|z|} \pi_C(dz) < \infty.$$

For each  $u \in C$  and each  $z$  in the support of  $\pi_C$ ,  $\lambda u \cdot z \leq \lambda|z|$ ; hence  $\lambda u \in \mathcal{D}_{\text{exp}}(\pi; C)$  and  $I_{\pi, C}^{\text{full}}(u) \geq \lambda$ . Therefore

$$\alpha_C^\pi \leq \inf_{u \in C} I_{\pi, C}^{\text{full}}(u).$$

Conversely, fix  $0 \leq \lambda < \inf_{u \in C} I_{\pi, C}^{\text{full}}(u)$ . For each  $u \in C$ , choose  $\theta_u \in \mathcal{D}_{\text{exp}}(\pi; C)$  with  $\theta_u \cdot u > \lambda$ . The set

$$U_u := \{v \in C : \theta_u \cdot v > \lambda\}$$

is relatively open in  $C$  and contains  $u$ . Choose a finite subcover  $C \subset U_{u_1} \cup \dots \cup U_{u_N}$  and write  $\theta_j := \theta_{u_j}$ . If  $z \neq 0$  and  $z/|z| \in C$ , choose  $j$  with  $z/|z| \in U_{u_j}$ ; then  $\theta_j \cdot z > \lambda|z|$ , and consequently

$$e^{\lambda|z|} \leq \sum_{j=1}^N e^{\theta_j \cdot z}.$$

Each summand is  $\pi_C$ -integrable, so the left-hand side is also  $\pi_C$ -integrable. Hence  $\lambda \leq \alpha_C^\pi$ . Letting  $\lambda$  increase to the infimum proves (6.24).  $\square$

**Proposition 6.13** (convex-dual formula for full projected abscissae). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ . Let  $\pi$  be a stationary distribution and let  $c \in S_\circ^\vee$ . Then*

$$(6.25) \quad \alpha_c^\pi = \inf \{ I_\pi^{\text{full}}(z) : z \in S, c \cdot z = 1 \}.$$

*Consequently, computing the exact projection logarithmic rate is equivalent to computing the support function of the full exponential-moment domain on the compact cross-section  $\{z \in S : c \cdot z = 1\}$ .*

*Proof.* Let  $D$  be the closure of  $\mathcal{D}_{\text{exp}}(\pi)$ . By proposition 5.1,  $D$  is closed and convex, contains 0, and is downward closed with respect to the dual cone. The same proposition gives the required downward closure of  $\mathcal{D}_{\text{exp}}(\pi)$ , so lemma 5.2, applied with  $A = \mathcal{D}_{\text{exp}}(\pi)$ , gives

$$(6.26) \quad \alpha_c^\pi = \sup \{ \lambda \geq 0 : \lambda c \in D \}.$$

This step uses both the order-ideal property and the strict positivity  $c \in S_\circ^\vee$ ; convexity alone would not justify replacing the moment domain by its closure along a positive ray.

Fix  $0 \leq \lambda < \alpha_c^\pi$ . By the definition of the supremum there is a finite  $\lambda' > \lambda$  such that

$$\lambda'c \in \mathcal{D}_{\text{exp}}(\pi).$$

Since

$$\lambda'c - \lambda c = (\lambda' - \lambda)c \in S^\vee,$$

the downward-closure statement of [proposition 5.1](#) gives  $\lambda c \in \mathcal{D}_{\text{exp}}(\pi)$ . Hence, for every  $z \in S$  with  $c \cdot z = 1$ ,

$$I_\pi^{\text{full}}(z) \geq \lambda c \cdot z = \lambda.$$

Letting  $\lambda \uparrow \alpha_c^\pi$  gives

$$\inf_{c \cdot z = 1} I_\pi^{\text{full}}(z) \geq \alpha_c^\pi.$$

If  $\alpha_c^\pi = \infty$ , this already proves the identity. Assume therefore that  $\alpha_c^\pi < \infty$ , and fix  $\lambda > \alpha_c^\pi$ . Then  $q := \lambda c \notin D$  by [\(6.26\)](#). To construct the separator without invoking a black-box separation theorem, let  $d := \text{dist}(q, D)$ . Since  $0 \in D$ ,  $d \leq |q|$ . Choose  $p_n \in D$  with  $|q - p_n| \leq d + n^{-1}$ . Then  $|p_n| \leq 2|q| + 1$ , so a subsequence converges to some  $p \in D$ , and  $|q - p| = d$ . Convexity makes this nearest point unique: if  $p' \neq p$  were another minimizer, then the midpoint identity would give

$$\left| q - \frac{p + p'}{2} \right|^2 = \frac{|q - p|^2 + |q - p'|^2}{2} - \frac{|p - p'|^2}{4} < d^2,$$

a contradiction. Put  $y := q - p \neq 0$ . For every  $\theta \in D$  and  $t \in [0, 1]$ , the point  $p + t(\theta - p)$  belongs to  $D$ . Differentiating at zero the squared-distance function

$$t \longmapsto |q - p - t(\theta - p)|^2$$

gives

$$y \cdot (\theta - p) \leq 0.$$

Consequently

$$(6.27) \quad \sup_{\theta \in D} \theta \cdot y = p \cdot y, \quad q \cdot y = p \cdot y + |y|^2 > \sup_{\theta \in D} \theta \cdot y.$$

In particular, the support function on the right is finite. Since  $-S^\vee \subset \mathcal{D}_{\text{exp}}(\pi) \subset D$ , this finiteness forces  $y \in (S^\vee)^\vee = S$ : otherwise there would be  $\eta \in S^\vee$  with  $\eta \cdot y < 0$ , and the points  $-\rho\eta \in D$  would make the support infinite. Moreover  $q \cdot y > \sup_D \theta \cdot y \geq 0$ , so  $c \cdot y > 0$ . Put  $z = y/(c \cdot y)$ . Then  $z \in S$ ,  $c \cdot z = 1$ , and

$$I_\pi^{\text{full}}(z) = \frac{1}{c \cdot y} \sup_{\theta \in D} \theta \cdot y < \lambda.$$

Taking the infimum over the cross-section and then letting  $\lambda \downarrow \alpha_c^\pi$  gives the reverse inequality. This proves [\(6.25\)](#).  $\square$

**Proposition 6.14** (interior Harnack lower tails). *Assume  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Let  $p_\pi$  be its strictly positive interior density. Fix an interior unit direction  $u \in S^\circ \cap \mathbb{S}^1$  and a radius  $\rho > 0$  such that the balls  $B_\rho(ru)$  are contained in  $S^\circ$  for all sufficiently large  $r$ . Then there exist constants  $C_u > 0$ ,  $\Gamma_u < \infty$ , and  $R_u < \infty$  such that*

$$(6.28) \quad \pi(B_\rho(ru)) \geq C_u e^{-\Gamma_u r}, \quad r \geq R_u.$$

*Proof.* By [proposition 3.7](#) and [corollary 3.8](#),  $p_\pi$  is a strictly positive smooth solution of

$$\frac{1}{2} \Delta p_\pi - \mu \cdot \nabla p_\pi = 0 \quad \text{in } S^\circ.$$

Because  $u$  is an interior direction, the Euclidean distance from  $ru$  to either face grows linearly in  $r$ . Hence there is  $r_0 < \infty$  such that

$$(6.29) \quad B_{2\rho}(su) \Subset S^\circ, \quad s \geq r_0.$$

The interior Harnack inequality for nonnegative solutions of uniformly elliptic equations with bounded lower-order coefficients [6, Chapter 9], applied on balls of radius  $2\rho$  to the fixed constant-coefficient operator, gives a constant  $H = H(\rho, \mu) \geq 1$  such that

$$(6.30) \quad \sup_{B_\rho(x)} p_\pi \leq H \inf_{B_\rho(x)} p_\pi \quad \text{whenever } B_{2\rho}(x) \Subset S^\circ.$$

Set

$$m_0 := \min_{\overline{B}_\rho(r_0 u)} p_\pi > 0.$$

Indeed, (6.29) implies  $\overline{B}_\rho(r_0 u) \Subset S^\circ$ ; the density is continuous and strictly positive on this compact set. Since the open ball is dense in its closure, this minimum is also  $\inf_{B_\rho(r_0 u)} p_\pi$ .

Fix  $r \geq r_0$ . If  $r = r_0$ , take  $N = 0$ . If  $r > r_0$ , set

$$N := \left\lceil \frac{r - r_0}{\rho} \right\rceil, \quad s_j := r_0 + \frac{j}{N}(r - r_0), \quad x_j := s_j u, \quad j = 0, \dots, N.$$

Then  $s_0 = r_0$ ,  $s_N = r$ , and

$$0 < s_{j+1} - s_j = \frac{r - r_0}{N} \leq \rho.$$

Moreover,

$$(6.31) \quad N = \left\lceil \frac{r - r_0}{\rho} \right\rceil \leq 1 + \frac{r - r_0}{\rho}.$$

The balls  $B_\rho(x_j)$  and  $B_\rho(x_{j+1})$  overlap. Choose  $y_j$  in their intersection and put  $m_j = \inf_{B_\rho(x_j)} p_\pi$ . Then

$$m_j \leq p_\pi(y_j) \leq \sup_{B_\rho(x_{j+1})} p_\pi \leq H m_{j+1},$$

where the last inequality is (6.30). Therefore  $m_{j+1} \geq H^{-1} m_j$ , and induction yields

$$\inf_{B_\rho(ru)} p_\pi = m_N \geq m_0 H^{-N}.$$

Using (6.31),

$$H^{-N} \geq H^{-1} \exp \left\{ -\frac{\log H}{\rho} (r - r_0) \right\}.$$

Thus (6.28) holds with, for example,

$$\Gamma_u = \frac{\log H}{\rho}, \quad C_u = |B_\rho(0)| m_0 H^{-1} e^{\Gamma_u r_0}, \quad R_u = r_0,$$

after integrating the pointwise lower bound over  $B_\rho(ru)$ .  $\square$

**Corollary 6.15** (finite full abscissae in genuine interior directions). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be a stationary distribution. Then the following hold.*

(i) *For every  $c \in S^\circ$ , the full projected abscissa is finite:*

$$\alpha_c^\pi < \infty.$$

*More quantitatively, for every  $u \in S^\circ \cap \mathbb{S}^1$ ,*

$$(6.32) \quad \alpha_c^\pi \leq \frac{\Gamma_u}{c \cdot u},$$

*where  $\Gamma_u$  is any constant for which (6.28) holds along the direction  $u$ .*

(ii) *The full radial abscissa  $\alpha_{\text{rad}}^\pi$  is finite.*

(iii) *If a compact angular set  $C \subset S \cap \mathbb{S}^1$  contains a nonempty open subarc of  $S^\circ \cap \mathbb{S}^1$ , then  $\alpha_C^\pi < \infty$ .*

*If, in addition,  $\mathcal{E}(\mu) \neq \emptyset$ , then the corresponding abscissae in (i), (ii), and (iii) are strictly positive whenever the inner rates in corollaries 6.3 and 6.5 are positive. In particular, in the classified existence regimes, all projection abscissae and the full radial abscissa lie in  $(0, \infty)$ .*

*Proof.* Fix  $c \in S^\circ$  and choose any  $u \in S^\circ \cap \mathbb{S}^1$ . Let  $\rho > 0$  be small enough for [proposition 6.14](#). For large  $x$ , set

$$r_x := \frac{x + \rho|c|}{c \cdot u}.$$

If  $z \in B_\rho(r_x u)$ , then

$$c \cdot z \geq r_x c \cdot u - \rho|c| = x.$$

Thus  $B_\rho(r_x u) \subset \{c \cdot z \geq x\}$ , and [proposition 6.14](#) gives

$$\Pi_c(x) \geq C e^{-\Gamma_u r_x}.$$

Using [proposition 6.2](#) yields (6.32), and in particular  $\alpha_c^\pi < \infty$ .

For the radial case, fix the same interior direction and a radius  $\rho$  for which

$$\pi(B_\rho(su)) \geq C e^{-\Gamma s}, \quad s \geq R.$$

For  $r \geq R$ , every  $z \in B_\rho((r + \rho)u)$  satisfies

$$|z| \geq |(r + \rho)u| - |z - (r + \rho)u| > r.$$

Hence

$$\pi\{|z| \geq r\} \geq \pi(B_\rho((r + \rho)u)) \geq C e^{-\Gamma(r + \rho)}.$$

Applying [proposition 6.4](#) with  $C = S \cap \mathbb{S}^1$  gives  $\alpha_{\text{rad}}^\pi \leq \Gamma < \infty$ .

Finally suppose that  $C$  contains a nonempty relative open subarc of  $S^\circ \cap \mathbb{S}^1$ . Choose  $u$  in that subarc and  $\eta > 0$  such that

$$\{v \in S \cap \mathbb{S}^1 : |v - u| < \eta\} \subset C.$$

Fix any  $\rho > 0$ . Since  $u$  is an interior direction,  $B_\rho((r + \rho)u) \Subset S^\circ$  for all sufficiently large  $r$ . For  $z \in B_\rho((r + \rho)u)$ , the reverse triangle inequality gives  $|z| \geq r$ . Once  $r > \rho$ , the elementary normalization estimate

$$\left| \frac{z}{|z|} - u \right| \leq \frac{2|z - (r + \rho)u|}{r + \rho} < \frac{2\rho}{r + \rho}$$

holds. Hence, for every

$$r > R_C := \max \left\{ \rho, \frac{2\rho}{\eta} \right\},$$

the entire ball lies in the angular sector over  $C$  and in  $\{|z| \geq r\}$ . Applying the Harnack-ball lower bound along this ball and then [proposition 6.4](#) gives  $\alpha_C^\pi < \infty$ . The strict positivity assertions follow from the inner lower bounds [corollaries 6.3](#) and [6.5](#).  $\square$

**Definition 6.16** (directional Harnack exponents). Let  $\pi$  be a stationary distribution, and let  $u \in S^\circ \cap \mathbb{S}^1$ . Define  $\mathfrak{G}_\pi(u)$  to be the infimum of all  $\Gamma \geq 0$  for which there exist constants  $\rho > 0$ ,  $C > 0$ , and  $R < \infty$  such that

$$B_\rho(ru) \subset S^\circ, \quad \pi(B_\rho(ru)) \geq C e^{-\Gamma r}, \quad r \geq R.$$

By [proposition 6.14](#),  $\mathfrak{G}_\pi(u) < \infty$  for every interior unit direction  $u$ . For  $c \in S^\circ$ , set

$$\mathfrak{H}_c^\pi := \inf_{u \in S^\circ \cap \mathbb{S}^1} \frac{\mathfrak{G}_\pi(u)}{c \cdot u}.$$

For radial tails set

$$\mathfrak{H}_{\text{rad}}^\pi := \inf_{u \in S^\circ \cap \mathbb{S}^1} \mathfrak{G}_\pi(u).$$

If  $C \subset S \cap \mathbb{S}^1$  is compact and contains a nonempty open angular subarc in  $S^\circ \cap \mathbb{S}^1$ , set

$$\mathfrak{H}_C^\pi := \inf \{ \mathfrak{G}_\pi(u) : u \in S^\circ \cap \mathbb{S}^1 \text{ and an angular neighborhood of } u \text{ is contained in } C \}.$$

**Proposition 6.17** (Harnack-envelope criterion for exact logarithmic tails). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be a stationary distribution.*

(i) For every  $c \in S_\circ^\vee$ ,

$$\limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi\{z : c \cdot z \geq x\} \leq \mathfrak{H}_c^\pi.$$

Consequently, if  $\mathfrak{H}_c^\pi = \alpha_c^\pi$ , then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z : c \cdot z \geq x\} = -\alpha_c^\pi.$$

(ii) For the radial tail,

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r\} \leq \mathfrak{H}_{\text{rad}}^\pi.$$

Thus equality  $\mathfrak{H}_{\text{rad}}^\pi = \alpha_{\text{rad}}^\pi$  implies the exact radial logarithmic tail.

(iii) If  $C \subset S \cap \mathbb{S}^1$  is compact and contains a nonempty open angular subarc in  $S^\circ \cap \mathbb{S}^1$ , then

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r, z/|z| \in C\} \leq \mathfrak{H}_C^\pi.$$

Thus equality  $\mathfrak{H}_C^\pi = \alpha_C^\pi$  implies the exact sector logarithmic tail.

*Proof.* Fix  $c \in S_\circ^\vee$ , an interior direction  $u$ , and a number  $\Gamma > \mathfrak{G}_\pi(u)$ . Choose  $\rho, C, R$  as in [definition 6.16](#). For large  $x$ , put

$$r_x := \frac{x + \rho|c|}{c \cdot u}.$$

If  $z \in B_\rho(r_x u)$ , then  $c \cdot z \geq x$ . Hence

$$\pi\{z : c \cdot z \geq x\} \geq C e^{-\Gamma r_x}.$$

It follows that

$$\limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi\{z : c \cdot z \geq x\} \leq \frac{\Gamma}{c \cdot u}.$$

Letting  $\Gamma \downarrow \mathfrak{G}_\pi(u)$  and then infimizing over  $u$  gives the first displayed inequality. Since [proposition 6.2](#) gives the corresponding liminf as  $\alpha_c^\pi$ , equality  $\mathfrak{H}_c^\pi = \alpha_c^\pi$  gives the projected logarithmic limit.

For the radial statement, the defining Harnack bound gives

$$\pi(B_\rho(su)) \geq C e^{-\Gamma s}, \quad s \geq R.$$

Taking  $s = r + \rho$  and using  $B_\rho((r + \rho)u) \subset \{|z| \geq r\}$  yields

$$\pi\{|z| \geq r\} \geq C e^{-\Gamma(r+\rho)}.$$

Therefore

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r\} \leq \Gamma.$$

Letting  $\Gamma \downarrow \mathfrak{G}_\pi(u)$  and then taking the infimum over  $u$  proves the radial inequality.

For the sector statement, choose an interior direction  $u$  having a relative angular neighborhood  $U \subset C$ , and choose  $\eta > 0$  with  $\{v \in S \cap \mathbb{S}^1 : |v - u| < \eta\} \subset U$ . For  $z \in B_\rho((r + \rho)u)$ ,

$$\left| \frac{z}{|z|} - u \right| \leq \frac{2\rho}{r + \rho}.$$

Thus, for  $r > \max\{R, 2\rho/\eta\}$ , the ball lies in the sector over  $C$  and in  $\{|z| \geq r\}$ . Consequently

$$\pi \left\{ |z| \geq r, \frac{z}{|z|} \in C \right\} \geq C e^{-\Gamma(r+\rho)},$$

Taking negative logarithms in the preceding inequality gives

$$-\frac{1}{r} \log \pi \left\{ |z| \geq r, \frac{z}{|z|} \in C \right\} \leq \Gamma \left( 1 + \frac{\rho}{r} \right) - \frac{\log C}{r}.$$

Hence

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \left\{ |z| \geq r, \frac{z}{|z|} \in C \right\} \leq \Gamma.$$

Letting  $\Gamma \downarrow \mathfrak{G}_\pi(u)$  and then taking the infimum over the directions allowed in  $\mathfrak{H}_C^\pi$  proves the sector inequality. Combining these limsup estimates with the corresponding liminf identities in [proposition 6.4](#) proves all exact-tail implications.  $\square$

*Remark 6.18* (meaning of the Harnack envelope criterion). The quantities  $\mathfrak{H}_c^\pi$ ,  $\mathfrak{H}_{\text{rad}}^\pi$ , and  $\mathfrak{H}_C^\pi$  are attached to the stationary law, not only to the drift. They convert interior elliptic positivity into a lower-tail mechanism on interior angular sectors. For projected tails one must also allow the possibility that the dominant exponential contribution lies in a boundary layer. The finite angular decomposition below separates these two issues.

For a Borel angular set  $A \subset S \cap \mathbb{S}^1$  and  $c \in S_\circ^\vee$ , define the restricted projected tail

$$\Pi_{c,A}(x) := \pi\{z \in S \setminus \{0\} : c \cdot z \geq x, z/|z| \in A\}, \quad x \geq 0,$$

and its scalar abscissa

$$\alpha_{c,A}^\pi := \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \Pi_{c,A}(x),$$

with the convention that the value is  $+\infty$  if  $\Pi_{c,A}$  vanishes eventually. Equivalently,  $\alpha_{c,A}^\pi$  is the abscissa of convergence of the push-forward of the restricted measure  $\mathbf{1}_{\{z \neq 0, z/|z| \in A\}} \pi(dz)$  under  $z \mapsto c \cdot z$ .

**Proposition 6.19** (finite angular decomposition of projected tail rates). *Let  $\pi$  be a stationary distribution, let  $c \in S_\circ^\vee$ , and let  $A_1, \dots, A_m \subset S \cap \mathbb{S}^1$  be Borel sets whose union is  $S \cap \mathbb{S}^1$ . Then*

$$(6.33) \quad \alpha_c^\pi = \min_{1 \leq j \leq m} \alpha_{c,A_j}^\pi.$$

Moreover, if the restricted projected logarithmic limits exist for all  $j$  in the extended sense,

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \Pi_{c,A_j}(x) = -\alpha_{c,A_j}^\pi,$$

where the value  $-\infty$  is allowed, then the unrestricted projected logarithmic limit exists in the same extended sense and

$$(6.34) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z : c \cdot z \geq x\} = -\min_{1 \leq j \leq m} \alpha_{c,A_j}^\pi.$$

*Proof.* The vertex does not affect tails with  $x > 0$ . Since

$$\Pi_c(x) \leq \sum_{j=1}^m \Pi_{c,A_j}(x) \leq m \max_{1 \leq j \leq m} \Pi_{c,A_j}(x),$$

and  $\Pi_c(x) \geq \max_j \Pi_{c,A_j}(x)$ , the quantities  $-x^{-1} \log \Pi_c(x)$  and  $-x^{-1} \log \max_j \Pi_{c,A_j}(x)$  have the same liminf and, whenever all component limits exist in the extended sense, the same limit. Because

$$-\frac{1}{x} \log \max_j \Pi_{c,A_j}(x) = \min_j \left( -\frac{1}{x} \log \Pi_{c,A_j}(x) \right),$$

the finite-minimum identities give [\(6.33\)](#) and [\(6.34\)](#).  $\square$

**Corollary 6.20** (interior–boundary reduction for projected logarithmic tails). *Fix  $c \in S_\circ^\vee$  and  $0 < \delta < \xi/2$ . Write  $u(\varphi) := (\cos \varphi, \sin \varphi)$ . Let*

$$A_1^\delta := \{u(\varphi) : 0 \leq \varphi \leq \delta\}, \quad A_0^\delta := \{u(\varphi) : \delta \leq \varphi \leq \xi - \delta\}, \quad A_2^\delta := \{u(\varphi) : \xi - \delta \leq \varphi \leq \xi\}.$$

Then

$$(6.35) \quad \alpha_c^\pi = \min\{\alpha_{c,A_1^\delta}^\pi, \alpha_{c,A_0^\delta}^\pi, \alpha_{c,A_2^\delta}^\pi\}.$$

Consequently, the projected logarithmic tail is exact once the three restricted projected tails are exact. On the middle angular block  $A_0^\delta$ , interior Harnack methods and sector-frontier arguments apply uniformly away from the two faces. The two boundary blocks require estimates adapted to the corresponding face neighborhoods.

*Proof.* This is [proposition 6.19](#) applied to the displayed three-set cover. The last sentence records only the geometric location of the three blocks:  $A_0^\delta$  has positive angular distance from both faces, whereas  $A_1^\delta$  and  $A_2^\delta$  are boundary layers.  $\square$

**Proposition 6.21** (angular coefficient squeeze). *Let  $\pi$  be a stationary distribution, let  $c \in S_\circ^\vee$ , and let  $A \subset S \cap \mathbb{S}^1$  be a nonempty compact angular set. Put*

$$m_{c,A} := \min_{u \in A} c \cdot u, \quad M_{c,A} := \max_{u \in A} c \cdot u.$$

Then  $0 < m_{c,A} \leq M_{c,A} < \infty$ . If

$$\Pi_A(r) := \pi\{z \neq 0 : |z| \geq r, z/|z| \in A\}, \quad \alpha_A^{\pi, \text{rad}} := \liminf_{r \rightarrow \infty} -\frac{1}{r} \log \Pi_A(r),$$

with the usual convention  $\alpha_A^{\pi, \text{rad}} = +\infty$  when  $\Pi_A$  vanishes eventually, then for every  $x \geq 0$ ,

$$(6.36) \quad \Pi_A\left(\frac{x}{m_{c,A}}\right) \leq \Pi_{c,A}(x) \leq \Pi_A\left(\frac{x}{M_{c,A}}\right),$$

and consequently

$$(6.37) \quad \frac{\alpha_A^{\pi, \text{rad}}}{M_{c,A}} \leq \alpha_{c,A}^\pi \leq \frac{\alpha_A^{\pi, \text{rad}}}{m_{c,A}}.$$

The same inequalities hold for the corresponding limsup logarithmic rates whenever these are used in place of the liminf rates.

*Proof.* For  $z = ru$ ,  $u \in A$ , one has

$$rm_{c,A} \leq c \cdot z \leq rM_{c,A}.$$

Thus  $|z| \geq x/m_{c,A}$  and  $z/|z| \in A$  imply  $c \cdot z \geq x$ , while  $c \cdot z \geq x$  and  $z/|z| \in A$  imply  $|z| \geq x/M_{c,A}$ . This proves (6.36). Taking  $\liminf$  of  $-x^{-1} \log$  of the two sides gives (6.37); the extended cases follow from the same monotonicity convention. The limsup version is identical.  $\square$

**Corollary 6.22** (shrinking face-layer reduction). *Let  $c \in S_\circ^\vee$ . For  $i = 1, 2$ , let  $A_i^\delta$  be the boundary angular blocks from [corollary 6.20](#), with  $A_1^\delta$  shrinking to the ray generated by  $u_1$  and  $A_2^\delta$  shrinking to the ray generated by  $u_2$ . Suppose that, for some  $i$ , the radial boundary-layer abscissae have a limit*

$$\beta_i^\pi := \lim_{\delta \downarrow 0} \alpha_{A_i^\delta}^{\pi, \text{rad}} \in [0, \infty].$$

Then

$$(6.38) \quad \lim_{\delta \downarrow 0} \alpha_{c, A_i^\delta}^\pi = \frac{\beta_i^\pi}{c \cdot u_i},$$

with the convention  $+\infty/(c \cdot u_i) = +\infty$ . Hence a boundary-layer contribution to the projected logarithmic rate is determined, in the shrinking-layer limit, by the corresponding radial face-layer rate and the geometric projection factor  $c \cdot u_i$ .

*Proof.* For  $A_i^\delta$ , continuity of  $u \mapsto c \cdot u$  gives

$$m_{c, A_i^\delta} \longrightarrow c \cdot u_i, \quad M_{c, A_i^\delta} \longrightarrow c \cdot u_i \quad (\delta \downarrow 0).$$

Since  $c \in S_\circ^\vee$ ,  $c \cdot u_i > 0$ . Applying [proposition 6.21](#) to  $A_i^\delta$  and letting  $\delta \downarrow 0$  gives (6.38).  $\square$

**Definition 6.23** (local angular radial abscissa). Let  $K := S \cap \mathbb{S}^1$ . For a nonempty Borel set  $A \subset K$ , define

$$\alpha_A^{\pi, \text{rad}} := \sup \left\{ \lambda \geq 0 : \int_{\{z \neq 0: z/|z| \in A\}} e^{\lambda|z|} \pi(dz) < \infty \right\}.$$

For  $u \in K$ , define

$$\beta_\pi(u) := \sup \left\{ \alpha_A^{\pi, \text{rad}} : A \text{ is a relative open angular neighborhood of } u \text{ in } K \right\} \in [0, \infty].$$

**Theorem 6.24** (local-angular representation of scalar abscissae). *Assume  $0 < \xi < \pi$ . Let  $\pi$  be a stationary distribution and let  $K = S \cap S^1$ . For  $u \in K$  and  $\delta > 0$ , put*

$$A_\delta(u) := \{v \in K : d_{\text{ang}}(u, v) \leq \delta\},$$

where  $d_{\text{ang}}$  is angular distance relative to the wedge. Then

$$(6.39) \quad \lim_{\delta \downarrow 0} \alpha_{A_\delta(u)}^{\pi, \text{rad}} = \beta_\pi(u).$$

For every  $c \in S_\circ^\vee$ ,

$$(6.40) \quad \lim_{\delta \downarrow 0} \alpha_{c, A_\delta(u)}^\pi = \frac{\beta_\pi(u)}{c \cdot u},$$

with the usual convention when  $\beta_\pi(u) = +\infty$ . Moreover,

$$(6.41) \quad \alpha_c^\pi = \inf_{u \in K} \frac{\beta_\pi(u)}{c \cdot u},$$

and

$$(6.42) \quad \alpha_{\text{rad}}^\pi = \inf_{u \in K} \beta_\pi(u).$$

*Proof.* The sets  $A_\delta(u)$  form a relative neighborhood basis at  $u$ . Since restriction to a smaller angular set can only increase the exponential-moment abscissa, the left-hand side of (6.39) is increasing as  $\delta \downarrow 0$ . If  $U$  is a relative open neighborhood of  $u$ , then  $A_\delta(u) \subset U$  for all sufficiently small  $\delta$ , and hence  $\alpha_{A_\delta(u)}^{\pi, \text{rad}} \geq \alpha_U^{\pi, \text{rad}}$ . Conversely, the relative open ball  $\{v : d_{\text{ang}}(u, v) < \delta\}$  is contained in  $A_\delta(u)$ , so its abscissa is at least  $\alpha_{A_\delta(u)}^{\pi, \text{rad}}$ . Taking the supremum over relative open neighborhoods proves (6.39). Equation (6.40) now follows from [proposition 6.21](#), because

$$\min_{v \in A_\delta(u)} c \cdot v, \quad \max_{v \in A_\delta(u)} c \cdot v \longrightarrow c \cdot u.$$

We next prove (6.41). Since the unrestricted projected tail dominates every restricted projected tail,

$$\alpha_c^\pi \leq \alpha_{c, A_\delta(u)}^\pi.$$

Letting  $\delta \downarrow 0$  and using (6.40) gives

$$\alpha_c^\pi \leq \inf_{u \in K} \frac{\beta_\pi(u)}{c \cdot u}.$$

For the reverse inequality there is nothing to prove if the infimum is zero. Otherwise let

$$0 \leq q < \inf_{u \in K} \frac{\beta_\pi(u)}{c \cdot u}.$$

Fix  $u \in K$ , and define the positive slack

$$\Delta_u := \beta_\pi(u) - q(c \cdot u) > 0.$$

Choose

$$a_u := q(c \cdot u) + \frac{\Delta_u}{2}, \quad q(c \cdot u) < a_u < \beta_\pi(u).$$

By the definition of  $\beta_\pi(u)$ , there is a relative open neighborhood  $V_u$  of  $u$  with  $\alpha_{V_u}^{\pi, \text{rad}} > a_u$ . Continuity of  $v \mapsto c \cdot v$  gives  $\delta_u > 0$  such that the closed relative angular ball

$$W_u := \{v \in K : d_{\text{ang}}(u, v) \leq \delta_u\}$$

satisfies

$$q \sup_{v \in W_u} c \cdot v < a_u.$$

Put  $U_u := V_u \cap \text{int}_K W_u$ . Then  $U_u$  is a relative open neighborhood of  $u$ , restriction gives  $\alpha_{U_u}^{\pi, \text{rad}} \geq \alpha_{V_u}^{\pi, \text{rad}} > a_u$ , and

$$qM_u < a_u < \alpha_{U_u}^{\pi, \text{rad}}, \quad M_u := \sup_{v \in U_u} c \cdot v < \infty.$$

Compactness of  $K$  gives a finite subcover  $U_{u_1}, \dots, U_{u_N}$ . On the cone over  $U_{u_j}$ ,

$$e^{qc \cdot z} \leq e^{qM_{u_j}|z|},$$

and  $qM_{u_j} < \alpha_{U_{u_j}}^{\pi, \text{rad}}$ . Summing the corresponding finite restricted moments yields

$$\int_S e^{qc \cdot z} \pi(dz) < \infty.$$

Thus  $q \leq \alpha_c^\pi$ . Letting  $q$  increase to the infimum proves (6.41).

The radial formula is proved in the same way. The unrestricted radial tail dominates each angular restriction, giving

$$\alpha_{\text{rad}}^\pi \leq \inf_{u \in K} \beta_\pi(u).$$

If  $q < \inf_{u \in K} \beta_\pi(u)$ , then for every  $u \in K$  the definition of  $\beta_\pi(u)$  supplies a relative open neighborhood  $U_u$  satisfying

$$\alpha_{U_u}^{\pi, \text{rad}} > q.$$

Choose a finite subcover  $U_{u_1}, \dots, U_{u_N}$ . Since

$$\int_{\{z \neq 0: z/|z| \in U_{u_j}\}} e^{q|z|} \pi(dz) < \infty \quad (j = 1, \dots, N),$$

and the union of these angular blocks covers every nonzero point of  $S$ ,

$$\int_S e^{q|z|} \pi(dz) \leq \pi(\{0\}) + \sum_{j=1}^N \int_{\{z \neq 0: z/|z| \in U_{u_j}\}} e^{q|z|} \pi(dz) < \infty.$$

Thus  $q \leq \alpha_{\text{rad}}^\pi$ . Letting  $q$  increase to the infimum proves (6.42).  $\square$

**Corollary 6.25** (scaling of the local angular profile). *Under the notation of theorem 6.6,*

$$\beta_{\pi_r}(u) = r^{-1} \beta_\pi(u), \quad u \in S \cap \mathbb{S}^1.$$

*Hence the variational formulas (6.41)–(6.42) are homogeneous of degree one along drift rays.*

*Proof.* Positive dilation leaves angular direction unchanged and sends  $|z|$  to  $r|z|$ . Therefore every restricted radial abscissa is multiplied by  $r^{-1}$ . Taking the supremum over angular neighborhoods proves the claim.  $\square$

**Corollary 6.26** (boundary-layer and interior contributions). *The full projected abscissa is determined by the local radial profile  $\beta_\pi$  through (6.41). In particular, face directions and interior directions enter the same variational formula. This is the intermediate angular reduction used by the later broken-path lower bounds, which establish the required no-gap estimate on the minimizing blocks throughout the classified regimes.*

*Remark 6.27* (role of the angular decomposition). The Harnack envelope criterion is an interior-sector mechanism. Corollary 6.20 prevents it from being used beyond its range: a projected tail may be governed by one of the two boundary layers. Proposition 6.21 and corollary 6.22 give the corresponding geometric conversion between projected boundary-layer rates and radial rates in shrinking face neighborhoods. The later one-face and broken-path estimates treat those boundary layers directly and complete the resulting finite angular decomposition.

**Proposition 6.28** (dominant-block principle for finite angular decompositions). *Let  $A_1, \dots, A_m$  be a finite Borel cover of  $S \cap \mathbb{S}^1$ , fix  $c \in S_\circ^\vee$ , and write*

$$a_j := \alpha_{c, A_j}^\pi, \quad a_* := \min_{1 \leq j \leq m} a_j.$$

*Assume  $a_* < \infty$ . If at least one block  $A_{j^*}$  with  $a_{j^*} = a_*$  has an exact restricted logarithmic tail,*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \Pi_{c, A_{j^*}}(x) = -a_*,$$

*then the unrestricted projected tail is exact and*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z : c \cdot z \geq x\} = -a_*.$$

*No logarithmic limit is required for any of the remaining blocks.*

*Proof.* The exact tail of the minimizing block gives

$$\Pi_c(x) \geq \Pi_{c, A_{j^*}}(x),$$

and hence the required logarithmic lower bound. For the upper bound, fix  $\varepsilon > 0$ . Since  $a_* - \varepsilon < a_j$  for every  $j$ , the definition

$$a_j = \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \Pi_{c, A_j}(x)$$

gives

$$\Pi_{c, A_j}(x) \leq e^{-(a_* - \varepsilon)x}$$

for all sufficiently large  $x$ , with the conclusion also valid when  $a_j = +\infty$ . Since the cover is finite and  $\Pi_c \leq \sum_j \Pi_{c, A_j}$ , the logarithmic upper bound follows. Letting  $\varepsilon \downarrow 0$  proves the result.  $\square$

**Proposition 6.29** (angular minimizers and localization of the projected frontier). *Assume  $0 < \xi < \pi$ , let  $\pi$  be a stationary distribution, and put*

$$K := S \cap \mathbb{S}^1.$$

*The local radial profile  $u \mapsto \beta_\pi(u)$  is lower semicontinuous on  $K$ . Consequently, for every  $c \in S_\circ^\vee$ , the extended-valued function*

$$\rho_c^\pi(u) := \frac{\beta_\pi(u)}{c \cdot u}, \quad u \in K,$$

*is lower semicontinuous. If  $\alpha_c^\pi < \infty$ , then the angular minimizing set*

$$\mathcal{A}_c^\pi := \{u \in K : \rho_c^\pi(u) = \alpha_c^\pi\}$$

*is nonempty and compact.*

*Let  $U \subset K$  be any relative open set containing  $\mathcal{A}_c^\pi$ , and put  $B := K \setminus U$ . With the convention that the abscissa of an empty angular block is  $+\infty$ , one has*

$$(6.43) \quad \alpha_{c, U}^\pi = \alpha_c^\pi, \quad \alpha_{c, B}^\pi > \alpha_c^\pi.$$

*Therefore, if the restricted tail on  $U$  has the exact logarithmic limit*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \Pi_{c, U}(x) = -\alpha_c^\pi,$$

*then the unrestricted projected tail has the same exact logarithmic rate.*

*Proof.* Let  $a < \beta_\pi(u)$ . By definition, there is a relative open neighborhood  $V$  of  $u$  such that

$$\alpha_V^{\pi, \text{rad}} > a.$$

For every  $v \in V$ , the same set  $V$  is a relative open neighborhood of  $v$ , and hence  $\beta_\pi(v) \geq \alpha_V^{\pi, \text{rad}} > a$ . Thus  $\{u : \beta_\pi(u) > a\}$  is open, proving lower semicontinuity. Since  $c \cdot u$  is continuous and strictly positive on the compact set  $K$ , the quotient  $\rho_c^\pi$  is also lower semicontinuous.

By [theorem 6.24](#),

$$\alpha_c^\pi = \inf_{u \in K} \rho_c^\pi(u).$$

If this number is finite, lower semicontinuity and compactness of  $K$  imply that the infimum is attained; its minimizing set is closed in  $K$  and therefore compact.

If  $B = \emptyset$ , the second assertion in (6.43) is immediate. Suppose  $B \neq \emptyset$ . Since  $B$  is compact and disjoint from  $\mathcal{A}_c^\pi$ , lower semicontinuity gives

$$m_B := \min_{u \in B} \rho_c^\pi(u) > \alpha_c^\pi.$$

Fix  $0 \leq q < m_B$ . For each  $u \in B$ , the argument used in the proof of [theorem 6.24](#) gives a relative open neighborhood  $V_u$  such that

$$q \max_{v \in V_u} c \cdot v < \alpha_{V_u}^{\pi, \text{rad}}.$$

A finite subcover of the compact set  $B$  then shows that

$$\int_{\{z \neq 0: z/|z| \in B\}} e^{qc \cdot z} \pi(dz) < \infty.$$

Hence  $q \leq \alpha_{c,B}^\pi$ . Letting  $q \uparrow m_B$  yields

$$\alpha_{c,B}^\pi \geq m_B > \alpha_c^\pi.$$

By [proposition 6.19](#), the two-set angular decomposition  $K = U \cup B$  gives

$$\alpha_c^\pi = \min\{\alpha_{c,U}^\pi, \alpha_{c,B}^\pi\},$$

so necessarily  $\alpha_{c,U}^\pi = \alpha_c^\pi$ . The final statement follows from [proposition 6.28](#).  $\square$

**Corollary 6.30** (invariance of angular minimizers along drift rays). *Under the notation of [theorem 6.6](#),*

$$\mathcal{A}_c^{\pi r} = \mathcal{A}_c^\pi, \quad c \in S_\circ^\vee.$$

*Thus positive dilation changes the value of the projected rate but not the angular set on which the variational formula is minimized.*

*Proof.* By [corollary 6.25](#),

$$\rho_c^{\pi r}(u) = r^{-1} \rho_c^\pi(u)$$

for every  $u \in K$ . Multiplication by the positive constant  $r^{-1}$  leaves the minimizing set unchanged.  $\square$

**Proposition 6.31** (local Harnack comparison at angular minimizers). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be a stationary distribution. For every*

$$u \in S^\circ \cap \mathbb{S}^1$$

*one has*

$$(6.44) \quad \beta_\pi(u) \leq \mathfrak{G}_\pi(u).$$

*Consequently, let  $c \in S_\circ^\vee$  with  $\alpha_c^\pi < \infty$ . If there is an interior angular minimizer*

$$u_* \in \mathcal{A}_c^\pi \cap S^\circ$$

*such that*

$$(6.45) \quad \mathfrak{G}_\pi(u_*) = \beta_\pi(u_*),$$

*then*

$$(6.46) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z : c \cdot z \geq x\} = -\alpha_c^\pi.$$

*Likewise, if  $u_* \in S^\circ \cap \mathbb{S}^1$  satisfies*

$$\beta_\pi(u_*) = \alpha_{\text{rad}}^\pi < \infty \quad \text{and} \quad \mathfrak{G}_\pi(u_*) = \beta_\pi(u_*),$$

*then*

$$(6.47) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log \pi\{|z| \geq r\} = -\alpha_{\text{rad}}^\pi.$$

*Proof.* Fix  $u \in S^\circ \cap \mathbb{S}^1$  and  $\Gamma > \mathfrak{G}_\pi(u)$ . Choose  $\rho, C, R$  as in [definition 6.16](#). Let  $U$  be any relative open angular neighborhood of  $u$  in  $K = S \cap \mathbb{S}^1$ . For all sufficiently large  $s$ , the ball  $B_\rho(su)$  is contained in the cone over  $U$ . Since every  $z \in B_\rho((r+\rho)u)$  satisfies  $|z| \geq r$ , one obtains

$$\pi\{z \neq 0 : |z| \geq r, z/|z| \in U\} \geq Ce^{-\Gamma(r+\rho)}$$

for all sufficiently large  $r$ . The one-dimensional tail-abscissa identity [lemma 6.1](#), applied to the restricted radial measure, therefore gives

$$\alpha_U^{\pi, \text{rad}} \leq \Gamma.$$

Taking the supremum over relative open neighborhoods  $U$  and then letting  $\Gamma \downarrow \mathfrak{G}_\pi(u)$  proves [\(6.44\)](#).

Now let  $u_* \in \mathcal{A}_c^\pi \cap S^\circ$  satisfy [\(6.45\)](#). By the variational formula [\(6.41\)](#),

$$\alpha_c^\pi = \frac{\beta_\pi(u_*)}{c \cdot u_*} = \frac{\mathfrak{G}_\pi(u_*)}{c \cdot u_*}.$$

Hence  $\mathfrak{H}_c^\pi \leq \alpha_c^\pi$ . The reverse inequality follows from the scalar abscissa identity together with the Harnack bound in [proposition 6.17](#). Thus  $\mathfrak{H}_c^\pi = \alpha_c^\pi$ , and [\(6.46\)](#) follows from that proposition.

For the radial assertion, the hypotheses give

$$\mathfrak{H}_{\text{rad}}^\pi = \inf_{u \in S^\circ \cap \mathbb{S}^1} \mathfrak{G}_\pi(u) \leq \mathfrak{G}_\pi(u_*) = \beta_\pi(u_*) = \alpha_{\text{rad}}^\pi.$$

On the other hand, the radial tail-abscissa identity and the radial Harnack-envelope inequality imply

$$\alpha_{\text{rad}}^\pi \leq \mathfrak{H}_{\text{rad}}^\pi.$$

Thus  $\mathfrak{H}_{\text{rad}}^\pi = \alpha_{\text{rad}}^\pi$ , and the radial part of [proposition 6.17](#) yields [\(6.47\)](#).  $\square$

**6.2. Kinematic outer bounds from the killed wedge kernel.** The local Harnack exponents in the preceding subsection are attached to the stationary law. The next argument gives an explicit upper bound depending only on the drift and the wedge. It uses trajectories that remain in the open wedge up to a prescribed time; reflection plays no role on this event.

**Definition 6.32** (kinematic directional cost and outer domain). Let  $\mu \neq 0$ , put  $m := |\mu|$ , and define

$$(6.48) \quad \mathfrak{J}_\mu(u) := m - \mu \cdot u, \quad u \in S \cap \mathbb{S}^1.$$

The associated closed convex set is

$$(6.49) \quad \mathcal{K}_{\text{kin}}(\mu) := \{\theta \in \mathbb{R}^2 : \theta \cdot u \leq \mathfrak{J}_\mu(u) \text{ for every } u \in S \cap \mathbb{S}^1\}.$$

For  $c \in S_\circ^\vee$ , set

$$(6.50) \quad \mathfrak{J}_c(\mu) := \min_{u \in S \cap \mathbb{S}^1} \frac{\mathfrak{J}_\mu(u)}{c \cdot u},$$

and define the radial kinematic rate by

$$(6.51) \quad \mathfrak{J}_{\text{rad}}(\mu) := \min_{u \in S \cap \mathbb{S}^1} \mathfrak{J}_\mu(u).$$

**Lemma 6.33** (killed heat kernel in the wedge). *Let  $x, y \in S^\circ$ , write*

$$x = (r_x, \theta_x), \quad y = (r_y, \theta_y),$$

and let

$$\tau_{\partial S} := \inf\{t \geq 0 : Z_t \in \partial S\}.$$

*Then the transition law of the reflected diffusion before  $\tau_{\partial S}$  is the transition law of Brownian motion with drift  $\mu$  killed on leaving  $S^\circ$ . Its density with respect to planar Lebesgue measure is*

$$(6.52) \quad p_S^\mu(t, x, y) = e^{\mu \cdot (y-x) - |\mu|^2 t / 2} p_S^0(t, x, y),$$

where

$$(6.53) \quad p_S^0(t, x, y) = \frac{2}{\xi t} e^{-(r_x^2 + r_y^2)/(2t)} \sum_{n=1}^{\infty} I_{n\pi/\xi} \left( \frac{r_x r_y}{t} \right) \sin \left( \frac{n\pi\theta_x}{\xi} \right) \sin \left( \frac{n\pi\theta_y}{\xi} \right).$$

Here  $I_\nu$  denotes the modified Bessel function of the first kind. The series converges locally uniformly on  $(0, \infty) \times S^\circ \times S^\circ$ , and the kernel is strictly positive there.

*Proof.* Let  $\tau = \tau_{\partial S}$ . If  $\varphi \in C_c^2(S^\circ)$ , then  $\varphi$  and  $-\varphi$  are both admissible tests. Applying the two submartingale inequalities shows that

$$M_t^\varphi := \varphi(Z_t) - \varphi(x) - \int_0^t \left( \frac{1}{2} \Delta \varphi + \mu \cdot \nabla \varphi \right) (Z_s) ds$$

is a martingale. Since  $\varphi$  and its generator are bounded, optional stopping at the bounded stopping time  $t \wedge \tau$  shows that

$$\varphi(Z_{t \wedge \tau}) - \varphi(x) - \int_0^{t \wedge \tau} \left( \frac{1}{2} \Delta \varphi + \mu \cdot \nabla \varphi \right) (Z_s) ds$$

is a martingale in  $t$ . Thus the coordinate process, stopped on first reaching  $\partial S$  and then sent to a cemetery state, solves the killed martingale problem for  $\frac{1}{2} \Delta + \mu \cdot \nabla$  on  $S^\circ$ . The ordinary SDE  $x + \mu t + B_t$ , killed at its first exit from  $S^\circ$ , solves the same problem. Local uniqueness of the Brownian martingale problem in  $\mathbb{R}^2$ , followed by stopping at the exit time, gives uniqueness of the killed problem. Hence the two killed laws agree.

We next derive the zero-drift kernel. In polar coordinates the Dirichlet angular eigenfunctions form the orthonormal basis

$$e_n(\theta) = \sqrt{\frac{2}{\xi}} \sin \left( \frac{n\pi\theta}{\xi} \right), \quad n \geq 1,$$

with angular eigenvalues  $(n\pi/\xi)^2$ . For  $\nu_n = n\pi/\xi$ , the radial heat kernel of

$$\frac{1}{2} (\partial_{rr} + r^{-1} \partial_r - \nu_n^2 r^{-2})$$

with respect to the measure  $r_y dr_y$  is

$$k_{\nu_n}(t, r_x, r_y) = \frac{1}{t} \exp \left\{ -\frac{r_x^2 + r_y^2}{2t} \right\} I_{\nu_n} \left( \frac{r_x r_y}{t} \right).$$

Expanding the heat semigroup in the basis  $\{e_n\}$  and using planar Lebesgue measure  $r_y dr_y d\theta_y$  gives

$$p_S^0(t, x, y) = \sum_{n=1}^{\infty} e_n(\theta_x) e_n(\theta_y) k_{\nu_n}(t, r_x, r_y),$$

which is exactly (6.53).

We verify local uniform convergence quantitatively. On a compact subset of  $(0, \infty) \times S^\circ \times S^\circ$ , the Bessel argument  $z = r_x r_y / t$  is bounded by some  $M < \infty$ . The power series for  $I_\nu$ , together with  $\Gamma(\nu + k + 1) \geq \Gamma(\nu + 1) k!$  for  $\nu \geq 0$ , gives

$$(6.54) \quad 0 \leq I_\nu(z) \leq \frac{(M/2)^\nu}{\Gamma(\nu + 1)} e^{M^2/4}, \quad 0 \leq z \leq M.$$

Set  $p := \pi/\xi > 1$  and  $A := (M/2)^p$ . Since  $np + 1 \geq n + 1 \geq 2$  and the gamma function is increasing on  $[2, \infty)$ ,

$$\Gamma(np + 1) \geq \Gamma(n + 1) = n!.$$

Consequently

$$\sum_{n=1}^{\infty} \frac{(M/2)^{np}}{\Gamma(np + 1)} \leq \sum_{n=1}^{\infty} \frac{A^n}{n!} = e^A - 1 < \infty.$$

The sine factors are bounded by one, so (6.54) and the Weierstrass test prove local uniform convergence, and hence continuity, of the kernel series on the stated compact sets. No termwise differentiation of the series is used below.

We now prove strict positivity without appealing to a one-line maximum-principle argument. Fix  $t > 0$ ,  $x \in S^\circ$ , and  $y_0 \in S^\circ$ . Since  $S = \{z : n_1 \cdot z \geq 0, n_2 \cdot z \geq 0\}$ , set

$$d_x := \min_{j=1,2} n_j \cdot x > 0, \quad d_y := \min_{j=1,2} n_j \cdot y_0 > 0,$$

and choose  $0 < \rho < d_y/2$ . Let  $V = \overline{B}_\rho(y_0)$ . Then  $V \subseteq S^\circ$ , because for  $y \in V$ ,

$$n_j \cdot y \geq n_j \cdot y_0 - |y - y_0| \geq d_y - \rho > \frac{d_y}{2}, \quad j = 1, 2.$$

For

$$\ell_y(s) := x + \frac{s}{t}(y - x), \quad 0 \leq s \leq t,$$

one has, writing  $\lambda = s/t$ ,

$$n_j \cdot \ell_y(s) = (1 - \lambda)n_j \cdot x + \lambda n_j \cdot y \geq \min\{d_x, d_y/2\}.$$

For every point  $z \in S$ , the distance from  $z$  to either boundary ray is at least its distance to the supporting line of that ray. Since  $n_j$  is the inward unit normal to the  $j$ th supporting line,

$$\text{dist}(z, \partial S) = \min_{j=1,2} \text{dist}(z, \partial S_j) \geq \min_{j=1,2} n_j \cdot z.$$

Applying this inequality to the preceding lower bounds gives

$$(6.55) \quad d_* := \inf_{y \in V} \inf_{0 \leq s \leq t} \text{dist}(\ell_y(s), \partial S) \geq \min\{d_x, d_y/2\} > 0.$$

Under the Brownian bridge from  $x$  to  $y$  in time  $t$ , the path has the representation  $\ell_y + \beta$ , where the centered bridge  $\beta$  has a law independent of  $x$  and  $y$ . If  $W$  is standard planar Brownian motion, then  $\beta(s) \stackrel{d}{=} W(s) - (s/t)W(t)$ . Therefore

$$\left\{ \sup_{0 \leq s \leq t} |W(s)| < \frac{d_*}{4} \right\} \subset \left\{ \sup_{0 \leq s \leq t} |\beta(s)| < \frac{d_*}{2} \right\}.$$

Set  $a_* := d_*/(4\sqrt{2})$ . For a one-dimensional standard Brownian motion  $W^{(1)}$ , the classical absorbing-interval expansion gives

$$p(a_*, t) := \mathbf{P} \left\{ \sup_{0 \leq s \leq t} |W^{(1)}(s)| < a_* \right\} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp \left\{ -\frac{(2k+1)^2 \pi^2 t}{8a_*^2} \right\} > 0.$$

The series is alternating with strictly decreasing absolute terms, and its first term exceeds the second, so its sum is positive. Independence of the two coordinates gives

$$\mathbf{P} \left\{ \sup_{0 \leq s \leq t} |W^{(j)}(s)| < a_*, \quad j = 1, 2 \right\} = p(a_*, t)^2 > 0.$$

On this event  $\sup_{s \leq t} |W(s)| < \sqrt{2}a_* = d_*/4$ . Consequently

$$p_* := p(a_*, t)^2 > 0$$

is independent of  $y \in V$ , and the centered bridge remains within distance  $d_*/2$  of  $\ell_y$ ; hence the bridge stays inside  $S^\circ$  with probability at least  $p_*$ .

Let

$$g_t(x, y) := (2\pi t)^{-1} \exp\{-|y - x|^2/(2t)\}$$

be the free planar heat kernel. Brownian-bridge disintegration gives, for bounded Borel  $f$ ,

$$\mathbb{E}_x^0[f(B_t); t < \tau] = \int_{S^\circ} f(y) g_t(x, y) \mathbb{P}_{x,y}^t \{B_s \in S^\circ \ (0 \leq s \leq t)\} dy.$$

Consequently the killed density is bounded below almost everywhere on  $V$  by

$$p_* \inf_{y \in V} g_t(x, y) > 0.$$

The series in (6.53) is a continuous version of that density by the local-uniform convergence proved above. Since it agrees almost everywhere with a nonnegative density, continuity first shows that it is nonnegative everywhere. A continuous nonnegative function that is bounded below by a positive constant almost everywhere on  $V$  has the same lower bound everywhere on  $V$ . In particular,  $p_S^0(t, x, y_0) > 0$ . Since  $y_0$  was arbitrary, the kernel is strictly positive throughout  $(0, \infty) \times S^\circ \times S^\circ$ .

Finally, Cameron–Martin–Girsanov on the survival event gives, for every bounded Borel  $g$ ,

$$\mathbb{E}_x^\mu[g(Z_t); t < \tau] = \mathbb{E}_x^0 \left[ g(Z_t) e^{\mu \cdot (Z_t - x) - |\mu|^2 t / 2}; t < \tau \right].$$

Substituting the zero-drift density proves (6.52).  $\square$

**Proposition 6.34** (kinematic lower bound along every interior direction). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for a nonzero drift  $\mu$ . Fix*

$$u \in S^\circ \cap \mathbb{S}^1.$$

*Then there exist  $\rho > 0$ ,  $C > 0$ , and  $R < \infty$  such that*

$$(6.56) \quad \pi(B_\rho(ru)) \geq Cr^{-1} e^{-r\mathfrak{J}_\mu(u)}, \quad r \geq R.$$

*Consequently,*

$$(6.57) \quad \mathfrak{G}_\pi(u) \leq \mathfrak{J}_\mu(u), \quad \beta_\pi(u) \leq \mathfrak{G}_\pi(u) \leq \mathfrak{J}_\mu(u).$$

*Proof.* Put  $m = |\mu| > 0$ . By [proposition 3.7](#) and [corollary 3.8](#), the stationary law has a strictly positive continuous density in  $S^\circ$ . Choose a closed ball  $K_0 \Subset S^\circ$  with  $\pi(K_0) > 0$ . Choose  $\rho > 0$  and a compact angular interval  $A \Subset (0, \xi)$  such that, for all sufficiently large  $r$ , every point of  $B_\rho(ru)$  has polar angle in  $A$ .

Set  $t_r = r/m$ . Since  $K_0$  is compact and separated from the vertex, there are constants  $0 < r_- < r_+ < \infty$  such that  $r_- \leq |x| \leq r_+$  for  $x \in K_0$ . For  $y \in B_\rho(ru)$ ,

$$\frac{|x||y|}{t_r} = m|x| \frac{|y|}{r}$$

therefore belongs, for all large  $r$ , to a fixed compact interval  $[z_-, z_+] \Subset (0, \infty)$ , uniformly over  $x \in K_0$  and  $y \in B_\rho(ru)$ . The angular variable of  $x$  also ranges in a compact subinterval of  $(0, \xi)$ . Define

$$\Phi(z, \theta, \varphi) := \sum_{n=1}^{\infty} I_{n\pi/\xi}(z) \sin\left(\frac{n\pi\theta}{\xi}\right) \sin\left(\frac{n\pi\varphi}{\xi}\right).$$

By the locally uniform convergence in [lemma 6.33](#),  $\Phi$  is continuous on the resulting compact parameter set. The positivity of the killed kernel and the positive prefactors in (6.53) imply  $\Phi > 0$  there. Hence its minimum on that compact set is a number  $c_\Phi > 0$ .

We also make the exponential remainder uniform. Write  $y = ru + e$ ,  $|e| \leq \rho$ . Since  $t_r = r/m$ ,

$$(6.58) \quad \begin{aligned} & \mu \cdot (y - x) - \frac{m^2 t_r}{2} - \frac{|x|^2 + |y|^2}{2t_r} \\ &= -r(m - \mu \cdot u) + (\mu - mu) \cdot e - \mu \cdot x - \frac{m}{2r}(|x|^2 + |e|^2). \end{aligned}$$

Let

$$r_+ := \max_{x \in K_0} |x|, \quad C_0 := \rho|\mu - mu| + |\mu|r_+ + \frac{m}{2}(r_+^2 + \rho^2).$$

For  $r \geq 1$ ,  $x \in K_0$ , and  $|e| \leq \rho$ , the remainder in (6.58) satisfies

$$\begin{aligned} & (\mu - mu) \cdot e - \mu \cdot x - \frac{m}{2r}(|x|^2 + |e|^2) \\ & \geq -\rho|\mu - mu| - |\mu|r_+ - \frac{m}{2}(r_+^2 + \rho^2) = -C_0. \end{aligned}$$

Combining this explicit bound,  $t_r^{-1} = m/r$ , and the compact minimum  $c_\Phi$  in (6.53) yields

$$(6.59) \quad p_S^\mu(t_r, x, y) \geq c_0 r^{-1} e^{-r\mathfrak{J}_\mu(u)}$$

for a constant  $c_0 > 0$ , all  $x \in K_0$ , all  $y \in B_\rho(ru)$ , and all sufficiently large  $r$ .

Stationarity and the killed-process identity now give

$$\begin{aligned} \pi(B_\rho(ru)) &= \int_S P_{t_r}^\mu(x, B_\rho(ru)) \pi(dx) \\ &\geq \int_{K_0} \int_{B_\rho(ru)} p_S^\mu(t_r, x, y) dy \pi(dx) \\ &\geq c_0 \pi(K_0) |B_\rho(0)| r^{-1} e^{-r\mathfrak{J}_\mu(u)}. \end{aligned}$$

This proves (6.56).

For every  $\varepsilon > 0$ , one has  $r^{-1} \geq e^{-\varepsilon r}$  for all sufficiently large  $r$ . Thus the preceding estimate is an admissible Harnack-ball lower bound with exponent  $\mathfrak{J}_\mu(u) + \varepsilon$ , and the definition of  $\mathfrak{G}_\pi(u)$  gives  $\mathfrak{G}_\pi(u) \leq \mathfrak{J}_\mu(u) + \varepsilon$ . Letting  $\varepsilon \downarrow 0$  and using proposition 6.31 proves (6.57).  $\square$

**Corollary 6.35** (kinematic outer bounds for the transform domain). *Under the hypotheses of proposition 6.34,*

$$(6.60) \quad D_\pi \subset \mathcal{K}_{\text{kin}}(\mu).$$

For every  $c \in S_\circ^\vee$ ,

$$(6.61) \quad \alpha_c^{\text{in}}(\mu) \leq \alpha_c^\pi \leq \mathfrak{H}_c^\pi \leq \mathfrak{J}_c(\mu),$$

and

$$(6.62) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^\pi \leq \mathfrak{H}_{\text{rad}}^\pi \leq \mathfrak{J}_{\text{rad}}(\mu).$$

*Proof.* Let  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$ , and fix  $u \in S^\circ \cap \mathbb{S}^1$ . For  $z \in B_\rho(ru)$ ,

$$\theta \cdot z \geq r\theta \cdot u - \rho|\theta|.$$

Together with (6.56), this gives

$$\int_S e^{\theta \cdot z} \pi(dz) \geq Cr^{-1} \exp\{r(\theta \cdot u - \mathfrak{J}_\mu(u)) - \rho|\theta|\}.$$

Finiteness of the left-hand side for every  $r$  forces  $\theta \cdot u \leq \mathfrak{J}_\mu(u)$ . Continuity extends the inequality to the two boundary directions, and closure gives (6.60).

The first inequalities in (6.61) and (6.62) are the inner-domain bounds. The middle inequalities are the Harnack-envelope bounds of proposition 6.17. Finally, (6.57) gives

$$\mathfrak{H}_c^\pi = \inf_{u \in S^\circ \cap \mathbb{S}^1} \frac{\mathfrak{G}_\pi(u)}{c \cdot u} \leq \inf_{u \in S^\circ \cap \mathbb{S}^1} \frac{\mathfrak{J}_\mu(u)}{c \cdot u} = \mathfrak{J}_c(\mu),$$

and, by the radial definitions,

$$\mathfrak{H}_{\text{rad}}^\pi = \inf_{u \in S^\circ \cap \mathbb{S}^1} \mathfrak{G}_\pi(u) \leq \inf_{u \in S^\circ \cap \mathbb{S}^1} \mathfrak{J}_\mu(u) = \mathfrak{J}_{\text{rad}}(\mu).$$

This proves the last inequality in (6.62).  $\square$

**Proposition 6.36** (convex geometry of the kinematic and inner domains). *Assume  $\mu \neq 0$  and  $\mathcal{E}(\mu) \neq \emptyset$ , and let*

$$\mathbb{B}_\mu := \overline{B}(-\mu, |\mu|).$$

Then

$$(6.63) \quad \mathcal{K}_{\text{kin}}(\mu) = \mathbb{B}_\mu - S^\vee.$$

Moreover,

$$(6.64) \quad \mathfrak{J}_c(\mu) = \sup\{\lambda \geq 0 : \lambda c \in \mathcal{K}_{\text{kin}}(\mu)\}, \quad c \in S^\vee_\circ.$$

Define the closed inner domain

$$(6.65) \quad \mathcal{K}_{\text{in}}(\mu) := \overline{\mathcal{E}(\mu)} - S^\vee.$$

Then

$$(6.66) \quad \mathcal{K}_{\text{in}}(\mu) \subset \mathcal{K}_{\text{kin}}(\mu),$$

and

$$(6.67) \quad \alpha_c^{\text{in}}(\mu) = \sup\{\lambda \geq 0 : \lambda c \in \mathcal{K}_{\text{in}}(\mu)\}.$$

Consequently,

$$(6.68) \quad \alpha_c^{\text{in}}(\mu) = \mathfrak{J}_c(\mu)$$

if and only if the kinematic endpoint  $\mathfrak{J}_c(\mu)c$  belongs to  $\mathcal{K}_{\text{in}}(\mu)$ , equivalently if and only if there exist

$$\vartheta \in \overline{\mathcal{E}(\mu)}, \quad \eta \in S^\vee,$$

such that

$$(6.69) \quad \vartheta - \eta = \mathfrak{J}_c(\mu)c.$$

*Proof.* For  $z \in S$ , the support function of  $\mathbb{B}_\mu$  is

$$h_{\mathbb{B}_\mu}(z) = |\mu||z| - \mu \cdot z.$$

The support function of  $-S^\vee$  is zero on  $S$  and is infinite outside  $S$ . Since the sum of a compact set and a closed convex cone is closed, the support-function characterization of closed convex sets gives

$$\mathbb{B}_\mu - S^\vee = \{\theta : \theta \cdot z \leq |\mu||z| - \mu \cdot z \text{ for every } z \in S\}.$$

For every nonzero  $z \in S$ , write

$$z = |z|u, \quad u := z/|z| \in S \cap \mathbb{S}^1.$$

Then

$$\theta \cdot z \leq |\mu||z| - \mu \cdot z \iff \theta \cdot u \leq |\mu| - \mu \cdot u = \mathfrak{J}_\mu(u).$$

The inequality at  $z = 0$  is automatic. Hence the displayed support representation is precisely (6.49), proving (6.63).

For  $c \in S^\vee_\circ$ , the condition  $\lambda c \in \mathcal{K}_{\text{kin}}(\mu)$  is equivalent to

$$\lambda \leq \frac{\mathfrak{J}_\mu(u)}{c \cdot u} \quad \text{for every } u \in S \cap \mathbb{S}^1.$$

This proves (6.64).

By lemma 5.5,

$$\overline{\mathcal{E}(\mu)} \subset \{\theta : |\theta + \mu| \leq |\mu|\} = \mathbb{B}_\mu.$$

Subtracting  $S^\vee$  gives (6.66).

Set  $A := \mathcal{E}(\mu) - S^\vee$ . The set  $A$  is downward closed with respect to  $S^\vee$ : if  $x = \vartheta - \eta \in A$  and  $x - y \in S^\vee$ , then

$$y = \vartheta - (\eta + x - y), \quad \eta + x - y \in S^\vee,$$

so  $y \in A$ . We next identify its closure. If  $\vartheta \in \overline{\mathcal{E}(\mu)}$  and  $\eta \in S^\vee$ , choose  $\vartheta_n \in \mathcal{E}(\mu)$  with  $\vartheta_n \rightarrow \vartheta$ ; then  $\vartheta_n - \eta \rightarrow \vartheta - \eta$ , and hence

$$\overline{\mathcal{E}(\mu)} - S^\vee \subset \overline{A}.$$

Conversely, the set on the left is closed. Indeed, if  $x_n = \vartheta_n - \eta_n \rightarrow x$ , with  $\vartheta_n \in \overline{\mathcal{E}(\mu)}$  and  $\eta_n \in S^\vee$ , compactness of  $\overline{\mathcal{E}(\mu)}$  gives, along a subsequence,  $\vartheta_n \rightarrow \vartheta \in \overline{\mathcal{E}(\mu)}$ . Then  $\eta_n = \vartheta_n - x_n \rightarrow \vartheta - x \in S^\vee$ , because  $S^\vee$  is closed, and therefore  $x = \vartheta - (\vartheta - x) \in \overline{\mathcal{E}(\mu)} - S^\vee$ . Thus

$$(6.70) \quad \overline{A} = \overline{\mathcal{E}(\mu)} - S^\vee = \mathcal{K}_{\text{in}}(\mu).$$

By the definition of  $\alpha_c^{\text{in}}(\mu)$ , its value is the ray extent of  $A$ . Applying [lemma 5.2](#) and then [\(6.70\)](#) proves [\(6.67\)](#).

Finally,  $\mathcal{K}_{\text{in}}(\mu)$  is a closed convex subset of  $\mathcal{K}_{\text{kin}}(\mu)$ , and both contain the origin. Their intersections with the ray  $\{\lambda c : \lambda \geq 0\}$  are therefore closed intervals beginning at the origin. Hence their ray extents are equal if and only if the outer endpoint  $\mathfrak{J}_c(\mu)c$  belongs to the inner interval. This membership is exactly the certificate [\(6.69\)](#).  $\square$

**Proposition 6.37** (active direction and reflection test for the kinematic frontier). *Assume  $0 < \xi < \pi$ , let  $\mu \neq 0$  satisfy  $\mathcal{E}(\mu) \neq \emptyset$ , and fix  $c \in S_\circ^\vee$ . Write*

$$u(\phi) := (\cos \phi, \sin \phi), \quad t(\phi) := (-\sin \phi, \cos \phi), \quad 0 \leq \phi \leq \xi,$$

and put  $m := |\mu|$ . Define

$$F_c(\phi) := \frac{m - \mu \cdot u(\phi)}{c \cdot u(\phi)}, \quad g_c(\phi) := \det(\mu, c) - m c \cdot t(\phi),$$

where  $\det(a, b) = a_1 b_2 - a_2 b_1$ . Then  $F_c$  has a unique minimizer  $\phi_c \in [0, \xi]$ , characterized by

$$(6.71) \quad \phi_c = \begin{cases} 0, & g_c(0) \geq 0, \\ \xi, & g_c(\xi) \leq 0, \\ \text{the unique zero of } g_c \text{ in } (0, \xi), & g_c(0) < 0 < g_c(\xi). \end{cases}$$

Consequently, with  $u_c := u(\phi_c)$ ,

$$(6.72) \quad \mathfrak{J}_c(\mu) = F_c(\phi_c).$$

In the interior case of [\(6.71\)](#),

$$(6.73) \quad \mathfrak{J}_c(\mu) = -\frac{2\mu \cdot c}{|c|^2}, \quad \mu + \mathfrak{J}_c(\mu)c = m u_c.$$

In the two endpoint cases, respectively,

$$(6.74) \quad \mathfrak{J}_c(\mu) = \frac{m - \mu \cdot u_1}{c \cdot u_1} \quad \text{or} \quad \mathfrak{J}_c(\mu) = \frac{m - \mu \cdot u_2}{c \cdot u_2}.$$

Define the active quadratic-boundary point

$$(6.75) \quad \vartheta_c^{\text{kin}} := m u_c - \mu.$$

Then

$$(6.76) \quad \vartheta_c^{\text{kin}} \in \partial \mathbb{B}_\mu, \quad \eta_c^{\text{kin}} := \vartheta_c^{\text{kin}} - \mathfrak{J}_c(\mu)c \in S^\vee.$$

Then

$$(6.77) \quad \alpha_c^{\text{in}}(\mu) = \mathfrak{J}_c(\mu) \iff \vartheta_c^{\text{kin}} \cdot v_i \leq 0 \quad (i = 1, 2).$$

Thus coincidence of the inner and kinematic projected frontiers is decided by two explicit oblique-reflection inequalities at the unique active point of the kernel circle.

*Proof.* A direct differentiation gives

$$F_c'(\phi) = \frac{g_c(\phi)}{(c \cdot u(\phi))^2}.$$

Since  $c \in S_\circ^\vee$ , one has  $c \cdot u(\phi) > 0$  on  $[0, \xi]$ , and

$$g_c'(\phi) = m c \cdot u(\phi) > 0.$$

Hence  $g_c$  is strictly increasing. The derivative of  $F_c$  is therefore either nonnegative on the whole interval, nonpositive on the whole interval, or changes sign exactly once from negative to positive. This proves the uniqueness and the three alternatives in (6.71). Formula (6.72) is the definition of  $\mathfrak{J}_c(\mu)$ .

Suppose that  $\phi_c \in (0, \xi)$ , and put  $\lambda = F_c(\phi_c)$ . Since  $\mathcal{E}(\mu) \neq \emptyset$ , [propositions 5.25](#) and [6.36](#) gives

$$0 < \alpha_c^{\text{in}}(\mu) \leq \mathfrak{J}_c(\mu) = \lambda.$$

The identities defining  $F_c$  and  $g_c$  give

$$(\mu + \lambda c) \cdot u_c = m, \quad (\mu + \lambda c) \cdot t(\phi_c) = 0.$$

Thus  $\mu + \lambda c = mu_c$ . Taking squared norms yields

$$0 = |\mu + \lambda c|^2 - |\mu|^2 = 2\lambda \mu \cdot c + \lambda^2 |c|^2.$$

Since  $\lambda > 0$ , division of the preceding identity by  $\lambda$  gives

$$\lambda = -\frac{2\mu \cdot c}{|c|^2}, \quad \mu + \lambda c = mu_c,$$

which is (6.73). If  $\phi_c = 0$ , then  $u(0) = u_1$  and

$$\mathfrak{J}_c(\mu) = F_c(0) = \frac{m - \mu \cdot u_1}{c \cdot u_1}.$$

If  $\phi_c = \xi$ , then  $u(\xi) = u_2$  and

$$\mathfrak{J}_c(\mu) = F_c(\xi) = \frac{m - \mu \cdot u_2}{c \cdot u_2}.$$

These are the two endpoint formulas.

By [proposition 6.36](#), the endpoint  $\mathfrak{J}_c(\mu)c$  belongs to  $\mathcal{K}_{\text{kin}}(\mu) = \mathbb{B}_\mu - S^\vee$ . Choose  $\vartheta \in \mathbb{B}_\mu$  and  $\eta \in S^\vee$  such that

$$\mathfrak{J}_c(\mu)c = \vartheta - \eta.$$

At the active direction  $u_c$ ,

$$\vartheta \cdot u_c \geq \mathfrak{J}_c(\mu)c \cdot u_c = m - \mu \cdot u_c = h_{\mathbb{B}_\mu}(u_c).$$

The reverse inequality holds because  $\vartheta \in \mathbb{B}_\mu$ . Equality in the support inequality for the Euclidean ball has the unique solution  $\vartheta = mu_c - \mu$ . Hence  $\vartheta = \vartheta_c^{\text{kin}}$  and  $\eta = \eta_c^{\text{kin}}$ , proving (6.76).

If  $\vartheta_c^{\text{kin}} \cdot v_i \leq 0$  for  $i = 1, 2$ , then  $\vartheta_c^{\text{kin}} \in S^\vee$  by (6.76), and

$$Q_\mu(\vartheta_c^{\text{kin}}) = 0.$$

The boundary description in [proposition 5.29](#) therefore gives  $\vartheta_c^{\text{kin}} \in \overline{\mathcal{E}(\mu)}$ . Together with (6.76), this shows that  $\mathfrak{J}_c(\mu)c \in \mathcal{K}_{\text{in}}(\mu)$ , and [proposition 6.36](#) yields  $\alpha_c^{\text{in}}(\mu) = \mathfrak{J}_c(\mu)$ .

Conversely, suppose that the two ray extents are equal. By [proposition 6.36](#), there exist  $\tilde{\vartheta} \in \overline{\mathcal{E}(\mu)} \subset \mathbb{B}_\mu$  and  $\tilde{\eta} \in S^\vee$  such that

$$\mathfrak{J}_c(\mu)c = \tilde{\vartheta} - \tilde{\eta}.$$

The preceding support-point argument at  $u_c$  forces  $\tilde{\vartheta} = \vartheta_c^{\text{kin}}$ . Since every point of  $\overline{\mathcal{E}(\mu)}$  satisfies the two closed oblique inequalities, the right-hand side of (6.77) follows.  $\square$

**Corollary 6.38** (geometric criterion for exact logarithmic tails). *Assume the hypotheses of [corollary 6.35](#) and suppose that  $\mathcal{E}(\mu) \neq \emptyset$ .*

(i) *If, for some  $c \in S_\circ^\vee$ ,*

$$(6.78) \quad \alpha_c^{\text{in}}(\mu) = \mathfrak{J}_c(\mu),$$

*then*

$$\alpha_c^{\text{in}}(\mu) = \alpha_c^\pi = \mathfrak{J}_c^\pi = \mathfrak{J}_c(\mu),$$

and

$$(6.79) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi \{z : c \cdot z \geq x\} = -\mathfrak{J}_c(\mu).$$

The equality in (6.78) is equivalent both to the finite-dimensional certificate (6.69) and to the two active-point inequalities

$$\vartheta_c^{\text{kin}} \cdot v_i \leq 0, \quad i = 1, 2,$$

of [proposition 6.37](#).

(ii) If

$$(6.80) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) = \mathfrak{J}_{\text{rad}}(\mu),$$

then

$$\alpha_{\text{rad}}^{\text{in}}(\mu) = \alpha_{\text{rad}}^{\pi} = \mathfrak{H}_{\text{rad}}^{\pi} = \mathfrak{J}_{\text{rad}}(\mu),$$

and

$$(6.81) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log \pi \{|z| \geq r\} = -\mathfrak{J}_{\text{rad}}(\mu).$$

Both conclusions hold for every stationary law at the prescribed drift; no uniqueness assumption is required.

*Proof.* Under (6.78), every inequality in (6.61) is an equality. The projected conclusion follows from the Harnack-envelope criterion [proposition 6.17](#). The two equivalent finite-dimensional formulations follow from [propositions 6.36](#) and [6.37](#).

Under (6.80), the chain (6.62) becomes

$$\mathfrak{J}_{\text{rad}}(\mu) = \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^{\pi} \leq \mathfrak{H}_{\text{rad}}^{\pi} \leq \mathfrak{J}_{\text{rad}}(\mu).$$

All four quantities are therefore equal. The radial part of [proposition 6.17](#) then gives (6.81).  $\square$

**6.3. Reflection-assisted one-face kinematic bounds.** The killed-wedge comparison uses paths that do not touch the boundary. Near a single open face, the reflected process has an additional low-cost mechanism: the Brownian control may push toward that face while the regulator supplies the reflection direction. The next results quantify this mechanism without using a global semimartingale decomposition at the vertex.

For  $i \in \{1, 2\}$ , let  $u_i$  be the unit tangent on  $\partial S_i$  pointing away from the vertex and let  $n_i$  be the inward unit normal. Recall that  $v_i \cdot n_i = 1$ .

**Definition 6.39** (one-face kinematic cost). For a drift  $\mu$  and a face  $i$ , define

$$(6.82) \quad \mathfrak{J}_i^f(\mu) := \inf_{a > 0, \ell \geq 0} \frac{|a u_i - \mu - \ell v_i|^2}{2a}$$

$$(6.83) \quad = \inf_{\ell \geq 0} \{|\mu + \ell v_i| - (\mu + \ell v_i) \cdot u_i\}.$$

The second equality follows by minimizing first over  $a > 0$ .

**Lemma 6.40** (relative Brownian tube estimate). Let  $B$  be a standard Brownian motion in  $\mathbb{R}^2$ . If  $h_r \rightarrow \infty$ , then for every  $\varepsilon > 0$ ,

$$(6.84) \quad \mathbb{P} \left\{ \sup_{s \geq 0} |B(s)| / (h_r + s) > \varepsilon \right\} \rightarrow 0.$$

The same conclusion holds for Brownian increments after an arbitrary deterministic time. Moreover, replacing the supremum over  $[0, \infty)$  by the supremum over a finite interval enlarges the tube event  $\{\sup |B(s)| / (h_r + s) \leq \varepsilon\}$  and therefore cannot decrease its probability.

*Proof.* Almost surely,  $|B(s)|/s \rightarrow 0$  as  $s \rightarrow \infty$ . Fix a sample path with this property and  $\varepsilon > 0$ . Choose  $M < \infty$  such that  $|B(s)| \leq \varepsilon s/2$  for  $s \geq M$ . On  $[0, M]$ , continuity gives

$C_M := \sup_{0 \leq s \leq M} |B(s)| < \infty$ , and therefore

$$\sup_{0 \leq s \leq M} \frac{|B(s)|}{h_r + s} \leq \frac{C_M}{h_r} \rightarrow 0.$$

For  $s \geq M$ , one has  $|B(s)|/(h_r + s) \leq \varepsilon/2$ . Thus the supremum in (6.84) converges to zero almost surely, and the probability statement follows. Brownian increments after a deterministic time have the same law.  $\square$

**Lemma 6.41** (stopped one-face Skorokhod representation). *Assume  $\alpha < 2$  and fix  $i \in \{1, 2\}$ . Let  $G$  be a relatively open subset of  $S \setminus \{0\}$  whose closure meets  $\partial S$  only in the open face  $\partial S_i$ , and let*

$$\tau_G := \inf\{t \geq 0 : Z(t) \notin G\}.$$

*Let  $X$  and  $Y = Z - X$  be the Brownian-ESP decomposition supplied by [9, Theorem 2.3]. If  $Z(0) = X(0) = x \in G$ , then there is a continuous nondecreasing process  $L$ , with  $L(0) = 0$ , such that, for every  $0 \leq t \leq \tau_G$  when  $\tau_G < \infty$ , and for every finite  $t$  when  $\tau_G = \infty$ ,*

$$(6.85) \quad Z(t) = X(t) + v_i L(t), \quad L(t) = \sup_{0 \leq s \leq t} [-X(s) \cdot n_i]_+.$$

*In particular, up to and including the stopping time  $\tau_G$ , the Lakner-Liu-Reed process is the usual obliquely reflected Brownian motion in the half-plane  $\{z : z \cdot n_i \geq 0\}$ , driven by the same Brownian path.*

*Proof.* For  $0 \leq s < t < \tau_G$ , the increment condition in the extended Skorokhod problem reads

$$(6.86) \quad Y(t) - Y(s) \in \overline{\text{cone}} \left( \bigcup_{r \in (s, t]} d(Z(r)) \right),$$

where  $\overline{\text{cone}}$  denotes the closed convex cone generated by its argument, and where  $d(z) = \{0\}$  in  $S^\circ$  and, away from the vertex,  $d(z) = \text{cone}\{v_j : j \in I(z)\}$  with  $I(z)$  the set of active faces. By the hypothesis on  $G$ , only the interior and the open face  $\partial S_i$  can be visited before  $\tau_G$ . The union on the right-hand side of (6.86) is therefore contained in  $\text{cone}\{v_i\}$ , a closed convex cone, and hence

$$Y(t) - Y(s) \in \text{cone}\{v_i\}.$$

Taking  $s = 0$  and using  $Y(0) = 0$  shows that  $Y(t)$  is a nonnegative multiple of  $v_i$ . Since  $v_i \cdot n_i = 1$ , this multiple is uniquely identified as

$$L(t) := Y(t) \cdot n_i.$$

The increment statement above shows that  $L$  is continuous and nondecreasing. Fix  $T < \tau_G$  and put

$$N(t) := Z(t) \cdot n_i = X(t) \cdot n_i + L(t), \quad 0 \leq t \leq T.$$

The set  $O_T := \{t \in [0, T] : N(t) > 0\}$  is relatively open. Each of its connected components is an interval on which  $Z$  lies in  $S^\circ$ . On every compact subinterval  $[a, b]$  of such a component, (6.86) has right-hand side  $\{0\}$ , so  $L(b) = L(a)$ . Exhausting the component by compact subintervals and using continuity shows that  $L$  is constant on the whole component. Therefore the Lebesgue-Stieltjes measure generated by  $L$  satisfies

$$(6.87) \quad dL(O_T) = 0, \quad \text{equivalently} \quad \int_0^T \mathbf{1}_{\{N(t) > 0\}} dL(t) = 0.$$

Thus  $dL$  is supported on the contact set  $\{N = 0\}$ .

Define

$$K(t) := \sup_{0 \leq s \leq t} [-X(s) \cdot n_i]_+.$$

Because  $N(s) = X(s) \cdot n_i + L(s) \geq 0$  and  $L$  is nondecreasing,

$$L(t) \geq L(s) \geq [-X(s) \cdot n_i]_+ \quad (0 \leq s \leq t),$$

so  $L(t) \geq K(t)$ . To prove the reverse inequality, suppose that  $L(t) > K(t)$  for some  $t \leq T$ , and choose  $a \in (K(t), L(t))$ . Since  $L(0) = 0 \leq K(t) < a$  and  $L$  is continuous,

$$\sigma := \inf\{s \in [0, t] : L(s) \geq a\}$$

is well defined and satisfies  $L(\sigma) = a$ . By the definition of the first hitting time, every left neighborhood of  $\sigma$  carries positive  $L$ -increment. Hence  $\sigma$  belongs to the support of  $dL$ , and (6.87) gives  $N(\sigma) = 0$ . Consequently

$$a = L(\sigma) = -X(\sigma) \cdot n_i \leq K(t),$$

a contradiction. Therefore  $L(t) = K(t)$  for every  $t \leq T$ . Since  $T < \tau_G$  was arbitrary, the supremum formula and the vector identity hold on  $[0, \tau_G)$ . If  $\tau_G < \infty$ , continuity of  $X, Z, L$ , and the running-supremum map lets  $t \uparrow \tau_G$  and proves the same identities at the stopping time itself.  $\square$

**Lemma 6.42** (finite-horizon stability under equivalent changes of measure). *Fix an initial state  $x \in S$  and a finite time  $T$ . Let  $(X, Z, Y)$  be the Brownian-ESP decomposition of [9, Theorem 2.3] under  $\mathbb{P}_x$ , so that  $Y = Z - X$  and  $(Z, Y)$  solves the extended Skorokhod problem for  $X$  on  $[0, T]$ . If  $\mathbb{Q}$  is equivalent to  $\mathbb{P}_x$  on  $\mathcal{F}_T$ , then the same pathwise ESP relation holds  $\mathbb{Q}$ -almost surely on  $[0, T]$ .*

*In particular, let  $w : [0, T] \rightarrow \mathbb{R}^2$  be deterministic and square integrable, and suppose*

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}_x} \right|_{\mathcal{F}_T} = \exp \left\{ \int_0^T w(s) \cdot dB(s) - \frac{1}{2} \int_0^T |w(s)|^2 ds \right\}.$$

*Then*

$$\tilde{B}(t) := B(t) - \int_0^t w(s) ds, \quad 0 \leq t \leq T,$$

*is Brownian motion under  $\mathbb{Q}$ , and the same reflected path is the ESP image of*

$$X(t) = x + \int_0^t (\mu + w(s)) ds + \tilde{B}(t).$$

*Consequently, for every one-face localization set  $G$  as in lemma 6.41, the representation (6.85) remains valid under  $\mathbb{Q}$  up to  $\tau_G \wedge T$ .*

*Proof.* The ESP relation is a pathwise property of the restrictions of  $(X, Z, Y)$  to  $[0, T]$ : it consists of  $Z = X + Y$ , the state constraint, and the closed-convex-cone increment condition in the definition of the extended Skorokhod problem. By the Brownian-ESP decomposition in [9, Theorem 2.3], there is an  $\mathcal{F}_T$ -measurable event  $\Omega_T$  of  $\mathbb{P}_x$ -probability one on which all of these relations hold simultaneously throughout  $[0, T]$ . No stochastic calculus is involved in the relation once the three paths are fixed.

Because  $\mathbb{Q}$  and  $\mathbb{P}_x$  are equivalent on  $\mathcal{F}_T$ , they have exactly the same null sets there. Hence  $\mathbb{Q}(\Omega_T) = 1$ , so the identical pathwise ESP relation holds under  $\mathbb{Q}$ . The asserted algebraic form of  $X$  and the Brownian property of  $\tilde{B}$  are the finite-horizon Cameron-Martin-Girsanov theorem. Applying lemma 6.41 pathwise on  $\Omega_T$  gives the final assertion.  $\square$

**Lemma 6.43** (uniform start-up tube for a one-face control). *Fix  $i \in \{1, 2\}$  and write  $v_i = \beta_i u_i + n_i$ . Let*

$$G_i = \{qu_i + sn_i : q > r_0, 0 \leq s < \kappa q\}$$

*be a one-face conical strip whose closure meets no true face other than  $\partial S_i$ . Fix  $a > 0$  and  $\ell \geq 0$ . There exist a compact ball  $K \Subset G_i \cap S^\circ$  and a number  $\eta_0 > 0$  with the following property. For*

every  $t_0 > 0$ , every  $x \in K$ , and every continuous path  $b$  satisfying

$$b(0) = 0, \quad \sup_{0 \leq t \leq t_0} |b(t)| \leq \eta_0,$$

the one-face Skorokhod image of

$$x + (au_i - lv_i)t + b(t)$$

remains in  $G_i$  on  $[0, t_0]$ . Its tangential coordinate at time  $t_0$  is bounded below by

$$at_0 + q_*,$$

where  $q_* > r_0$  is independent of  $x, t_0$ , and  $b$ . Consequently, if  $W$  is standard planar Brownian motion, the event

$$E_0(t_0) := \left\{ \sup_{0 \leq t \leq t_0} |W(t)| \leq \eta_0 \right\}$$

has probability  $p_0(t_0) > 0$ , and the preceding conclusions hold uniformly for all  $x \in K$  on this event.

*Proof.* For  $x = q_x u_i + s_x n_i$  and  $b \equiv 0$ , the normal one-dimensional Skorokhod formula is explicit. If  $\ell > 0$ , then

$$L_x^0(t) = (\ell t - s_x)_+,$$

whereas  $L_x^0 \equiv 0$  when  $\ell = 0$ . In both cases the reflected normal coordinate lies in  $[0, s_x]$ , and the tangential coordinate is

$$q_x + (a - \ell \beta_i)t + \beta_i(\ell t - s_x)_+ \geq q_x - |\beta_i|s_x + at.$$

We now construct the compact ball and its margins explicitly. Set

$$q_* := r_0 + 1$$

and choose

$$(6.88) \quad 0 < m_* < \frac{\kappa q_*}{4}.$$

Then  $\kappa q_* - 2m_* > 0$ . Put

$$s_* := \min \left\{ \frac{m_*}{4(1 + |\beta_i|)}, \frac{\kappa q_* - 2m_*}{4} \right\} > 0, \quad x_* := (q_* + 4m_*)u_i + s_* n_i,$$

and choose

$$0 < d_* < \min \left\{ \frac{s_*}{2}, \frac{m_*}{4(1 + |\beta_i|)}, \frac{\kappa q_* - 2m_*}{4} \right\}.$$

Let  $K := \overline{B}_{d_*}(x_*)$ . If  $x = q_x u_i + s_x n_i \in K$ , orthonormality of  $(u_i, n_i)$  gives  $|q_x - (q_* + 4m_*)| \leq d_*$  and  $|s_x - s_*| \leq d_*$ . In particular,  $s_x \geq s_* - d_* > 0$ , and

$$\begin{aligned} q_x - |\beta_i|s_x &\geq q_* + 4m_* - d_* - |\beta_i|(s_* + d_*) \\ &= q_* + 4m_* - |\beta_i|s_* - (1 + |\beta_i|)d_* \\ &> q_* + \frac{7}{2}m_* > q_* + 2m_*, \\ s_x &\leq s_* + d_* < \frac{\kappa q_* - 2m_*}{2} < \kappa q_* - 2m_* \\ &= \kappa(q_* + 2m_*) - 2(1 + \kappa)m_*. \end{aligned}$$

These inequalities imply  $q_x > q_* > r_0$  and  $s_x < \kappa q_x$ , so indeed  $K \Subset G_i \cap S^\circ$ , and they supply precisely the two uniform margins needed below. If  $(q_x^0(t), s_x^0(t))$  denotes the deterministic reflected path, the preceding explicit formulas give, for every  $t \geq 0$ ,

$$q_x^0(t) \geq q_* + 2m_* + at, \quad 0 \leq s_x^0(t) \leq s_x,$$

and therefore

$$q_x^0(t) - r_0 \geq q_* - r_0 + 2m_*, \quad \kappa q_x^0(t) - s_x^0(t) \geq 2(1 + \kappa)m_* + \kappa at.$$

Thus the deterministic path has explicit, uniform coordinate margins from the lower cutoff  $q = r_0$  and the artificial side  $s = \kappa q$ ; the true side  $s = 0$  is part of the state space and is not an exit boundary.

The one-dimensional regulator map is Lipschitz in the uniform norm: if  $L_b$  and  $L_0$  correspond to normal inputs differing by  $b \cdot n_i$ , then

$$\sup_{t \leq t_0} |L_b(t) - L_0(t)| \leq \sup_{t \leq t_0} |b(t) \cdot n_i|.$$

Hence the full oblique images differ by at most  $d_b := (1 + |v_i|) \sup_{t \leq t_0} |b(t)|$ . Choose  $\eta_0 > 0$  so that

$$(1 + |v_i|)\eta_0 < m_*.$$

A perturbation of Euclidean size at most  $d_b$  changes  $q$  by at most  $d_b$  and changes  $\kappa q - s$  by at most  $(1 + \kappa)d_b$ . The displayed deterministic margins therefore imply  $q(t) > r_0$ ,  $s(t) < \kappa q(t)$ , and  $q(t) \geq at + q_*$  throughout  $[0, t_0]$ . Finally, a Brownian small-ball event in  $C([0, t_0], \mathbb{R}^2)$  has strictly positive probability, which gives  $p_0(t_0) > 0$ .  $\square$

**Lemma 6.44** (localized one-face tube estimate). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be a stationary distribution for the drift  $\mu$ . Fix a face  $i$ , numbers  $a > 0$  and  $\ell \geq 0$ , and a relative open angular neighborhood  $U$  of  $u_i$  in  $S \cap \mathbb{S}^1$ . Put*

$$w := au_i - \mu - \ell v_i.$$

*For every  $\delta > 0$  and every  $\gamma > 0$ , there is a compact set  $K \Subset S^\circ$ , which may depend on  $\delta$  and  $\gamma$ , such that  $\pi(K) > 0$  and, with*

$$T_r := \frac{(1 + \delta)r}{a},$$

*one has*

$$(6.89) \quad \liminf_{r \rightarrow \infty} \frac{1}{r} \log \inf_{x \in K} \mathbb{P}_x \left\{ |Z(T_r)| \geq r, \frac{Z(T_r)}{|Z(T_r)|} \in U \right\} \geq -\frac{1 + \delta}{2a} |w|^2 - \gamma.$$

*Proof.* Write

$$v_i = \beta_i u_i + n_i, \quad \beta_i := v_i \cdot u_i.$$

Fix  $r_0 > 0$ , and choose

$$0 < \kappa < \frac{\sin \xi}{2(1 + |\cos \xi|)}.$$

For  $j = 3 - i$ , the angle geometry gives

$$n_j \cdot u_i = \sin \xi, \quad n_j \cdot n_i = -\cos \xi.$$

Hence every point  $z = qu_i + sn_i$  with  $q \geq r_0$  and  $0 \leq s \leq \kappa q$  satisfies

$$n_j \cdot z = q \sin \xi - s \cos \xi \geq q(\sin \xi - \kappa |\cos \xi|) > \frac{1}{2} q \sin \xi > 0.$$

Thus the closed conical strip

$$\overline{G}_i := \{qu_i + sn_i : q \geq r_0, 0 \leq s \leq \kappa q\}$$

is separated from the opposite face and from the vertex; its only true boundary side is  $s = 0$ . Let

$$G_i := \{qu_i + sn_i : q > r_0, 0 \leq s < \kappa q\}.$$

Apply [lemma 6.43](#) to obtain a compact ball  $K \Subset G_i \cap S^\circ$ , a start-up radius  $\eta_0 > 0$ , and the associated constant  $q_* > r_0$ . These choices are independent of the start-up time and of the relative-tube tolerance. Strict positivity of the stationary density gives  $\pi(K) > 0$ .

For a finite horizon  $T$ , make the Cameron–Martin change of measure with constant control

$$w = au_i - \mu - \ell v_i.$$

Under the changed measure the Brownian input is  $\tilde{B}(t) = B(t) - wt$ , and, by [lemmas 6.41](#) and [6.42](#), the reflected path, up to  $T \wedge \tau_i$ , has the representation

$$(6.90) \quad \hat{Z}(t) = X(t) + v_i L(t), \quad L(t) = \sup_{0 \leq s \leq t} [-X(s) \cdot n_i]_+,$$

where  $\tau_i$  is the exit time from  $G_i$  and

$$X(t) = x + (au_i - \ell v_i)t + \tilde{B}(t).$$

In particular,

$$(6.91) \quad L(t) = \sup_{0 \leq s \leq t} \{\ell s - x \cdot n_i - \tilde{B}(s) \cdot n_i\}_+.$$

For  $t \leq T \wedge \tau_i$ , put  $M_t := \sup_{0 \leq s \leq t} |\tilde{B}(s)|$  and  $C_K := \sup_{x \in K} |x \cdot n_i|$ . The supremum formula gives

$$L(t) \leq \ell t + C_K + M_t.$$

For the opposite inequality, if  $\ell t \leq C_K + M_t$ , then  $\ell t - L(t) \leq C_K + M_t$  because  $L(t) \geq 0$ . If  $\ell t > C_K + M_t$ , the expression inside the positive part at  $s = t$  is positive, and hence

$$L(t) \geq \ell t - x \cdot n_i - \tilde{B}(t) \cdot n_i \geq \ell t - C_K - M_t.$$

Thus, in both cases,

$$(6.92) \quad |L(t) - \ell t| \leq C_K + M_t.$$

Since

$$\hat{Z}(t) - atu_i = x + \tilde{B}(t) + v_i(L(t) - \ell t),$$

we obtain the explicit bound

$$(6.93) \quad |\hat{Z}(t) - atu_i| \leq C_0 + C_1 M_t, \quad C_0 := \sup_{x \in K} |x| + |v_i| C_K, \quad C_1 := 1 + |v_i|.$$

Put

$$C_2 := C_0 + C_1 \eta_0, \quad C_3 := C_1.$$

Choose  $\rho_U > 0$  so that every unit vector in  $S \cap \mathbb{S}^1$  at Euclidean distance less than  $\rho_U$  from  $u_i$  belongs to  $U$ . Choose  $\varepsilon > 0$  once and for all so that

$$(6.94) \quad \frac{2(1+\delta)}{a} \varepsilon |w| < \gamma,$$

$$(6.95) \quad (1+\kappa)C_3 \varepsilon < \frac{\kappa a}{2},$$

$$(6.96) \quad \frac{(1+\delta)C_3}{a} \varepsilon < \frac{\delta}{2}, \quad \frac{4C_3}{a} \varepsilon < \rho_U.$$

The first condition is void when  $w = 0$ . Notice that all geometric and likelihood requirements on  $\varepsilon$  have now been imposed; the tolerance will not be decreased after the start-up time is selected.

By [lemma 6.40](#), choose  $t_0$  so large that

$$(6.97) \quad \mathbb{P} \left\{ \sup_{s \geq 0} \frac{|W(s)|}{t_0 + s} \leq \varepsilon \right\} \geq \frac{1}{2}$$

for a standard planar Brownian motion  $W$ . Increase  $t_0$ , if necessary, so that

$$(6.98) \quad (1+\kappa)C_2 < \frac{\kappa a t_0}{2}, \quad at_0 - C_2 - C_3 \varepsilon t_0 > r_0.$$

The event in [\(6.97\)](#) is monotone in  $t_0$ , so this increase preserves its probability bound. The compact set  $K$  and  $\eta_0$  remain unchanged by the uniform start-up lemma.

For every sufficiently large  $r$ , so that  $T_r \geq t_0$ , define  $\mathbb{Q}_{x,r}$  on  $\mathcal{F}_{T_r}$  by

$$\frac{d\mathbb{Q}_{x,r}}{d\mathbb{P}_x} \Big|_{\mathcal{F}_{T_r}} = \exp \left\{ w \cdot B(T_r) - \frac{1}{2}|w|^2 T_r \right\}.$$

Then  $\tilde{B}_r(t) := B(t) - wt$ ,  $0 \leq t \leq T_r$ , is standard Brownian motion under  $\mathbb{Q}_{x,r}$ . Define

$$(6.99) \quad E_0 := \left\{ \sup_{0 \leq t \leq t_0} |\tilde{B}_r(t)| \leq \eta_0 \right\}$$

and

$$E_{1,r} := \left\{ \sup_{0 \leq s \leq T_r - t_0} \frac{|\tilde{B}_r(t_0 + s) - \tilde{B}_r(t_0)|}{t_0 + s} \leq \varepsilon \right\}.$$

The start-up lemma gives

$$(6.100) \quad \inf_{x \in K} \mathbb{Q}_{x,r}(E_0) =: p_0(t_0) > 0,$$

and the infinite-horizon event in (6.97) gives  $\mathbb{Q}_{x,r}(E_{1,r}) \geq 1/2$ . The two events concern disjoint Brownian increments and are independent. Consequently,

$$(6.101) \quad \inf_{x \in K} \mathbb{Q}_{x,r}(E_0 \cap E_{1,r}) \geq \frac{p_0(t_0)}{2}.$$

On  $E_0 \cap E_{1,r}$ , for  $t_0 \leq t \leq T_r \wedge \tau_i$ ,

$$\sup_{0 \leq s \leq t} |\tilde{B}_r(s)| \leq \eta_0 + \varepsilon t.$$

Hence (6.93) gives

$$(6.102) \quad |\hat{Z}(t) - atu_i| \leq C_2 + C_3 \varepsilon t.$$

Writing  $\hat{Z}(t) = q(t)u_i + s(t)n_i$  and using  $s(t) \geq 0$ , we obtain

$$q(t) \geq at - C_2 - C_3 \varepsilon t, \quad s(t) \leq C_2 + C_3 \varepsilon t.$$

We verify the two artificial-boundary inequalities explicitly. First, (6.95) implies  $C_3 \varepsilon < a/2$ . Hence, for  $t \geq t_0$ ,

$$q(t) \geq (a - C_3 \varepsilon)t - C_2 \geq at_0 - C_2 - C_3 \varepsilon t_0 > r_0$$

by (6.98). Second,

$$\begin{aligned} \kappa q(t) - s(t) &\geq \kappa at - (1 + \kappa)(C_2 + C_3 \varepsilon t) \\ &> \frac{\kappa at}{2} - (1 + \kappa)C_2 \\ &\geq \frac{\kappa at_0}{2} - (1 + \kappa)C_2 > 0, \end{aligned}$$

where the first strict inequality uses  $(1 + \kappa)C_3 \varepsilon < \kappa a/2$ , and the last uses (6.98). Thus  $q(t) > r_0$  and  $s(t) < \kappa q(t)$  throughout  $[t_0, T_r \wedge \tau_i]$ . On  $[0, t_0]$ , the same strict inequalities follow from  $E_0$  and lemma 6.43. If  $\tau_i \leq T_r$ , the continuity of the path and the validity of the stopped representation at  $\tau_i$  would force one of the equalities  $q(\tau_i) = r_0$  or  $s(\tau_i) = \kappa q(\tau_i)$ , contradicting the strict estimates just proved. Therefore  $\tau_i > T_r$ , and the one-face representation agrees with the original diffusion on the whole interval  $[0, T_r]$ .

At the terminal time, put

$$R_r := aT_r = (1 + \delta)r, \quad E_r := C_2 + C_3 \varepsilon T_r.$$

Then (6.102) gives  $|Z(T_r) - R_r u_i| \leq E_r$ . By (6.96),

$$C_3 \varepsilon T_r = \frac{(1 + \delta)C_3}{a} \varepsilon r < \frac{\delta}{2} r.$$

For all sufficiently large  $r$ , also  $C_2 < \delta r/2$ ; hence  $E_r < \delta r$ , and the reverse triangle inequality gives

$$|Z(T_r)| \geq (1 + \delta)r - E_r \geq r.$$

Moreover  $E_r < R_r/2$  for all large  $r$ , and for any nonzero vector  $z$  with  $|z - R_r u_i| \leq E_r$ ,

$$\left| \frac{z}{|z|} - u_i \right| \leq \frac{2E_r}{R_r}.$$

Since

$$\limsup_{r \rightarrow \infty} \frac{2E_r}{R_r} \leq \frac{2C_3}{a} \varepsilon < \frac{\rho_U}{2}$$

by (6.96), the terminal direction belongs to  $U$  for all sufficiently large  $r$ . Thus the controlled terminal event has probability at least  $p_0(t_0)/2$  under every  $\mathbb{Q}_{x,r}$ , uniformly for  $x \in K$ .

The inverse likelihood ratio is

$$\frac{d\mathbb{P}_x}{d\mathbb{Q}_{x,r}} \Big|_{\mathcal{F}_{T_r}} = \exp \left\{ -\frac{1}{2} |w|^2 T_r - w \cdot \tilde{B}_r(T_r) \right\}.$$

On  $E_0 \cap E_{1,r}$ ,  $|\tilde{B}_r(T_r)| \leq \eta_0 + \varepsilon T_r$ , and therefore

$$\frac{d\mathbb{P}_x}{d\mathbb{Q}_{x,r}} \geq \exp \left\{ -\left( \frac{1}{2} |w|^2 + \varepsilon |w| \right) T_r - |w| \eta_0 \right\}.$$

Combining this with (6.101) and the terminal inclusion gives, uniformly for  $x \in K$ ,

$$\begin{aligned} & \mathbb{P}_x \left\{ |Z(T_r)| \geq r, \frac{Z(T_r)}{|Z(T_r)|} \in U \right\} \\ & \geq \frac{p_0(t_0)}{2} \exp \left\{ -\left( \frac{1}{2} |w|^2 + \varepsilon |w| \right) T_r - |w| \eta_0 \right\}. \end{aligned}$$

Taking logarithms, dividing by  $r$ , and using  $T_r/r = (1 + \delta)/a$  yields

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \log \inf_{x \in K} \mathbb{P}_x(\cdots) \geq -\frac{1 + \delta}{2a} |w|^2 - \frac{1 + \delta}{a} \varepsilon |w|.$$

The last term is smaller than  $\gamma/2$ , and hence in particular smaller than  $\gamma$ , by (6.94). This proves (6.89). □

**Proposition 6.45** (one-face kinematic cost and its dual representation). *For each  $i \in \{1, 2\}$ , the infimum in (6.83) is attained and*

$$(6.103) \quad \mathfrak{J}_i^f(\mu) = \max \{ \theta \cdot u_i : Q_\mu(\theta) \leq 0, \theta \cdot v_i \leq 0 \}.$$

If  $\ell_i$  minimizes (6.83), put

$$q_i := \mu + \ell_i v_i.$$

When  $q_i \neq 0$ , the maximizing point in (6.103) is

$$(6.104) \quad \vartheta_i^f := |q_i| u_i - q_i.$$

It satisfies

$$Q_\mu(\vartheta_i^f) = 0, \quad \vartheta_i^f \cdot v_i \leq 0, \quad \vartheta_i^f \cdot u_i = \mathfrak{J}_i^f(\mu).$$

If  $\ell_i > 0$ , then  $\vartheta_i^f \cdot v_i = 0$ . In the degenerate case  $q_i = 0$ , the same conclusions hold with  $\vartheta_i^f = 0$  and  $\mathfrak{J}_i^f(\mu) = 0$ .

*Proof.* For fixed  $q \in \mathbb{R}^2$ ,

$$\inf_{a > 0} \frac{|a u_i - q|^2}{2a} = \inf_{a > 0} \left\{ \frac{a}{2} - q \cdot u_i + \frac{|q|^2}{2a} \right\} = |q| - q \cdot u_i.$$

This proves (6.83). We next verify attainment rather than appealing only to a geometric picture. Write

$$v_i = (v_i \cdot u_i)u_i + n_i,$$

where  $n_i \perp u_i$  and  $|n_i| = v_i \cdot n_i = 1$ . Hence

$$|v_i| = \sqrt{1 + (v_i \cdot u_i)^2} > v_i \cdot u_i.$$

For  $\ell \geq 0$ , the reverse triangle inequality gives

$$\begin{aligned} |\mu + \ell v_i| - (\mu + \ell v_i) \cdot u_i &\geq \ell |v_i| - |\mu| - \mu \cdot u_i - \ell v_i \cdot u_i \\ &= \ell (|v_i| - v_i \cdot u_i) - |\mu| - \mu \cdot u_i. \end{aligned}$$

The right-hand side tends to  $+\infty$  as  $\ell \rightarrow \infty$ . The objective in (6.83) is continuous on  $[0, \infty)$ , so it attains its infimum on a compact subinterval.

Let  $\theta$  satisfy the two constraints in (6.103). For every  $\ell \geq 0$ ,

$$Q_{\mu + \ell v_i}(\theta) = Q_\mu(\theta) + \ell \theta \cdot v_i \leq 0.$$

Thus  $\theta$  belongs to the closed ball

$$\{\zeta : Q_{\mu + \ell v_i}(\zeta) \leq 0\},$$

whose support in direction  $u_i$  equals  $|\mu + \ell v_i| - (\mu + \ell v_i) \cdot u_i$ . Hence the maximum in (6.103) is at most  $\mathfrak{J}_i^f(\mu)$ .

Let  $\ell_i$  be a minimizer and first assume  $q_i \neq 0$ . The point  $\vartheta_i^f = |q_i|u_i - q_i$  lies on the quadratic boundary for the drift  $q_i$ , and

$$\frac{d}{d\ell} (|\mu + \ell v_i| - (\mu + \ell v_i) \cdot u_i) = -\frac{(|\mu + \ell v_i|u_i - (\mu + \ell v_i)) \cdot v_i}{|\mu + \ell v_i|}.$$

The one-sided optimality condition therefore gives  $\vartheta_i^f \cdot v_i \leq 0$ , with equality if  $\ell_i > 0$ . If  $\ell_i = 0$ , then  $q_i = \mu$  and  $Q_\mu(\vartheta_i^f) = Q_{q_i}(\vartheta_i^f) = 0$ . If  $\ell_i > 0$ , the optimality condition gives  $\vartheta_i^f \cdot v_i = 0$ , and therefore

$$Q_\mu(\vartheta_i^f) = Q_{q_i}(\vartheta_i^f) - \ell_i \vartheta_i^f \cdot v_i = 0.$$

Finally,

$$\vartheta_i^f \cdot u_i = |q_i| - q_i \cdot u_i = \mathfrak{J}_i^f(\mu).$$

Thus  $\vartheta_i^f$  is feasible and attains the upper bound. If  $q_i = 0$ , the cost is zero and the origin gives the required maximizer.  $\square$

**Proposition 6.46** (explicit form of the one-face cost). *Let*

$$\tau_i := \tan \theta_i, \quad p_i := u_i + \tau_i n_i.$$

Then

$$p_i \cdot v_i = 0, \quad p_i \cdot u_i = 1, \quad |p_i|^2 = 1 + \tau_i^2.$$

Put

$$M := |\mu|, \quad \vartheta_i^0 := Mu_i - \mu, \quad J_i^0 := M - \mu \cdot u_i.$$

With  $x_+ := \max\{x, 0\}$ , the one-face cost is

$$(6.105) \quad \mathfrak{J}_i^f(\mu) = \begin{cases} J_i^0, & \vartheta_i^0 \cdot v_i \leq 0, \\ \left(-\frac{2\mu \cdot p_i}{|p_i|^2}\right)_+, & \vartheta_i^0 \cdot v_i > 0. \end{cases}$$

In the first case,  $\vartheta_i^0$  is a maximizing point in (6.103). In the second case, a maximizing point is

$$(6.106) \quad \vartheta_i^f = \mathfrak{J}_i^f(\mu) p_i.$$

*Proof.* The support point of the full kernel disk  $\mathbb{B}_\mu$  in direction  $u_i$  is  $\vartheta_i^0 = Mu_i - \mu$ , and its support value is  $J_i^0$ . If  $\vartheta_i^0 \cdot v_i \leq 0$ , this point is feasible in (6.103), proving the first case.

Suppose that  $\vartheta_i^0 \cdot v_i > 0$ , and let  $\theta_*$  maximize  $\theta \cdot u_i$  over  $\mathbb{B}_\mu \cap \{\theta : \theta \cdot v_i \leq 0\}$ , which is compact. We claim that  $\theta_* \cdot v_i = 0$ . If instead  $\theta_* \cdot v_i < 0$ , consider the segment

$$\theta_s := (1-s)\theta_* + s\vartheta_i^0, \quad 0 < s < 1.$$

Convexity of  $\mathbb{B}_\mu$  gives  $\theta_s \in \mathbb{B}_\mu$ . Because  $\theta_* \cdot v_i < 0 < \vartheta_i^0 \cdot v_i$ , one has  $\theta_s \cdot v_i < 0$  whenever

$$0 < s < \frac{-\theta_* \cdot v_i}{\vartheta_i^0 \cdot v_i - \theta_* \cdot v_i}.$$

But  $\vartheta_i^0$  is the unique maximizer of the strictly convex Euclidean disk in direction  $u_i$ , so  $\vartheta_i^0 \cdot u_i > \theta_* \cdot u_i$ ; hence every such  $s$  satisfies

$$\theta_s \cdot u_i = (1-s)\theta_* \cdot u_i + s\vartheta_i^0 \cdot u_i > \theta_* \cdot u_i,$$

contradicting maximality. Therefore  $\theta_* \cdot v_i = 0$ .

The nonzero vector  $p_i$  satisfies  $p_i \cdot v_i = 0$ ; in two dimensions it spans the line  $\{\theta : \theta \cdot v_i = 0\}$ . Write  $\theta = tp_i$ . Since  $p_i \cdot u_i = 1$ , the objective equals  $t$ , and the disk constraint is exactly

$$Q_\mu(tp_i) = t\mu \cdot p_i + \frac{1}{2}t^2|p_i|^2 \leq 0.$$

The point  $t = 0$  is feasible. For  $t > 0$ , the inequality is equivalent to

$$t \leq -\frac{2\mu \cdot p_i}{|p_i|^2},$$

which has a positive solution precisely when  $\mu \cdot p_i < 0$ . Thus the maximal objective value is

$$\left(-\frac{2\mu \cdot p_i}{|p_i|^2}\right)_+,$$

and the maximizing point is the vector in (6.106).  $\square$

**Proposition 6.47** (stationary lower bounds in face neighborhoods). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be stationary for the drift  $\mu$ . For each face  $i$  and every relative open angular neighborhood  $U$  of  $u_i$ ,*

$$(6.107) \quad \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in U \right\} \leq \mathfrak{J}_i^f(\mu).$$

Consequently,

$$(6.108) \quad \beta_\pi(u_i) \leq \mathfrak{J}_i^f(\mu), \quad i = 1, 2.$$

*Proof.* Fix  $a > 0$ ,  $\ell \geq 0$ ,  $\delta > 0$ , and  $\gamma > 0$ . Apply lemma 6.44 and use stationarity:

$$\pi(A_r) = \int_S P_{T_r}(x, A_r) \pi(dx) \geq \pi(K) \inf_{x \in K} P_{T_r}(x, A_r),$$

where

$$A_r := \{z \neq 0 : |z| \geq r, z/|z| \in U\}.$$

Since  $\pi(K) > 0$  is independent of  $r$ ,

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi(A_r) \leq \frac{1+\delta}{2a} |au_i - \mu - \ell v_i|^2 + \gamma.$$

First let  $\gamma \downarrow 0$ , then  $\delta \downarrow 0$ , and finally minimize over  $a > 0$  and  $\ell \geq 0$ . This proves (6.107). For a fixed relative open neighborhood  $U$ , the one-dimensional tail–abscissa identity proposition 6.4 gives

$$\alpha_U^{\pi, \text{rad}} = \liminf_{r \rightarrow \infty} -\frac{1}{r} \log \pi(A_r) \leq \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi(A_r) \leq \mathfrak{J}_i^f(\mu).$$

Taking the supremum over all such neighborhoods  $U$  in the definition of  $\beta_\pi(u_i)$  proves (6.108).  $\square$

**Definition 6.48** (one-face sharpened projected and radial rates). For  $c \in S_\circ^\vee$ , define

$$(6.109) \quad \mathfrak{J}_c^{\text{1f}}(\mu) := \min \left\{ \mathfrak{J}_c(\mu), \frac{\mathfrak{J}_1^{\text{f}}(\mu)}{c \cdot u_1}, \frac{\mathfrak{J}_2^{\text{f}}(\mu)}{c \cdot u_2} \right\},$$

and

$$(6.110) \quad \mathfrak{J}_{\text{rad}}^{\text{1f}}(\mu) := \min \left\{ \mathfrak{J}_{\text{rad}}(\mu), \mathfrak{J}_1^{\text{f}}(\mu), \mathfrak{J}_2^{\text{f}}(\mu) \right\}.$$

Also set

$$(6.111) \quad \mathcal{K}_{\text{kin}}^{\text{1f}}(\mu) := \mathcal{K}_{\text{kin}}(\mu) \cap \bigcap_{i=1}^2 \{ \theta : \theta \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu) \}.$$

**Corollary 6.49** (one-face sharpened outer bounds). Under the hypotheses of [proposition 6.47](#),

$$(6.112) \quad D_\pi \subset \mathcal{K}_{\text{kin}}^{\text{1f}}(\mu).$$

For every  $c \in S_\circ^\vee$ ,

$$(6.113) \quad \alpha_c^{\text{in}}(\mu) \leq \alpha_c^\pi \leq \mathfrak{J}_c^{\text{1f}}(\mu),$$

and

$$(6.114) \quad \limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi \{ z : c \cdot z \geq x \} \leq \mathfrak{J}_c^{\text{1f}}(\mu).$$

The corresponding radial bounds are

$$(6.115) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^\pi \leq \mathfrak{J}_{\text{rad}}^{\text{1f}}(\mu),$$

and

$$(6.116) \quad \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \{ |z| \geq r \} \leq \mathfrak{J}_{\text{rad}}^{\text{1f}}(\mu).$$

If  $\mathcal{E}(\mu) \neq \emptyset$ , then

$$(6.117) \quad \mathcal{K}_{\text{in}}(\mu) \subset \mathcal{K}_{\text{kin}}^{\text{1f}}(\mu),$$

and  $\mathfrak{J}_c^{\text{1f}}(\mu)$  is the ray extent of the right-hand side in direction  $c$ .

*Proof.* The interior killed-wedge lower bound gives the constraints defining  $\mathcal{K}_{\text{kin}}(\mu)$ . We verify the two face constraints without suppressing the tail-to-moment step. Fix  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$  and a face  $i$ . Suppose, for a contradiction, that

$$\theta \cdot u_i > \mathfrak{J}_i^{\text{f}}(\mu).$$

Choose  $\varepsilon > 0$  such that  $\theta \cdot u_i \geq \mathfrak{J}_i^{\text{f}}(\mu) + 3\varepsilon$ . By continuity there is a relative open angular neighborhood  $U$  of  $u_i$  such that

$$\theta \cdot u \geq \mathfrak{J}_i^{\text{f}}(\mu) + 2\varepsilon, \quad u \in U.$$

Let

$$A_r := \{ z \neq 0 : |z| \geq r, z/|z| \in U \}.$$

The bound [\(6.107\)](#) implies that, for all sufficiently large  $r$ ,

$$\pi(A_r) \geq \exp\{ -(\mathfrak{J}_i^{\text{f}}(\mu) + \varepsilon)r \}.$$

On  $A_r$  one has  $\theta \cdot z \geq (\mathfrak{J}_i^{\text{f}}(\mu) + 2\varepsilon)r$ ; hence

$$\int_S e^{\theta \cdot z} \pi(dz) \geq e^{(\mathfrak{J}_i^{\text{f}}(\mu) + 2\varepsilon)r} \pi(A_r) \geq e^{\varepsilon r}$$

for every sufficiently large  $r$ , contradicting  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$ . Therefore

$$\theta \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu), \quad i = 1, 2.$$

Together with the killed-wedge constraints this places  $\mathcal{D}_{\text{exp}}(\pi)$  in  $\mathcal{K}_{\text{kin}}^{\text{lf}}(\mu)$ ; closure proves (6.112). The ray extent of that intersection is the minimum in (6.109), which yields the upper bound in (6.113).

For (6.114), the interior mechanism gives the bound  $\mathfrak{J}_c(\mu)$ . Fix a face  $i$  and  $0 < \delta < c \cdot u_i$ . Continuity of  $u \mapsto c \cdot u$  supplies a relative open angular neighborhood  $U_\delta$  of  $u_i$  such that

$$|c \cdot u - c \cdot u_i| < \delta, \quad u \in U_\delta.$$

Put

$$m_\delta := \inf_{u \in U_\delta} c \cdot u \geq c \cdot u_i - \delta > 0.$$

Then

$$\{|z| \geq x/m_\delta, z/|z| \in U_\delta\} \subset \{c \cdot z \geq x\}.$$

Writing  $s = x/m_\delta$  in (6.107) gives

$$\limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi\{z : c \cdot z \geq x\} \leq \frac{\mathfrak{J}_i^f(\mu)}{m_\delta} \leq \frac{\mathfrak{J}_i^f(\mu)}{c \cdot u_i - \delta}.$$

Letting  $\delta \downarrow 0$  proves that the  $i$ -th face mechanism contributes  $\mathfrak{J}_i^f(\mu)/(c \cdot u_i)$ . Taking the minimum of the interior mechanism and the two face mechanisms proves the displayed limsup. For the radial tail, the killed-wedge mechanism gives

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r\} \leq \mathfrak{J}_{\text{rad}}(\mu).$$

For each face  $i$  and every relative angular neighborhood  $U$  of  $u_i$ , the event in (6.107) is contained in  $\{|z| \geq r\}$ . Hence

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r\} \leq \mathfrak{J}_i^f(\mu), \quad i = 1, 2.$$

Taking the minimum of these three bounds gives (6.116). The moment-domain inequality in (6.115) follows from  $\alpha_{\text{rad}}^{\text{in}} \leq \alpha_{\text{rad}}^\pi$  and the tail-abscissa identity proposition 6.4.

Finally, let  $\theta \in \mathcal{E}(\mu)$  and  $\eta \in S^\vee$ . The point  $\theta - \eta$  belongs to  $\mathcal{K}_{\text{kin}}(\mu)$ . Moreover,

$$(\theta - \eta) \cdot u_i \leq \theta \cdot u_i \leq \mathfrak{J}_i^f(\mu),$$

because  $\theta$  is feasible in the dual problem (6.103). This proves (6.117). The ray-extent assertion follows directly from (6.111).  $\square$

**Corollary 6.50** (one-face geometric criterion for exact logarithmic tails). *Assume  $\mathcal{E}(\mu) \neq \emptyset$  and let  $\pi$  be any stationary law for the drift  $\mu$ .*

(i) *If, for some  $c \in S_\circ^\vee$ ,*

$$(6.118) \quad \alpha_c^{\text{in}}(\mu) = \mathfrak{J}_c^{\text{lf}}(\mu),$$

*then*

$$(6.119) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z : c \cdot z \geq x\} = -\mathfrak{J}_c^{\text{lf}}(\mu).$$

*The equality in (6.118) holds if and only if the endpoint  $\mathfrak{J}_c^{\text{lf}}(\mu)c$  belongs to  $\mathcal{K}_{\text{in}}(\mu)$ .*

(ii) *If*

$$\alpha_{\text{rad}}^{\text{in}}(\mu) = \mathfrak{J}_{\text{rad}}^{\text{lf}}(\mu),$$

*then*

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log \pi\{|z| \geq r\} = -\mathfrak{J}_{\text{rad}}^{\text{lf}}(\mu).$$

*Both conclusions hold for every stationary law at the prescribed drift.*

*Proof.* For projected tails, the inner moment bound gives

$$\alpha_c^{\text{in}}(\mu) \leq \liminf_{x \rightarrow \infty} -\frac{1}{x} \log \pi\{z : c \cdot z \geq x\}.$$

The limsup is bounded above by  $\mathfrak{J}_c^{1f}(\mu)$  by [corollary 6.49](#). Under [\(6.118\)](#), the two bounds coincide. The endpoint characterization follows from the inclusion [\(6.117\)](#), closedness and convexity of the two sets, and the fact that both contain the origin. The radial proof is the same.  $\square$

**Corollary 6.51** (explicit face-active certificate). *Assume  $\mathcal{E}(\mu) \neq \emptyset$ . Fix  $c \in S_o^\vee$ , and suppose that for some  $i \in \{1, 2\}$ ,*

$$\lambda := \mathfrak{J}_c^{1f}(\mu) = \frac{\mathfrak{J}_i^f(\mu)}{c \cdot u_i}.$$

Let  $\vartheta_i^f$  be a maximizing point from [proposition 6.45](#). If

$$(6.120) \quad \vartheta_i^f \cdot v_{3-i} \leq 0, \quad \vartheta_i^f - \lambda c \in S^\vee,$$

then  $\alpha_c^{\text{in}}(\mu) = \lambda$ , and the exact projected tail [\(6.119\)](#) holds for every stationary law at drift  $\mu$ .

*Proof.* The second condition in [\(6.120\)](#), together with  $\lambda c \in S^\vee$ , implies  $\vartheta_i^f \in S^\vee$ . By [proposition 6.45](#), the point lies on  $Q_\mu = 0$  and satisfies its own face inequality  $\vartheta_i^f \cdot v_i \leq 0$ . The first condition supplies the other oblique inequality. Hence

$$\vartheta_i^f \in \overline{\mathcal{E}(\mu)}.$$

The decomposition

$$\lambda c = \vartheta_i^f - (\vartheta_i^f - \lambda c)$$

therefore places the one-face endpoint in  $\mathcal{K}_{\text{in}}(\mu)$ . Apply [corollary 6.50](#).  $\square$

**Lemma 6.52** (at most one active one-face clip in the classified geometry). *Assume*

$$0 < \xi < \pi, \quad 1 \leq \alpha < 2,$$

and let  $\mu \neq 0$ . Put  $M = |\mu|$  and

$$\vartheta_i^0 := Mu_i - \mu, \quad i = 1, 2.$$

Then the two strict inequalities

$$\vartheta_1^0 \cdot v_1 > 0, \quad \vartheta_2^0 \cdot v_2 > 0$$

cannot hold simultaneously. Consequently at least one one-face cost is unclipped:

$$\mathfrak{J}_i^f(\mu) = h_{\mathbb{B}_\mu}(u_i)$$

for some  $i \in \{1, 2\}$ , and for that index

$$\mathbb{C}_i(\mu) = \mathbb{B}_\mu.$$

*Proof.* Write  $\mu = Mu(\varphi)$  with  $\varphi \in [0, 2\pi)$ . With the angle normalization of [section 2](#),

$$\cos \theta_1 v_1 = (-\sin \theta_1, \cos \theta_1),$$

and

$$\cos \theta_2 v_2 = (\sin(\xi - \theta_2), -\cos(\xi - \theta_2)).$$

We derive the two sign formulas explicitly. Since  $u_1 = (1, 0)$ ,  $u_2 = (\cos \xi, \sin \xi)$ , and  $\mu/M = (\cos \varphi, \sin \varphi)$ ,

$$\begin{aligned} \frac{\cos \theta_1}{M} \vartheta_1^0 \cdot v_1 &= (1 - \cos \varphi)(-\sin \theta_1) - \sin \varphi \cos \theta_1 \\ &= \sin(\theta_1 - \varphi) - \sin \theta_1 \\ &= -2 \sin \frac{\varphi}{2} \cos \left( \frac{\varphi}{2} - \theta_1 \right). \end{aligned}$$

Likewise,

$$\begin{aligned} \frac{\cos \theta_2}{M} \vartheta_2^0 \cdot v_2 &= (\cos \xi - \cos \varphi) \sin(\xi - \theta_2) - (\sin \xi - \sin \varphi) \cos(\xi - \theta_2) \\ &= \sin(\varphi - \xi + \theta_2) - \sin \theta_2 \\ &= 2 \sin \frac{\varphi - \xi}{2} \cos \left( \frac{\varphi - \xi}{2} + \theta_2 \right). \end{aligned}$$

Because  $\theta_1 \in (-\pi/2, \pi/2)$ , the first expression is positive exactly when

$$\varphi \in (\pi + 2\theta_1, 2\pi).$$

For the second expression, the condition  $\theta_1 + \theta_2 \geq \xi$ , together with  $\theta_1 < \pi/2$ , gives  $\theta_2 > \xi - \pi/2$ . Hence the cosine factor is positive on  $[0, \xi]$ , while the sine factor is nonpositive there. On  $(\xi, 2\pi)$  the sine factor is positive, and the cosine factor remains positive exactly until its first zero. Thus the second expression is positive exactly when

$$\varphi \in (\xi, \xi + \pi - 2\theta_2).$$

Both intervals lie in  $[0, 2\pi)$ , and

$$(\pi + 2\theta_1) - (\xi + \pi - 2\theta_2) = 2(\theta_1 + \theta_2) - \xi = (2\alpha - 1)\xi > 0.$$

They are therefore disjoint, proving the first assertion.

If  $\vartheta_i^0 \cdot v_i \leq 0$ , the unrestricted support point of the kernel disk in direction  $u_i$  is feasible in the one-face dual problem of [proposition 6.45](#). Hence the one-face support value is the full-disk support value. The defining half-space of  $\mathbb{C}_i(\mu)$  is then a supporting half-space of the whole disk and does not remove any point of  $\mathbb{B}_\mu$ .  $\square$

**6.4. One-face broken paths and a sharper outer frontier.** The preceding face cost describes motion whose macroscopic endpoint lies on a boundary ray. A projected tail can be cheaper when the path first uses one face and then leaves it along an interior segment. This leads to an infimal convolution of the face cost with the unconstrained Brownian quasipotential.

For  $x \in \mathbb{R}^2$ , put

$$(6.121) \quad \mathfrak{J}_\mu(x) := |\mu| |x| - \mu \cdot x = h_{\mathbb{B}_\mu}(x), \quad \mathbb{B}_\mu = \overline{B}(-\mu, |\mu|).$$

Equivalently,

$$\mathfrak{J}_\mu(x) = \inf_{t>0} \frac{|x - \mu t|^2}{2t}.$$

**Definition 6.53** (one-face broken-path cost). For  $i \in \{1, 2\}$  and  $x \in \mathbb{R}^2$ , define

$$(6.122) \quad \mathfrak{J}_{i,\text{br}}(x; \mu) := \inf_{y \geq 0} \left\{ y \mathfrak{J}_i^f(\mu) + \mathfrak{J}_\mu(x - y u_i) \right\}.$$

For a unit direction  $u \in S \cap \mathbb{S}^1$ , the parameter  $y$  is the macroscopic distance traveled along face  $i$  before the path enters the interior. The extension to all  $x \in \mathbb{R}^2$  makes the positive homogeneity and the support-function representation below literal.

**Lemma 6.54** (uniform tubular containment of a broken segment). *Let  $S^\sharp \subset S$  be a closed convex subwedge with one boundary ray equal to  $\partial S_i$  and the other boundary ray  $\Gamma^\sharp \subset S^\circ$ . Let  $\gamma : [0, 1] \rightarrow S^\sharp$  be continuous and suppose that*

$$\gamma([0, 1]) \cap (\{0\} \cup \Gamma^\sharp) = \emptyset, \quad \gamma(1) = u \in S^\circ \cap \mathbb{S}^1.$$

Put

$$m_\gamma := \min_{s \in [0, 1]} |\gamma(s)| > 0, \quad d_\gamma := \min_{s \in [0, 1]} \text{dist}(\gamma(s), \Gamma^\sharp) > 0.$$

Let  $r_0 > 0$ . Suppose that  $R_r \geq 0$  satisfies  $R_r/r \rightarrow \rho$  with

$$(6.123) \quad 0 \leq \rho < \frac{1}{2} \min\{m_\gamma, d_\gamma\}.$$

Let  $\sigma_r \in (0, 1]$ . If a continuous path  $z_r : [0, \sigma_r] \rightarrow S$  obeys

$$(6.124) \quad \sup_{0 \leq s \leq \sigma_r} |z_r(s) - r\gamma(s)| \leq R_r,$$

then, for all sufficiently large  $r$ , the path neither reaches  $\{|z| = r_0\}$  nor intersects  $\Gamma^\sharp$ . If in addition  $\sigma_r = 1$ , then

$$(6.125) \quad |z_r(1)| \geq r - R_r, \quad \left| \frac{z_r(1)}{|z_r(1)|} - u \right| \leq \frac{2R_r}{r}$$

whenever  $R_r < r$ .

*Proof.* Distance from a point to the origin and distance from a point to the closed ray  $\Gamma^\sharp$  are both 1-Lipschitz functions. Hence (6.124) gives, uniformly for  $0 \leq s \leq \sigma_r$ ,

$$|z_r(s)| \geq rm_\gamma - R_r, \quad \text{dist}(z_r(s), \Gamma^\sharp) \geq rd_\gamma - R_r.$$

Condition (6.123) makes both right-hand sides positive for all large  $r$ , and the first tends to infinity; this proves the non-exit assertions on every possible prefix. Assume now that  $\sigma_r = 1$ . Since  $\gamma(1) = u$  and  $|u| = 1$ , the first estimate in (6.125) is the reverse triangle inequality. Finally,

$$\begin{aligned} \left| \frac{z_r(1)}{|z_r(1)|} - u \right| &\leq \left| \frac{z_r(1)}{|z_r(1)|} - \frac{z_r(1)}{r} \right| + \left| \frac{z_r(1)}{r} - u \right| \\ &= \frac{||z_r(1)| - r|}{r} + \frac{|z_r(1) - ru|}{r} \leq \frac{2R_r}{r}, \end{aligned}$$

which proves the terminal direction estimate.  $\square$

**Proposition 6.55** (stationary lower bound for one-face broken paths). *Assume  $0 < \xi < \pi$  and  $\alpha < 2$ , and let  $\pi$  be stationary for the drift  $\mu$ . Fix  $i \in \{1, 2\}$ , an interior direction  $u \in S^\circ \cap \mathbb{S}^1$ , and a relative open angular neighborhood  $U$  of  $u$ . Then*

$$(6.126) \quad \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in U \right\} \leq \mathfrak{J}_{i, \text{br}}(u; \mu).$$

Consequently,

$$(6.127) \quad \beta_\pi(u) \leq \min_{i=1,2} \mathfrak{J}_{i, \text{br}}(u; \mu), \quad u \in S^\circ \cap \mathbb{S}^1.$$

*Proof.* Fix  $y > 0$ ,  $\ell \geq 0$ , and speeds  $a_1, a_2 > 0$ . Put

$$d := u - yu_i, \quad T_{1,r} := \frac{yr}{a_1}, \quad T_{2,r} := \frac{|d|r}{a_2}, \quad T_r := T_{1,r} + T_{2,r}.$$

Since  $u$  is an interior direction,  $d \neq 0$  and  $d \cdot n_i = u \cdot n_i > 0$ .

Choose a closed convex subwedge  $S_{i,u}^\sharp \subset S$  with one boundary ray equal to  $\partial S_i$ , with its second boundary ray lying strictly between  $u$  and the opposite face, and with  $u$  in its relative interior. Let  $\Gamma_{i,u}^\sharp$  be the auxiliary boundary ray. For a fixed  $r_0 > 0$ , set

$$G_{i,u}^\sharp := (S_{i,u}^\sharp \cap \{|z| > r_0\}) \setminus \Gamma_{i,u}^\sharp.$$

This relatively open set contains the true reflecting side  $\partial S_i \cap \{|z| > r_0\}$ , meets no other true face, and is separated from the vertex. Choose a one-face conical strip  $H_i$  whose closure, apart from its true side, is contained in  $G_{i,u}^\sharp$ , and denote its aperture parameter by  $\kappa$ .

Apply lemma 6.43 to  $H_i$ , with speed  $a_1$  and regulator rate  $\ell$ . Fix the resulting compact ball  $K \Subset H_i \cap S^\circ$ , start-up radius  $\eta_0 > 0$ , and constant  $q_* > r_0$ . These choices are uniform in the start-up time and in the relative-tube tolerance. Strict positivity of the stationary density gives  $\pi(K) > 0$ .

The deterministic second-phase segment, at macroscopic scale one, is

$$\gamma(s) := yu_i + s(u - yu_i), \quad 0 \leq s \leq 1.$$

It is a compact subset of  $S_{i,u}^\sharp$ , meets the true face only at  $s = 0$ , and is disjoint from the vertex and the auxiliary ray. Hence

$$(6.128) \quad m_\gamma := \min_{0 \leq s \leq 1} |\gamma(s)| > 0, \quad d_\gamma := \min_{0 \leq s \leq 1} \text{dist}(\gamma(s), \Gamma_{i,u}^\sharp) > 0.$$

Choose  $\rho_U > 0$  so that every unit vector within Euclidean distance  $\rho_U$  of  $u$  belongs to  $U$ .

Define

$$w_1 := a_1 u_i - \mu - \ell v_i, \quad w_2 := a_2 \frac{d}{|d|} - \mu, \quad \gamma_i := a_2 \frac{d \cdot n_i}{|d|} > 0.$$

Apply the path estimate (6.93) with  $a = a_1$  and the present value of  $\ell$ . There are constants  $D_0, D_1 < \infty$ , depending only on  $K$ , the reflection vector, and the fixed first-phase control, such that, before exit from  $H_i$ ,

$$|Z(t) - a_1 t u_i| \leq D_0 + D_1 \sup_{0 \leq q \leq t} |\tilde{B}_r(q)|.$$

On the first two tube events below, the running Brownian supremum is at most  $\eta_0 + \varepsilon t$ . We therefore fix

$$A_0 = A_2 := D_0 + D_1 \eta_0, \quad A_1 := \frac{D_1 y}{a_1}, \quad A_3 := D_1.$$

These constants give exactly the switching and coordinate estimates (6.140)–(6.141). During the second phase, the switching error contributes at most  $A_0 + A_1 \varepsilon r$ , the Brownian increment contributes at most  $\varepsilon(h_r + T_{2,r})$ , and the additional reflection term contributes at most  $|v_i| \varepsilon h_r$ . Since  $T_{2,r} = |d|r/a_2$ , define

$$(6.129) \quad C_* := A_1 + \frac{|d|}{a_2} > 0$$

and

$$(6.130) \quad R_{r,\varepsilon} := A_0 + (1 + |v_i|) \varepsilon h_r.$$

Then the total deviation is bounded by  $C_* \varepsilon r + R_{r,\varepsilon}$ , and, for every fixed  $\varepsilon > 0$ ,

$$\frac{R_{r,\varepsilon}}{r} = \frac{A_0}{r} + (1 + |v_i|) \varepsilon r^{-1/2} \longrightarrow 0.$$

All constants are independent of the starting point  $x \in K$ .

Choose  $\varepsilon > 0$  before choosing the start-up time so that

$$(6.131) \quad (1 + \kappa) A_3 \varepsilon < \frac{\kappa a_1}{2},$$

$$(6.132) \quad \varepsilon < \frac{\gamma_i}{2},$$

$$(6.133) \quad 4C_* \varepsilon < \min\{m_\gamma, d_\gamma, 1\}, \quad 8C_* \varepsilon < \rho_U.$$

All geometric smallness requirements have now been imposed. By lemma 6.40, choose  $t_0$  so large that

$$\mathbb{P} \left\{ \sup_{s \geq 0} \frac{|W(s)|}{t_0 + s} \leq \varepsilon \right\} \geq \frac{1}{2}$$

for standard planar Brownian motion  $W$ . Increase  $t_0$ , if necessary, so that

$$(6.134) \quad (1 + \kappa) A_2 < \frac{\kappa a_1 t_0}{2}, \quad a_1 t_0 - A_2 - A_3 \varepsilon t_0 > r_0.$$

Increasing  $t_0$  preserves the relative-tube probability bound, and the uniform start-up lemma leaves  $K$  and  $\eta_0$  unchanged.

For every sufficiently large  $r$ , so that  $t_0 < T_{1,r}$ , define on  $\mathcal{F}_{T_r}$  the Cameron–Martin law  $\mathbb{Q}_{x,r}$  associated with the piecewise constant control

$$w_r(t) = \begin{cases} w_1, & 0 \leq t \leq T_{1,r}, \\ w_2, & T_{1,r} < t \leq T_r. \end{cases}$$

Under  $\mathbb{Q}_{x,r}$ ,

$$\tilde{B}_r(t) := B(t) - \int_0^t w_r(s) ds$$

is standard Brownian motion. By [lemmas 6.41](#) and [6.42](#), the same reflected path is represented by the one-face Skorokhod map on  $[0, T_r \wedge \tau_{i,u}^\sharp]$ , where  $\tau_{i,u}^\sharp$  is the exit time from  $G_{i,u}^\sharp$ .

Use the three events

$$(6.135) \quad \begin{aligned} E_0 &:= \left\{ \sup_{0 \leq t \leq t_0} |\tilde{B}_r(t)| \leq \eta_0 \right\}, \\ E_{1,r} &:= \left\{ \sup_{0 \leq s \leq T_{1,r} - t_0} \frac{|\tilde{B}_r(t_0 + s) - \tilde{B}_r(t_0)|}{t_0 + s} \leq \varepsilon \right\}, \end{aligned}$$

and, with  $h_r = r^{1/2}$ ,

$$(6.136) \quad E_{2,r} := \left\{ \sup_{0 \leq s \leq T_{2,r}} \frac{|\tilde{B}_r(T_{1,r} + s) - \tilde{B}_r(T_{1,r})|}{h_r + s} \leq \varepsilon \right\}.$$

The start-up lemma gives

$$(6.137) \quad \inf_{x \in K} \mathbb{Q}_{x,r}(E_0) = p_0(t_0) > 0.$$

The choice of  $t_0$  gives  $\mathbb{Q}_{x,r}(E_{1,r}) \geq 1/2$ , uniformly in  $r$  and  $x$ , while [lemma 6.40](#) gives  $\mathbb{Q}_{x,r}(E_{2,r}) \rightarrow 1$ . The three events are measurable with respect to Brownian increments on the disjoint intervals  $[0, t_0]$ ,  $[t_0, T_{1,r}]$ , and  $[T_{1,r}, T_r]$ , respectively; they are therefore independent under  $\mathbb{Q}_{x,r}$ . Choose  $r_1$  so large that  $\mathbb{Q}_{x,r}(E_{2,r}) \geq 1/2$  for  $r \geq r_1$ . Then

$$(6.138) \quad \inf_{x \in K} \mathbb{Q}_{x,r}(E_0 \cap E_{1,r} \cap E_{2,r}) \geq \frac{p_0(t_0)}{4}, \quad r \geq r_1.$$

Since probabilities are at most one, this constant lower bound proves

$$(6.139) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log \inf_{x \in K} \mathbb{Q}_{x,r}(E_0 \cap E_{1,r} \cap E_{2,r}) = 0.$$

During the first phase the controlled input drift is  $a_1 u_i - \ell v_i$ . The explicit one-face formula gives, on  $E_0 \cap E_{1,r}$  and up to  $T_{1,r} \wedge \tau_{i,u}^\sharp$ ,

$$(6.140) \quad Z(T_{1,r}) = y r u_i + R_{1,r}, \quad |R_{1,r}| \leq A_0 + A_1 \varepsilon r,$$

provided the exit time exceeds  $T_{1,r}$ , and, in the conical-strip coordinates  $Z(t) = q(t)u_i + s(t)n_i$ ,

$$(6.141) \quad q(t) \geq a_1 t - A_2 - A_3 \varepsilon t, \quad 0 \leq s(t) \leq A_2 + A_3 \varepsilon t$$

for  $t_0 \leq t \leq T_{1,r} \wedge \tau_{i,u}^\sharp$ . The first-phase non-exit follows from explicit inequalities. From [\(6.131\)](#),  $A_3 \varepsilon < a_1/2$ , and hence, for  $t \geq t_0$ ,

$$q(t) \geq (a_1 - A_3 \varepsilon)t - A_2 \geq a_1 t_0 - A_2 - A_3 \varepsilon t_0 > r_0.$$

Moreover,

$$\begin{aligned} \kappa q(t) - s(t) &\geq \kappa a_1 t - (1 + \kappa)(A_2 + A_3 \varepsilon t) \\ &> \frac{\kappa a_1 t}{2} - (1 + \kappa)A_2 \\ &\geq \frac{\kappa a_1 t_0}{2} - (1 + \kappa)A_2 > 0, \end{aligned}$$

where the last inequality is [\(6.134\)](#). On  $[0, t_0]$ , the same strict strip inequalities follow from  $E_0$  and the start-up lemma. Let  $\sigma_i$  be the first time the path meets either artificial boundary of  $H_i$ . If  $\sigma_i \leq T_{1,r} \wedge \tau_{i,u}^\sharp$ , continuity would force  $q(\sigma_i) = r_0$  or  $s(\sigma_i) = \kappa q(\sigma_i)$ , contradicting the strict bounds. Thus  $\sigma_i > T_{1,r} \wedge \tau_{i,u}^\sharp$ . We now exclude  $\tau_{i,u}^\sharp \leq T_{1,r}$  by a first-exit contradiction rather than by an implicit geometric assertion. Suppose that this inequality held. Then  $\sigma_i > \tau_{i,u}^\sharp$ , so

the path has not met an artificial boundary of  $H_i$  by time  $\tau_{i,u}^\sharp$ . The coordinate bounds, which hold up to the stopped time, give

$$q(\tau_{i,u}^\sharp) > r_0, \quad 0 \leq s(\tau_{i,u}^\sharp) < \kappa q(\tau_{i,u}^\sharp).$$

Consequently

$$Z(\tau_{i,u}^\sharp) = q(\tau_{i,u}^\sharp)u_i + s(\tau_{i,u}^\sharp)n_i \in H_i.$$

The true side  $s = 0$  is part of  $G_{i,u}^\sharp$ , and by construction  $\bar{H}_i$  away from that side is contained in  $G_{i,u}^\sharp$ . Hence in either case  $Z(\tau_{i,u}^\sharp) \in G_{i,u}^\sharp$ , contradicting the definition of the first exit time from this relatively open set. Therefore  $\tau_{i,u}^\sharp > T_{1,r}$ . This proves the first-phase non-exit and validates (6.140).

For  $0 \leq s \leq T_{2,r} \wedge (\tau_{i,u}^\sharp - T_{1,r})$ , set  $\Delta L_r(s) := L(T_{1,r} + s) - L(T_{1,r})$ . The shifted one-dimensional Skorokhod formula gives

$$\Delta L_r(s) = \sup_{0 \leq q \leq s} \{-Z(T_{1,r}) \cdot n_i - \gamma_i q - [\tilde{B}_r(T_{1,r} + q) - \tilde{B}_r(T_{1,r})] \cdot n_i\}_+.$$

By (6.132), the event  $E_{2,r}$  yields

$$(6.142) \quad 0 \leq \Delta L_r(s) \leq \varepsilon h_r, \quad 0 \leq s \leq T_{2,r} \wedge (\tau_{i,u}^\sharp - T_{1,r}).$$

Indeed, the expression inside the running supremum is bounded above by  $\varepsilon h_r - (\gamma_i - \varepsilon)q$ .

The vector representation during the second phase is

$$(6.143) \quad Z(T_{1,r} + s) = Z(T_{1,r}) + a_2 \frac{d}{|d|} s + \tilde{B}_r(T_{1,r} + s) - \tilde{B}_r(T_{1,r}) + v_i \Delta L_r(s).$$

Since

$$yru_i + a_2 \frac{d}{|d|} s = r\gamma \left( \frac{s}{T_{2,r}} \right),$$

subtracting this deterministic segment from (6.143) and using (6.140) gives

$$\begin{aligned} & \left| Z(T_{1,r} + s) - r\gamma \left( \frac{s}{T_{2,r}} \right) \right| \\ & \leq A_0 + A_1 \varepsilon r + \varepsilon(h_r + s) + |v_i| \varepsilon h_r \\ & \leq \left( A_1 + \frac{|d|}{a_2} \right) \varepsilon r + A_0 + (1 + |v_i|) \varepsilon h_r \\ & = C_* \varepsilon r + R_{r,\varepsilon} \end{aligned}$$

for  $0 \leq s \leq T_{2,r} \wedge (\tau_{i,u}^\sharp - T_{1,r})$ . Thus

$$(6.144) \quad \sup_{0 \leq s \leq T_{2,r} \wedge (\tau_{i,u}^\sharp - T_{1,r})} \left| Z(T_{1,r} + s) - r\gamma \left( \frac{s}{T_{2,r}} \right) \right| \leq C_* \varepsilon r + R_{r,\varepsilon}.$$

By (6.133),  $C_* \varepsilon < \frac{1}{4} \min\{m_\gamma, d_\gamma\}$ . Since  $R_{r,\varepsilon}/r \rightarrow 0$ , for all sufficiently large  $r$ ,

$$\frac{C_* \varepsilon r + R_{r,\varepsilon}}{r} < \frac{1}{2} \min\{m_\gamma, d_\gamma\}.$$

We turn the preceding tube estimate into an explicit first-exit contradiction. Suppose that  $\tau_{i,u}^\sharp \leq T_r$ . The first-phase argument already gives  $\tau_{i,u}^\sharp > T_{1,r}$ , so

$$\sigma_r := \frac{\tau_{i,u}^\sharp - T_{1,r}}{T_{2,r}} \in (0, 1].$$

Define the stopped, rescaled second-phase path

$$z_r(q) := Z(T_{1,r} + qT_{2,r}), \quad 0 \leq q \leq \sigma_r.$$

Substituting  $s = qT_{2,r}$  into (6.144) gives

$$\sup_{0 \leq q \leq \sigma_r} |z_r(q) - r\gamma(q)| \leq C_*\varepsilon r + R_{r,\varepsilon}.$$

The ratio of the right-hand side to  $r$  is, for all large  $r$ , smaller than  $\frac{1}{2} \min\{m_\gamma, d_\gamma\}$ . Hence lemma 6.54, applied with this  $\sigma_r$ , shows that  $z_r$  neither reaches the inner circle nor meets the auxiliary ray up to and including  $q = \sigma_r$ . Those are the only exit portions of  $G_{i,u}^\sharp$ ; the true reflecting face belongs to the relative state space and is not an exit boundary. This contradicts the definition of  $\tau_{i,u}^\sharp$ . Therefore  $\tau_{i,u}^\sharp > T_r$  on the three tube events.

At  $s = T_{2,r}$ , (6.144) gives

$$(6.145) \quad Z(T_r) = ru + R_{2,r}, \quad |R_{2,r}| \leq C_*\varepsilon r + R_{r,\varepsilon}.$$

For all sufficiently large  $r$ , the convergence  $R_{r,\varepsilon}/r \rightarrow 0$  makes the right-hand side at most  $2C_*\varepsilon r$ . The reverse triangle inequality then gives

$$|Z(T_r)| \geq (1 - 2C_*\varepsilon)r.$$

The terminal direction estimate in lemma 6.54 gives

$$\left| \frac{Z(T_r)}{|Z(T_r)|} - u \right| \leq \frac{2|R_{2,r}|}{r} \leq 4C_*\varepsilon < \frac{\rho U}{2}$$

by (6.133). Hence, for all sufficiently large  $r$ ,

$$(6.146) \quad |Z(T_r)| \geq (1 - 2C_*\varepsilon)r, \quad \frac{Z(T_r)}{|Z(T_r)|} \in U.$$

On  $\mathcal{F}_{T_r}$ ,

$$\frac{d\mathbb{P}_x}{d\mathbb{Q}_{x,r}} = \exp \left\{ - \int_0^{T_r} w_r(s) \cdot d\tilde{B}_r(s) - \frac{1}{2} \int_0^{T_r} |w_r(s)|^2 ds \right\}.$$

Since the control is constant on each phase,

$$\int_0^{T_r} w_r(s) \cdot d\tilde{B}_r(s) = w_1 \cdot \tilde{B}_r(T_{1,r}) + w_2 \cdot [\tilde{B}_r(T_r) - \tilde{B}_r(T_{1,r})].$$

On the controlled event its absolute value is at most

$$(6.147) \quad \varepsilon r C_{\text{like}} + \rho_{r,\varepsilon}, \quad C_{\text{like}} := \frac{y}{a_1} |w_1| + \frac{|d|}{a_2} |w_2|, \quad \rho_{r,\varepsilon} := |w_1| \eta_0 + |w_2| \varepsilon h_r.$$

For fixed  $\varepsilon$ ,  $\rho_{r,\varepsilon}/r \rightarrow 0$ . The deterministic action is exactly

$$(6.148) \quad \frac{1}{2} \int_0^{T_r} |w_r(s)|^2 ds = r A_{\text{ctrl}},$$

where

$$A_{\text{ctrl}} := \frac{y}{2a_1} |a_1 u_i - \mu - \ell v_i|^2 + \frac{|d|}{2a_2} \left| a_2 \frac{d}{|d|} - \mu \right|^2.$$

It follows from (6.147)–(6.148) that, on the controlled event,

$$(6.149) \quad \frac{d\mathbb{P}_x}{d\mathbb{Q}_{x,r}} \geq \exp\{-r(A_{\text{ctrl}} + \varepsilon C_{\text{like}}) - \rho_{r,\varepsilon}\}.$$

Let

$$A_{r,\varepsilon} := \left\{ z \neq 0 : |z| \geq (1 - 2C_*\varepsilon)r, \frac{z}{|z|} \in U \right\}.$$

By stationarity and  $\pi(K) > 0$ ,

$$(6.150) \quad \pi(A_{r,\varepsilon}) \geq \pi(K) \inf_{x \in K} \mathbb{P}_x\{Z(T_r) \in A_{r,\varepsilon}\}.$$

The terminal inclusion, the constant controlled-event lower bound (6.138), and (6.149) yield, for all large  $r$ ,

$$\pi(A_{r,\varepsilon}) \geq \pi(K) \frac{p_0(t_0)}{4} \exp\{-r(A_{\text{ctrl}} + \varepsilon C_{\text{like}}) - \rho_{r,\varepsilon}\}.$$

Therefore

$$(6.151) \quad \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi(A_{r,\varepsilon}) \leq A_{\text{ctrl}} + \varepsilon C_{\text{like}}.$$

Put  $s = (1 - 2C_*\varepsilon)r$ . The choice of  $\varepsilon$  ensures  $1 - 2C_*\varepsilon > 0$ , and (6.151) becomes

$$\begin{aligned} \limsup_{s \rightarrow \infty} -\frac{1}{s} \log \pi \left\{ z \neq 0 : |z| \geq s, \frac{z}{|z|} \in U \right\} \\ \leq \frac{A_{\text{ctrl}} + \varepsilon C_{\text{like}}}{1 - 2C_*\varepsilon}. \end{aligned}$$

Letting  $\varepsilon \downarrow 0$ , then minimizing over  $a_1 > 0$  and  $\ell \geq 0$ , gives  $y\mathfrak{J}_i^f(\mu)$ ; minimizing over  $a_2 > 0$  gives  $\mathfrak{J}_\mu(u - yu_i)$ . Finally minimize over  $y > 0$ . The function

$$y \mapsto y\mathfrak{J}_i^f(\mu) + \mathfrak{J}_\mu(u - yu_i)$$

is continuous on  $[0, \infty)$ , because  $\mathfrak{J}_\mu$  is continuous. Since 0 is a limit point of  $(0, \infty)$ , the infimum over  $y > 0$  equals the infimum over  $y \geq 0$ . This proves (6.126). For a fixed relative open neighborhood  $U$ , the radial tail–abscissa identity proposition 6.4 gives

$$\alpha_U^{\pi, \text{rad}} \leq \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in U \right\} \leq \mathfrak{J}_{i, \text{br}}(u; \mu).$$

This holds for each face  $i$ . Taking the minimum over  $i$  and then the supremum over all relative open angular neighborhoods  $U$  of  $u$  in the definition of  $\beta_\pi(u)$  proves (6.127).  $\square$

**Proposition 6.56** (dual representation of the broken-path cost). *For  $i \in \{1, 2\}$ , define the compact convex set*

$$(6.152) \quad \mathbb{C}_i(\mu) := \mathbb{B}_\mu \cap \left\{ \theta : \theta \cdot u_i \leq \mathfrak{J}_i^f(\mu) \right\}.$$

Then

$$(6.153) \quad \mathfrak{J}_{i, \text{br}}(u; \mu) = h_{\mathbb{C}_i(\mu)}(u) = \max_{\theta \in \mathbb{C}_i(\mu)} \theta \cdot u, \quad u \in \mathbb{R}^2.$$

In particular,

$$(6.154) \quad \mathfrak{J}_{i, \text{br}}(u; \mu) \leq \mathfrak{J}_\mu(u), \quad \mathfrak{J}_{i, \text{br}}(u_i; \mu) = \mathfrak{J}_i^f(\mu).$$

*Proof.* Let  $f = h_{\mathbb{B}_\mu} = \mathfrak{J}_\mu$ , and define the proper convex function

$$k_i(x) = \begin{cases} y\mathfrak{J}_i^f(\mu), & x = yu_i, y \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the left-hand side of (6.153) is the infimal convolution  $(f \square k_i)(u)$ . Both summands are nonnegative, and the choice  $y = 0$  shows that the infimal convolution is finite on all of  $\mathbb{R}^2$ . It is convex as an infimal convolution of proper convex functions. Every finite convex function on  $\mathbb{R}^2$  is continuous; hence  $f \square k_i$  is proper, closed, and convex.

Moreover,

$$f^* = \delta_{\mathbb{B}_\mu}, \quad k_i^* = \delta_{\{\theta : \theta \cdot u_i \leq \mathfrak{J}_i^f(\mu)\}}.$$

Since  $f$  is finite and continuous on  $\mathbb{R}^2$ , the conjugacy rule for infimal convolution gives

$$(f \square k_i)^* = \delta_{\mathbb{C}_i(\mu)}.$$

The Fenchel–Moreau theorem now yields  $f \square k_i = h_{\mathbb{C}_i(\mu)}$ , proving (6.153). The first inequality in (6.154) follows by taking  $y = 0$  in (6.122).

For the endpoint identity, let  $\vartheta_i^f$  be a maximizer in the one-face dual problem of [proposition 6.45](#). Then

$$\vartheta_i^f \in \mathbb{B}_\mu, \quad \vartheta_i^f \cdot u_i = \mathfrak{J}_i^f(\mu).$$

The second equality implies  $\vartheta_i^f \in \mathbb{C}_i(\mu)$ , and therefore

$$h_{\mathbb{C}_i(\mu)}(u_i) \geq \vartheta_i^f \cdot u_i = \mathfrak{J}_i^f(\mu).$$

Conversely, every  $\theta \in \mathbb{C}_i(\mu)$  satisfies  $\theta \cdot u_i \leq \mathfrak{J}_i^f(\mu)$  by the defining half-space in [\(6.152\)](#). Thus

$$h_{\mathbb{C}_i(\mu)}(u_i) \leq \mathfrak{J}_i^f(\mu).$$

The two inequalities prove  $\mathfrak{J}_{i,\text{br}}(u_i; \mu) = \mathfrak{J}_i^f(\mu)$ .  $\square$

**Proposition 6.57** (explicit clipped-kernel formula for broken paths). *Fix a face  $i$ , and write*

$$M := |\mu|, \quad \mu_{i,t} := \mu \cdot u_i, \quad \mu_{i,n} := \mu \cdot n_i, \quad J_i := \mathfrak{J}_i^f(\mu).$$

Set

$$(6.155) \quad \kappa_i := J_i + \mu_{i,t}, \quad \sigma_i := -\mu_{i,n} + \sqrt{M^2 - \kappa_i^2}.$$

Then  $-M \leq \kappa_i \leq M$ . For  $x \in S$ , write

$$x = a_i(x)u_i + b_i(x)n_i, \quad b_i(x) \geq 0.$$

For  $x \neq 0$ , one has

$$(6.156) \quad \mathfrak{J}_{i,\text{br}}(x; \mu) = \begin{cases} M|x| - \mu \cdot x, & M \frac{a_i(x)}{|x|} - \mu_{i,t} \leq J_i, \\ J_i a_i(x) + \sigma_i b_i(x), & M \frac{a_i(x)}{|x|} - \mu_{i,t} > J_i. \end{cases}$$

The value at  $x = 0$  is zero. In the first case a supporting point is

$$M \frac{x}{|x|} - \mu,$$

whereas in the second case a supporting point is

$$(6.157) \quad J_i u_i + \sigma_i n_i.$$

Consequently  $\mathfrak{J}_{i,\text{br}}(\cdot; \mu)$  is continuous, convex, and positively homogeneous. It is real analytic on the relative interior of each region defined by the two strict alternatives in [\(6.156\)](#); when both regions are nonempty, their common boundary is a ray.

*Proof.* By [proposition 6.56](#), the broken-path cost is the support function of

$$\mathbb{C}_i(\mu) = \overline{\mathbb{B}}(-\mu, M) \cap \{\theta : \theta \cdot u_i \leq J_i\}.$$

Since  $0 \leq J_i \leq M - \mu_{i,t}$ , the number  $\kappa_i = J_i + \mu_{i,t}$  belongs to  $[-M, M]$ , so  $\sigma_i$  is real.

The support point of the full disk in direction  $x \neq 0$  is  $Mx/|x| - \mu$ . If its  $u_i$ -coordinate does not exceed  $J_i$ , it remains feasible and gives the first line of [\(6.156\)](#). Otherwise the clipping constraint is active. In the orthonormal coordinates  $(u_i, n_i)$ , the active chord is  $\theta \cdot u_i = J_i$ , and its upper endpoint is precisely  $J_i u_i + \sigma_i n_i$ . Because  $x \cdot n_i = b_i(x) \geq 0$  for  $x \in S$ , this endpoint maximizes  $\theta \cdot x$  on the chord and gives the second line. The remaining assertions follow from the support function representation and from agreement of the two formulas on the switching ray.  $\square$

**Definition 6.58** (broken-path transform domain and scalar rates). Set

$$(6.158) \quad \mathcal{K}_{\text{br}}(\mu) := \bigcap_{i=1}^2 (\mathbb{C}_i(\mu) - S^\vee).$$

For  $c \in S_\circ^\vee$ , define

$$(6.159) \quad \mathfrak{J}_{c,i}^{\text{br}}(\mu) := \min_{u \in S \cap \mathbb{S}^1} \frac{\mathfrak{J}_{i,\text{br}}(u; \mu)}{c \cdot u},$$

$$(6.160) \quad \mathfrak{J}_c^{\text{br}}(\mu) := \min_{i=1,2} \mathfrak{J}_{c,i}^{\text{br}}(\mu),$$

$$(6.161) \quad \mathfrak{J}_{\text{rad}}^{\text{br}}(\mu) := \min_{i=1,2} \min_{u \in S \cap \mathbb{S}^1} \mathfrak{J}_{i,\text{br}}(u; \mu).$$

**Lemma 6.59** (conversion of localized radial lower bounds to projected lower bounds). *Let  $\pi$  be a probability measure on  $S$ , let  $c \in S_\circ^\vee$ , and let  $J : S \cap \mathbb{S}^1 \rightarrow [0, \infty)$  be continuous. Suppose that for every  $u \in S^\circ \cap \mathbb{S}^1$  and every relative open angular neighborhood  $U$  of  $u$ ,*

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in U \right\} \leq J(u).$$

Then

$$(6.162) \quad \limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi \{z : c \cdot z \geq x\} \leq \min_{u \in S \cap \mathbb{S}^1} \frac{J(u)}{c \cdot u}.$$

Also,

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \{|z| \geq r\} \leq \min_{u \in S \cap \mathbb{S}^1} J(u).$$

*Proof.* Fix an interior direction  $u$ . Since  $c \cdot u > 0$ , for every  $0 < \delta < c \cdot u$ , continuity of  $v \mapsto c \cdot v$  supplies a relative open angular neighborhood  $U_\delta$  of  $u$  such that

$$|c \cdot v - c \cdot u| < \delta, \quad v \in U_\delta.$$

Consequently

$$m_\delta := \inf_{v \in U_\delta} c \cdot v \geq c \cdot u - \delta > 0,$$

and

$$\left\{ z \neq 0 : |z| \geq x/m_\delta, \frac{z}{|z|} \in U_\delta \right\} \subset \{z : c \cdot z \geq x\}.$$

With  $R = x/m_\delta$ , the assumed sector estimate gives

$$\begin{aligned} \limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi \{c \cdot z \geq x\} &\leq \frac{1}{m_\delta} \limsup_{R \rightarrow \infty} -\frac{1}{R} \log \pi \left\{ z \neq 0 : |z| \geq R, \frac{z}{|z|} \in U_\delta \right\} \\ &\leq \frac{J(u)}{m_\delta} \leq \frac{J(u)}{c \cdot u - \delta}. \end{aligned}$$

Letting  $\delta \downarrow 0$  yields

$$\limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi \{c \cdot z \geq x\} \leq \frac{J(u)}{c \cdot u}$$

for every interior direction  $u$ . The function  $u \mapsto J(u)/(c \cdot u)$  is continuous on the compact angular interval, because  $c \cdot u$  has a strictly positive minimum there. For any boundary direction  $u_\partial$ , choose interior directions  $u_n \rightarrow u_\partial$ ; continuity gives

$$\frac{J(u_n)}{c \cdot u_n} \longrightarrow \frac{J(u_\partial)}{c \cdot u_\partial}.$$

Thus the infimum over the interior equals the minimum over the closed angular interval, proving (6.162). For the radial assertion, the sector event itself is contained in  $\{|z| \geq r\}$ , so the hypothesis gives

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \{|z| \geq r\} \leq J(u)$$

for each interior  $u$ . For a boundary direction  $u_\partial \in \{u_1, u_2\}$ , choose interior directions  $u_n \rightarrow u_\partial$ . Continuity of  $J$  gives

$$J(u_n) \longrightarrow J(u_\partial).$$

Therefore

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r\} \leq \inf_{u \in S^\circ \cap \mathbb{S}^1} J(u) = \min_{u \in S \cap \mathbb{S}^1} J(u),$$

which is the radial assertion.  $\square$

**Lemma 6.60** (ray extent of a compact convex core). *Let  $\mathbb{C} \subset \mathbb{R}^2$  be a nonempty compact convex set with  $0 \in \mathbb{C}$ , and put*

$$\mathcal{K}_{\mathbb{C}} := \mathbb{C} - S^\vee.$$

*Then  $\mathcal{K}_{\mathbb{C}}$  is closed and convex and admits the support representation*

$$(6.163) \quad \mathcal{K}_{\mathbb{C}} = \{x \in \mathbb{R}^2 : x \cdot u \leq h_{\mathbb{C}}(u) \text{ for every } u \in S\}.$$

*For every  $c \in S^\vee$ , its ray extent is*

$$(6.164) \quad \rho_{\mathbb{C}}(c) := \sup\{\lambda \geq 0 : \lambda c \in \mathcal{K}_{\mathbb{C}}\} = \min_{u \in S \cap \mathbb{S}^1} \frac{h_{\mathbb{C}}(u)}{c \cdot u}.$$

*The minimum is attained, the endpoint  $\rho_{\mathbb{C}}(c)c$  belongs to  $\mathcal{K}_{\mathbb{C}}$ , and*

$$\mathcal{K}_{\mathbb{C}} \cap \{\lambda c : \lambda \geq 0\} = \{\lambda c : 0 \leq \lambda \leq \rho_{\mathbb{C}}(c)\}.$$

*Proof.* If  $x_1 = \zeta_1 - \eta_1$  and  $x_2 = \zeta_2 - \eta_2$  belong to  $\mathcal{K}_{\mathbb{C}}$ , then for  $t \in [0, 1]$ ,

$$tx_1 + (1-t)x_2 = (t\zeta_1 + (1-t)\zeta_2) - (t\eta_1 + (1-t)\eta_2) \in \mathcal{K}_{\mathbb{C}},$$

because both  $\mathbb{C}$  and  $S^\vee$  are convex. Thus  $\mathcal{K}_{\mathbb{C}}$  is convex. To prove closedness, let  $x_n = \zeta_n - \eta_n \rightarrow x$ , with  $\zeta_n \in \mathbb{C}$  and  $\eta_n \in S^\vee$ . After passing to a subsequence, compactness gives  $\zeta_n \rightarrow \zeta \in \mathbb{C}$ , and then  $\eta_n = \zeta_n - x_n \rightarrow \zeta - x \in S^\vee$ . Thus  $x = \zeta - (\zeta - x) \in \mathcal{K}_{\mathbb{C}}$ .

Every point of  $\mathbb{C} - S^\vee$  satisfies the inequalities in (6.163). Conversely, suppose that  $x \notin \mathcal{K}_{\mathbb{C}}$ . We construct the separating vector from the nearest point. Let  $d = \text{dist}(x, \mathcal{K}_{\mathbb{C}})$ . Since  $0 \in \mathcal{K}_{\mathbb{C}}$ , one has  $d \leq |x|$ . Choose  $p_n \in \mathcal{K}_{\mathbb{C}}$  with  $|x - p_n| \leq d + n^{-1}$ . Then  $|p_n| \leq 2|x| + 1$ , so closedness of  $\mathcal{K}_{\mathbb{C}}$  and compactness of the corresponding ball give, along a subsequence, a nearest point  $p \in \mathcal{K}_{\mathbb{C}}$ . Put  $y = x - p \neq 0$ . For every  $z \in \mathcal{K}_{\mathbb{C}}$ , convexity gives  $p + t(z - p) \in \mathcal{K}_{\mathbb{C}}$  for  $0 \leq t \leq 1$ . Differentiating at zero the squared distance from  $x$  gives

$$y \cdot (z - p) \leq 0.$$

The variational inequality gives  $y \cdot z \leq y \cdot p$  for every  $z \in \mathcal{K}_{\mathbb{C}}$ , and equality is attained at  $z = p$ . Consequently

$$(6.165) \quad \sup_{z \in \mathcal{K}_{\mathbb{C}}} y \cdot z = y \cdot p, \quad y \cdot x = y \cdot p + |y|^2 > y \cdot p.$$

The support on the left is finite. Because  $-S^\vee \subset \mathcal{K}_{\mathbb{C}}$ , this forces  $y \in (S^\vee)^\vee = S$ ; otherwise some  $\eta \in S^\vee$  would satisfy  $y \cdot \eta < 0$ , and the points  $-r\eta \in \mathcal{K}_{\mathbb{C}}$  would make the support infinite. For  $y \in S$ ,

$$\sup_{z \in \mathcal{K}_{\mathbb{C}}} y \cdot z = \sup_{\zeta \in \mathbb{C}, \eta \in S^\vee} y \cdot (\zeta - \eta) = h_{\mathbb{C}}(y),$$

because  $y \cdot \eta \geq 0$  and  $\eta = 0$  is allowed. Thus (6.165) says  $x \cdot y > h_{\mathbb{C}}(y)$ , so  $x$  violates one of the inequalities in (6.163). This proves the support representation.

Because  $c \in S^\vee$ , one has  $c \cdot u > 0$  on the compact set  $S \cap \mathbb{S}^1$ . Applying (6.163) to  $\lambda c$  therefore gives

$$\lambda c \in \mathcal{K}_{\mathbb{C}} \iff \lambda \leq \frac{h_{\mathbb{C}}(u)}{c \cdot u} \text{ for every } u \in S \cap \mathbb{S}^1.$$

This proves (6.164). Compactness of the angular set gives attainment, and closedness gives inclusion of the endpoint. Since  $0 \in \mathbb{C}$ , all ratios are nonnegative, and the final ray identity follows.  $\square$

**Lemma 6.61** (localized sector lower bounds force transform half-spaces). *Let  $\pi$  be a probability measure on  $S$ , and let  $J : S \cap \mathbb{S}^1 \rightarrow [0, \infty)$  be continuous. Suppose that, for every  $u \in S^\circ \cap \mathbb{S}^1$*

and every relative open angular neighborhood  $U$  of  $u$ ,

$$(6.166) \quad \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in U \right\} \leq J(u).$$

Then every  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$  satisfies

$$(6.167) \quad \theta \cdot u \leq J(u), \quad u \in S \cap \mathbb{S}^1.$$

The same inequalities hold for every  $\theta \in \overline{\mathcal{D}_{\text{exp}}(\pi)}$ .

In particular, if  $\mathbb{C} \subset \mathbb{R}^2$  is compact and convex,  $0 \in \mathbb{C}$ , and  $J(u) = h_{\mathbb{C}}(u)$  on  $S \cap \mathbb{S}^1$ , then

$$(6.168) \quad \overline{\mathcal{D}_{\text{exp}}(\pi)} \subset \mathbb{C} - S^\vee.$$

*Proof.* Fix  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$  and an interior direction  $u \in S^\circ \cap \mathbb{S}^1$ . Suppose, for contradiction, that  $g := \theta \cdot u - J(u) > 0$ , and set  $\varepsilon = g/5$ . Then

$$\theta \cdot u = J(u) + 5\varepsilon.$$

By continuity of  $v \mapsto \theta \cdot v$ , there is a relative open angular neighborhood  $U$  of  $u$  such that

$$|\theta \cdot v - \theta \cdot u| < \varepsilon, \quad v \in U.$$

Therefore

$$(6.169) \quad \inf_{v \in U} \theta \cdot v \geq \theta \cdot u - \varepsilon = J(u) + 4\varepsilon.$$

Let

$$A_r(U) := \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in U \right\}.$$

The limsup assumption (6.166) implies that, for all sufficiently large  $r$ ,

$$(6.170) \quad \pi(A_r(U)) \geq \exp\{-(J(u) + \varepsilon)r\}.$$

Indeed, an infinite sequence of radii at which the reverse strict inequality holds would force the limsup in (6.166) to be at least  $J(u) + \varepsilon$ ; in particular, the events cannot vanish at large radii.

On  $A_r(U)$ , (6.169) gives  $\theta \cdot z \geq (J(u) + 4\varepsilon)r$ . Consequently,

$$\int_S e^{\theta \cdot z} \pi(dz) \geq e^{(J(u) + 4\varepsilon)r} \pi(A_r(U)) \geq e^{3\varepsilon r}$$

for every sufficiently large  $r$ . This contradicts  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$ . Hence  $\theta \cdot u \leq J(u)$  for every interior direction. Density of the interior angular directions and continuity of both sides extend the inequality to the two boundary directions. Since each constraint in (6.167) is closed in  $\theta$ , it also holds on the closure of the moment domain.

For the final assertion, positive homogeneity extends (6.167) to

$$\theta \cdot z \leq h_{\mathbb{C}}(z), \quad z \in S.$$

The support representation (6.163), proved in lemma 6.60 by a closedness argument and an explicit metric-projection separator, identifies the set of all such  $\theta$  with  $\mathbb{C} - S^\vee$ . This proves (6.168).  $\square$

**Theorem 6.62** (broken-path outer frontier and exact logarithmic tails). *Assume  $0 < \xi < \pi$ ,  $\alpha < 2$ , and let  $\pi$  be stationary for the drift  $\mu$ . Then*

$$(6.171) \quad D_\pi \subset \mathcal{K}_{\text{br}}(\mu) \subset \mathcal{K}_{\text{kin}}^{\text{lf}}(\mu).$$

For every  $c \in S_\circ^\vee$ ,

$$(6.172) \quad \alpha_c^{\text{in}}(\mu) \leq \alpha_c^{2\text{exp}}(\mu) \leq \alpha_c^{\text{fn}}(\mu) \leq \alpha_c^\pi \leq \mathfrak{J}_c^{\text{br}}(\mu) \leq \mathfrak{J}_c^{\text{lf}}(\mu),$$

and

$$(6.173) \quad \limsup_{x \rightarrow \infty} -\frac{1}{x} \log \pi \{z : c \cdot z \geq x\} \leq \mathfrak{J}_c^{\text{br}}(\mu).$$

The radial bounds are

$$(6.174) \quad \alpha_{\text{rad}}^{\text{in}}(\mu) \leq \alpha_{\text{rad}}^{2\text{exp}}(\mu) \leq \alpha_{\text{rad}}^{\text{fin}}(\mu) \leq \alpha_{\text{rad}}^{\pi} \leq \mathfrak{J}_{\text{rad}}^{\text{br}}(\mu),$$

and

$$(6.175) \quad \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \pi\{|z| \geq r\} \leq \mathfrak{J}_{\text{rad}}^{\text{br}}(\mu).$$

If  $\mathcal{E}(\mu) \neq \emptyset$ , then

$$(6.176) \quad \mathcal{K}_{\text{in}}(\mu) \subset \mathcal{K}_{\text{br}}(\mu).$$

Independently of this last inclusion, if

$$(6.177) \quad \alpha_c^{\text{fin}}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu),$$

then, for every stationary law at the prescribed drift,

$$(6.178) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z : c \cdot z \geq x\} = -\mathfrak{J}_c^{\text{br}}(\mu).$$

If

$$(6.179) \quad \alpha_{\text{rad}}^{\text{fin}}(\mu) = \mathfrak{J}_{\text{rad}}^{\text{br}}(\mu),$$

then the radial logarithmic tail is exact with this common rate.

*Proof.* For each fixed face  $i$ , combine [propositions 6.55](#) and [6.56](#): for every interior direction and every relative angular neighborhood, the localized radial lower bound has continuous rate  $J_i(u) = h_{\mathbb{C}_i(\mu)}(u)$ . Applying [lemma 6.61](#) gives

$$D_\pi \subset \mathbb{C}_i(\mu) - S^\vee, \quad i = 1, 2.$$

Intersecting the two inclusions proves the first inclusion in [\(6.171\)](#). Since  $\mathbb{C}_i(\mu) \subset \mathbb{B}_\mu$  and every point of  $\mathbb{C}_i(\mu) - S^\vee$  satisfies  $\theta \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu)$ , the second inclusion follows.

By [lemma 6.60](#), for each face  $i$ ,

$$\sup\{\lambda \geq 0 : \lambda c \in \mathbb{C}_i(\mu) - S^\vee\} = \min_{u \in S \cap \mathbb{S}^1} \frac{h_{\mathbb{C}_i(\mu)}(u)}{c \cdot u} = \mathfrak{J}_{c,i}^{\text{br}}(\mu).$$

Moreover,

$$\lambda c \in \bigcap_{i=1}^2 (\mathbb{C}_i(\mu) - S^\vee) \iff 0 \leq \lambda \leq \min_{i=1,2} \mathfrak{J}_{c,i}^{\text{br}}(\mu).$$

Hence the ray extent of the intersection in [\(6.158\)](#) is  $\mathfrak{J}_c^{\text{br}}(\mu)$ , which proves the transform-domain part of [\(6.172\)](#). For each  $i$ , the function  $u \mapsto h_{\mathbb{C}_i(\mu)}(u)$  is continuous. Applying [lemma 6.59](#) to [\(6.126\)](#), and then minimizing over  $i$ , gives [\(6.173\)](#), including the case of a boundary minimizing direction. The radial assertion follows from the radial part of the same lemma.

If  $\vartheta \in \overline{\mathcal{E}(\mu)}$ , then  $\vartheta \in \mathbb{B}_\mu$  and, by [proposition 6.45](#),  $\vartheta \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu)$ . Hence  $\overline{\mathcal{E}(\mu)} \subset \mathbb{C}_i(\mu)$  for both faces. Subtracting  $S^\vee$  proves [\(6.176\)](#). Assume [\(6.177\)](#) and write  $J = \mathfrak{J}_c^{\text{br}}(\mu)$ . Because  $\mathcal{M}_{\text{fin}}(\mu)$  is convex, contains the origin, and is stable under subtraction of  $S^\vee$ , every strict subcritical point  $\lambda c$ ,  $0 \leq \lambda < J$ , belongs to  $\mathcal{M}_{\text{fin}}(\mu)$ . By [proposition 5.12](#), its exponential moment is finite. Markov's inequality therefore gives

$$\liminf_{x \rightarrow \infty} -\frac{1}{x} \log \pi\{z : c \cdot z \geq x\} \geq \lambda.$$

Letting  $\lambda \uparrow J$  and combining with [\(6.173\)](#) proves [\(6.178\)](#). In particular, either of the stronger sufficient conditions

$$\alpha_c^{2\text{exp}}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu) \quad \text{or} \quad \alpha_c^{\text{in}}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu)$$

remains valid. Under [\(6.179\)](#), fix  $q < \mathfrak{J}_{\text{rad}}^{\text{br}}(\mu) = \alpha_{\text{rad}}^{\text{fin}}(\mu)$ . By the definition of the last supremum, there exists  $\theta \in \mathcal{M}_{\text{fin}}(\mu)$  such that

$$\min_{u \in S \cap \mathbb{S}^1} \theta \cdot u > q.$$

The moment  $\int e^{\theta \cdot z} d\pi$  is finite by [proposition 5.12](#), and  $\theta \cdot z \geq q|z|$  on  $S$ . Hence the radial  $q$ -moment is finite. Markov's inequality and then  $q \uparrow \mathfrak{J}_{\text{rad}}^{\text{br}}(\mu)$  give the matching logarithmic upper-tail bound. Combining it with [\(6.175\)](#) proves the exact radial logarithmic tail.  $\square$

**Corollary 6.63** (homogeneity of the finite-exponential and broken-path constructions). *For every  $\rho > 0$ ,*

$$\begin{aligned} \mathfrak{J}_i^{\text{f}}(\rho\mu) &= \rho\mathfrak{J}_i^{\text{f}}(\mu), & \mathbb{C}_i(\rho\mu) &= \rho\mathbb{C}_i(\mu), \\ \mathfrak{J}_{i,\text{br}}(x; \rho\mu) &= \rho\mathfrak{J}_{i,\text{br}}(x; \mu), & \mathfrak{J}_c^{\text{br}}(\rho\mu) &= \rho\mathfrak{J}_c^{\text{br}}(\mu), \end{aligned}$$

and the same homogeneity holds for the radial broken-path rate. Hence the endpoint condition

$$\alpha_c^{\text{fin}}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu)$$

is invariant under positive rescaling of the drift and may be checked on one normalized representative of each nonzero drift ray.

*Proof.* The first identity follows from [\(6.82\)](#) after the change of variables  $a = \rho a'$ ,  $\ell = \rho \ell'$ . Since  $\mathbb{B}_{\rho\mu} = \rho\mathbb{B}_\mu$ , the clipped-kernel definition then gives  $\mathbb{C}_i(\rho\mu) = \rho\mathbb{C}_i(\mu)$ . For every  $x \in \mathbb{R}^2$ ,

$$h_{\mathbb{C}_i(\rho\mu)}(x) = h_{\rho\mathbb{C}_i(\mu)}(x) = \sup_{\theta \in \mathbb{C}_i(\mu)} (\rho\theta) \cdot x = \rho h_{\mathbb{C}_i(\mu)}(x).$$

Thus

$$\mathfrak{J}_{i,\text{br}}(x; \rho\mu) = \rho\mathfrak{J}_{i,\text{br}}(x; \mu),$$

and taking the minima in the definitions of the projected and radial broken-path rates gives the remaining broken-path identities. The finite-exponential rates scale by [proposition 5.15](#); the final assertion follows.  $\square$

**Proposition 6.64** (simultaneous-clipping bound for the finite-exponential hull). *Define*

$$\mathbb{C}_{12}(\mu) := \mathbb{C}_1(\mu) \cap \mathbb{C}_2(\mu), \quad \mathcal{K}_{12}(\mu) := \mathbb{C}_{12}(\mu) - S^\vee.$$

Then

$$(6.180) \quad \mathcal{M}_{\text{fin}}(\mu) \subset \mathcal{K}_{12}(\mu) \subset \mathcal{K}_{\text{br}}(\mu).$$

For  $c \in S_\circ^\vee$ , let

$$(6.181) \quad \mathfrak{J}_c^{12}(\mu) := \min_{u \in S \cap \mathbb{S}^1} \frac{h_{\mathbb{C}_{12}(\mu)}(u)}{c \cdot u},$$

which is the ray extent of  $\mathcal{K}_{12}(\mu)$  in direction  $c$ . Then

$$(6.182) \quad \alpha_c^{\text{fin}}(\mu) \leq \mathfrak{J}_c^{12}(\mu) \leq \mathfrak{J}_c^{\text{br}}(\mu).$$

The two geometric ray extents in the last display are proved equal for all positive dual rays in [proposition 6.67](#).

*Proof.* Consider first  $\theta \in \mathcal{E}_{\text{fin}}(\mu)$ , and choose a witnessing family. Put

$$\bar{\zeta} := \theta + \eta = \sum_{k=1}^N t_k \zeta_k.$$

Because  $Q_\mu(\zeta_k) < 0$  for every  $k$ , convexity of the open kernel disk gives  $Q_\mu(\bar{\zeta}) < 0$ . On face  $i$ , let  $m_i$  be the largest tangential exponent and  $I_i$  the corresponding index set. The strict inequality

$$\sum_{k \in I_i} a_k \zeta_k \cdot v_i < 0$$

implies that some  $k_i \in I_i$  satisfies  $\zeta_{k_i} \cdot v_i < 0$ . The dual formula [\(6.103\)](#) therefore gives

$$m_i = \zeta_{k_i} \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu).$$

Every exponent in the family has  $u_i$ -coordinate at most  $m_i$ , so

$$\bar{\zeta} \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu), \quad i = 1, 2.$$

Thus  $\bar{\zeta} \in \mathbb{C}_{12}(\mu)$ , and  $\theta = \bar{\zeta} - \eta \in \mathcal{K}_{12}(\mu)$ .

The same conclusion holds for every  $\theta \in \mathcal{E}(\mu)$ , because such a point satisfies  $Q_\mu(\theta) < 0$  and is feasible in both one-face dual problems. The origin belongs to  $\mathbb{C}_{12}(\mu)$  as well. Hence every generator in the convex hull defining  $\mathcal{M}_{\text{fin}}(\mu)$  belongs to  $\mathcal{K}_{12}(\mu) = \mathbb{C}_{12}(\mu) - S^\vee$ . This latter set is convex and is stable under subtraction of  $S^\vee$ ; the definition of  $\mathcal{M}_{\text{fin}}(\mu)$  therefore proves the first inclusion in (6.180). The second follows from  $\mathbb{C}_{12}(\mu) \subset \mathbb{C}_i(\mu)$ ,  $i = 1, 2$ .

Since  $0 \in \mathbb{C}_{12}(\mu)$ , lemma 6.60 applied to  $\mathbb{C} = \mathbb{C}_{12}(\mu)$  gives (6.181). Taking ray extents in (6.180) proves (6.182).  $\square$

**Lemma 6.65** (strict-feasibility criterion for the clipped-kernel interior). *Let*

$$\mathbb{C}_{12}(\mu) = \mathbb{B}_\mu \cap \{\theta : \theta \cdot u_1 \leq \mathfrak{J}_1^f(\mu)\} \cap \{\theta : \theta \cdot u_2 \leq \mathfrak{J}_2^f(\mu)\}.$$

*If a point  $\theta_*$  satisfies*

$$Q_\mu(\theta_*) < 0, \quad \theta_* \cdot u_i < \mathfrak{J}_i^f(\mu), \quad i = 1, 2,$$

*then  $\theta_* \in \text{int } \mathbb{C}_{12}(\mu)$ . More precisely, if*

$$d_0 := \text{dist}(\theta_*, \partial \mathbb{B}_\mu) > 0, \quad \sigma_i := \mathfrak{J}_i^f(\mu) - \theta_* \cdot u_i > 0,$$

*then every*

$$0 < r_* < \min\{d_0, \sigma_1, \sigma_2\}$$

*satisfies  $B_{r_*}(\theta_*) \subset \mathbb{C}_{12}(\mu)$ . In particular, with  $m_i^{12} := h_{\mathbb{C}_{12}(\mu)}(u_i)$ ,*

$$(6.183) \quad m_i^{12} \geq \theta_* \cdot u_i + \frac{r_*}{2} > \theta_* \cdot u_i, \quad i = 1, 2.$$

*Proof.* The strict kernel inequality says exactly that  $\theta_*$  lies in the open disk  $\text{int } \mathbb{B}_\mu$ , so  $d_0 > 0$ . If  $|h| < r_*$ , then  $\theta_* + h \in \mathbb{B}_\mu$ , and, because  $|u_i| = 1$ ,

$$(\theta_* + h) \cdot u_i \leq \theta_* \cdot u_i + |h| < \theta_* \cdot u_i + \sigma_i = \mathfrak{J}_i^f(\mu).$$

Thus the whole ball lies in all three defining constraints. Finally,  $\theta_* + (r_*/2)u_i \in \mathbb{C}_{12}(\mu)$ , and evaluating the support function at this point proves (6.183).  $\square$

**Lemma 6.66** (two-coordinate clipping lemma). *Let  $C \subset \mathbb{R}^2$  be convex and contain the origin. Let*

$$\ell = (\ell_1, \ell_2) \in (0, \infty)^2, \quad J = (J_1, J_2) \in [\ell_1, \infty) \times [\ell_2, \infty).$$

*Suppose that there are points  $x^{(1)}, x^{(2)} \in C$  such that*

$$x^{(1)} \geq \ell, \quad x_1^{(1)} \leq J_1,$$

$$x^{(2)} \geq \ell, \quad x_2^{(2)} \leq J_2,$$

*where vector inequalities are coordinatewise. Then*

$$C \cap [\ell_1, J_1] \times [\ell_2, J_2] \neq \emptyset.$$

*Proof.* If  $x_2^{(1)} \leq J_2$  or  $x_1^{(2)} \leq J_1$ , one of the two points already belongs to the required rectangle. Assume therefore that

$$x_2^{(1)} > J_2, \quad x_1^{(2)} > J_1.$$

Put  $r = \ell_2/\ell_1$ . If  $x_2^{(1)}/x_1^{(1)} \leq r$ , then the point

$$y = \frac{\ell_2}{x_2^{(1)}} x^{(1)}$$

belongs to  $C$  by convexity and  $0 \in C$ , and it satisfies  $y_2 = \ell_2$ ,  $\ell_1 \leq y_1 \leq J_1$ . Thus  $y$  lies in the rectangle. In the second ratio case  $x_2^{(2)}/x_1^{(2)} \geq r$ , set

$$y = \frac{\ell_1}{x_1^{(2)}} x^{(2)}.$$

Because  $C$  is convex and contains the origin, the multiplier  $\ell_1/x_1^{(2)} \in (0, 1]$  gives  $y \in C$ . Moreover,

$$y_1 = \ell_1, \quad y_2 = \ell_1 \frac{x_2^{(2)}}{x_1^{(2)}} \geq \ell_1 r = \ell_2, \quad y_2 \leq x_2^{(2)} \leq J_2.$$

Hence  $y$  lies in the required rectangle in this case as well.

In the remaining case,

$$\frac{x_2^{(1)}}{x_1^{(1)}} > \frac{\ell_2}{\ell_1} > \frac{x_2^{(2)}}{x_1^{(2)}}.$$

Thus

$$\det(x^{(1)}, x^{(2)}) < 0, \quad \det(\ell, x^{(2)}) < 0, \quad \det(x^{(1)}, \ell) < 0.$$

Cramer's rule gives the unique coefficients in  $ax^{(1)} + bx^{(2)} = \ell$ :

$$a = \frac{\det(\ell, x^{(2)})}{\det(x^{(1)}, x^{(2)})} > 0, \quad b = \frac{\det(x^{(1)}, \ell)}{\det(x^{(1)}, x^{(2)})} > 0.$$

Because both points dominate  $\ell$  coordinatewise, either coordinate of this identity gives  $a + b \leq 1$ . Therefore

$$\ell = ax^{(1)} + bx^{(2)} + (1 - a - b)0 \in C,$$

and  $\ell$  itself belongs to the required rectangle.  $\square$

**Proposition 6.67** (raywise equality of simultaneous and broken clipping). *For every drift  $\mu$  and every  $c \in S_\circ^\vee$ ,*

$$(6.184) \quad \mathfrak{J}_c^{12}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu).$$

More precisely, for every  $\lambda > 0$ ,

$$(6.185) \quad \lambda c \in \mathcal{K}_{\text{br}}(\mu) \iff \lambda c \in \mathcal{K}_{12}(\mu).$$

Thus the two sets may differ away from the positive dual cone, but they have identical intersections with every ray generated by  $S_\circ^\vee$ .

*Proof.* Only the reverse implication in (6.185) requires proof. Let

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T\vartheta = (\vartheta \cdot u_1, \vartheta \cdot u_2).$$

Since  $0 < \xi < \pi$ , the map  $T$  is invertible, and

$$T(S^\vee) = [0, \infty)^2.$$

Put  $\ell = T(\lambda c) \in (0, \infty)^2$  and  $J_i = \mathfrak{J}_i^{\text{f}}(\mu)$ . Membership of  $\lambda c$  in  $\mathbb{C}_i(\mu) - S^\vee$  gives a point  $\vartheta^{(i)} \in \mathbb{C}_i(\mu)$  such that

$$T\vartheta^{(i)} \geq \ell.$$

Moreover,

$$(T\vartheta^{(1)})_1 \leq J_1, \quad (T\vartheta^{(2)})_2 \leq J_2.$$

The set  $T\mathbb{B}_\mu$  is convex and contains the origin. The hypotheses of lemma 6.66 therefore hold with  $C = T\mathbb{B}_\mu$ . We obtain a point  $x \in T\mathbb{B}_\mu$  satisfying

$$\ell_i \leq x_i \leq J_i, \quad i = 1, 2.$$

Let  $\vartheta = T^{-1}x$ . Then  $\vartheta \in \mathbb{C}_{12}(\mu)$  and  $\vartheta - \lambda c \in S^\vee$ , which proves  $\lambda c \in \mathcal{K}_{12}(\mu)$ . Equality of the ray extents follows.  $\square$

**Corollary 6.68** (global equality of the simultaneous and broken domains). *Assume*

$$0 < \xi < \pi, \quad 1 \leq \alpha < 2, \quad \mu \neq 0.$$

Then

$$(6.186) \quad \mathcal{K}_{12}(\mu) = \mathcal{K}_{\text{br}}(\mu).$$

More precisely, either both one-face clips are inactive and  $\mathbb{C}_{12}(\mu) = \mathbb{B}_\mu$ , or exactly one clip is active and  $\mathbb{C}_{12}(\mu)$  equals the corresponding singly clipped disk.

*Proof.* By lemma 6.52, at least one of the sets  $\mathbb{C}_i(\mu)$  equals  $\mathbb{B}_\mu$ . If, for example,  $\mathbb{C}_2(\mu) = \mathbb{B}_\mu$ , then

$$\mathbb{C}_{12}(\mu) = \mathbb{C}_1(\mu)$$

and, because  $\mathbb{C}_1(\mu) \subset \mathbb{B}_\mu$ ,

$$(\mathbb{C}_1(\mu) - S^\vee) \cap (\mathbb{B}_\mu - S^\vee) = \mathbb{C}_1(\mu) - S^\vee.$$

This is exactly (6.186). If instead  $\mathbb{C}_1(\mu) = \mathbb{B}_\mu$ , then

$$\mathbb{C}_{12}(\mu) = \mathbb{C}_2(\mu), \quad (\mathbb{B}_\mu - S^\vee) \cap (\mathbb{C}_2(\mu) - S^\vee) = \mathbb{C}_2(\mu) - S^\vee.$$

If both clips are inactive, then  $\mathbb{C}_1(\mu) = \mathbb{C}_2(\mu) = \mathbb{B}_\mu$ , so both sides of (6.186) equal  $\mathbb{B}_\mu - S^\vee$ . These alternatives exhaust the conclusion of lemma 6.52.  $\square$

**Definition 6.69** (closed finite-exponential endpoint certificate). Fix  $c \in S^\vee$  and  $\lambda > 0$ . A *closed finite-exponential endpoint certificate* for  $(c, \lambda)$  consists of an integer  $N \geq 1$ , positive coefficients  $a_k$ , exponent vectors  $\zeta_k \in \mathbb{R}^2$ , target weights  $t_k \geq 0$  with  $\sum_{k=1}^N t_k = 1$ , and a cone shift  $\eta \in S^\vee$  such that

$$(6.187) \quad \lambda c + \eta = \sum_{k=1}^N t_k \zeta_k, \quad Q_\mu(\zeta_k) \leq 0, \quad k = 1, \dots, N.$$

For  $i = 1, 2$ , write

$$m_i := \max_{1 \leq k \leq N} \zeta_k \cdot u_i, \quad I_i := \{k : \zeta_k \cdot u_i = m_i\}.$$

The endpoint family is required to satisfy the strict leading boundary conditions

$$(6.188) \quad \sum_{k \in I_i} a_k \zeta_k \cdot v_i < 0, \quad i = 1, 2.$$

The adjective *closed* refers only to the nonstrict kernel inequalities in (6.187); the boundary sums remain strict.

**Proposition 6.70** (five-exponential endpoint reduction and exact logarithmic tails). *Assume*

$$0 < \xi < \pi, \quad \alpha < 2, \quad \mathcal{E}(\mu) \neq \emptyset.$$

Fix  $c \in S^\vee$ , put  $\lambda = \mathfrak{J}_c^{\text{br}}(\mu)$ , and suppose that  $(c, \lambda)$  has a closed finite-exponential endpoint certificate. Then the certificate may be replaced by one using at most five exponent vectors. Moreover,

$$(6.189) \quad \alpha_c^{\text{fn}}(\mu) = \alpha_c^\pi = \mathfrak{J}_c^{\text{br}}(\mu),$$

and every stationary law at drift  $\mu$  satisfies

$$(6.190) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z \in S : c \cdot z \geq x\} = -\mathfrak{J}_c^{\text{br}}(\mu).$$

Thus ties among the leading exponents are allowed, provided their weighted leading oblique sums are strictly negative.

*Proof.* We first reduce the size of the endpoint family. Put

$$\bar{\zeta} := \lambda c + \eta = \sum_{k=1}^N t_k \zeta_k.$$

Carathéodory's theorem in  $\mathbb{R}^2$  gives a set  $J_t$  of at most three indices such that  $\bar{\zeta}$  is a convex combination of the corresponding exponent vectors. For the two faces define

$$q_k := (\mathbf{1}_{\{k \in I_1\}} \zeta_k \cdot v_1, \mathbf{1}_{\{k \in I_2\}} \zeta_k \cdot v_2) \in \mathbb{R}^2.$$

By (6.188),

$$q := \sum_{k=1}^N a_k q_k \in (-\infty, 0)^2.$$

Conic Carathéodory in dimension two gives a set  $J_b$  of at most two indices and nonnegative numbers  $b_k$  such that

$$q = \sum_{k \in J_b} b_k q_k.$$

Delete indices with zero coefficient and put  $J = J_t \cup J_b$ , so  $|J| \leq 5$ . Since both coordinates of  $q$  are negative,  $J_b \cap I_i \neq \emptyset$  for  $i = 1, 2$ . Hence the maximal tangential rates of the restricted family are still  $m_1, m_2$ . Give the retained exponents coefficients

$$\widehat{a}_k = \begin{cases} b_k + \varepsilon, & k \in J_b, \\ \varepsilon, & k \in J \setminus J_b, \end{cases}$$

where, with  $w := \sum_{k \in J} q_k$ , we choose

$$0 < \varepsilon < \min_{i=1,2} \frac{-q_i}{2(1 + |w_i|)}.$$

The two leading boundary sums are the coordinates of

$$q + \varepsilon w.$$

For each  $i$ ,

$$q_i + \varepsilon w_i \leq q_i + \varepsilon |w_i| < \frac{q_i}{2} < 0,$$

so both remain strictly negative. The target weights supplied by Carathéodory on  $J_t$ , extended by zero on the other retained indices, preserve (6.187). This proves the five-exponential reduction. Relabel the retained family as  $k = 1, \dots, N$ , where  $N \leq 5$ , and let  $I_i$  denote its maximizing index sets.

Choose  $\vartheta_* \in \mathcal{E}(\mu)$ . For  $0 < s < 1$ , replace every retained exponent by the common interpolation

$$\zeta_{k,s} := s\zeta_k + (1-s)\vartheta_*.$$

Because  $\vartheta_*$  lies in the open kernel disk and each  $\zeta_k$  lies in the closed kernel disk,

$$Q_\mu(\zeta_{k,s}) < 0.$$

The common affine interpolation preserves every tangential ordering and every tie: the maximizing index sets on the two faces are still  $I_1, I_2$ . Their leading sums are

$$s \sum_{k \in I_i} \widehat{a}_k \zeta_k \cdot v_i + (1-s)(\vartheta_* \cdot v_i) \sum_{k \in I_i} \widehat{a}_k, \quad i = 1, 2.$$

The first term is strictly negative whenever  $s > 0$ , by the closed endpoint certificate. The second term is nonpositive because  $\vartheta_* \in \mathcal{E}(\mu)$  implies  $\vartheta_* \cdot v_i \leq 0$ . Hence the displayed leading sum is strictly negative for every  $0 < s < 1$ ; no strict oblique inequality is required of the interpolating point. Finally,

$$\sum_k t_k \zeta_{k,s} = s\lambda c + \{s\eta + (1-s)\vartheta_*\},$$

where the term in braces belongs to  $S^\vee$ . Thus  $s\lambda c \in \mathcal{E}_{\text{fin}}(\mu)$  for every  $0 < s < 1$ , and therefore

$$\alpha_c^{\text{fin}}(\mu) \geq \lambda.$$

The reverse chain

$$\alpha_c^{\text{fin}}(\mu) \leq \alpha_c^\pi \leq \mathfrak{J}_c^{\text{br}}(\mu) = \lambda$$

is [proposition 5.12](#) and [theorem 6.62](#). This proves (6.189); the matching moment upper bound and broken-path lower bound yield (6.190).  $\square$

**Proposition 6.71** (finite-dimensional margin formulation of endpoint certificates). *Fix  $c \in S^\vee$  and  $\lambda > 0$ . A closed finite-exponential endpoint certificate for  $(c, \lambda)$  exists if and only if there are an integer  $1 \leq N \leq 5$ , nonempty index sets  $I_1, I_2 \subset \{1, \dots, N\}$ , a number  $\delta > 0$ , exponent vectors  $\zeta_k \in \mathbb{R}^2$ , coefficient vectors  $a = (a_1, \dots, a_N)$  and  $t = (t_1, \dots, t_N)$ , and numbers  $m_1, m_2$  such that*

$$(6.191) \quad Q_\mu(\zeta_k) \leq 0, \quad k = 1, \dots, N,$$

$$(6.192) \quad a_k \geq \delta, \quad \sum_{k=1}^N a_k = 1, \quad t_k \geq 0, \quad \sum_{k=1}^N t_k = 1,$$

$$(6.193) \quad \sum_{k=1}^N t_k \zeta_k - \lambda c \in S^\vee,$$

$$(6.194) \quad \zeta_k \cdot u_i = m_i, \quad k \in I_i, \quad i = 1, 2,$$

$$(6.195) \quad \zeta_k \cdot u_i \leq m_i - \delta, \quad k \notin I_i, \quad i = 1, 2,$$

$$(6.196) \quad \sum_{k \in I_i} a_k \zeta_k \cdot v_i \leq -\delta, \quad i = 1, 2.$$

For fixed  $(\mu, c, \lambda)$ , positivity of the largest feasible margin  $\delta$ , maximized over the finitely many choices of  $N, I_1, I_2$ , is therefore equivalent to existence of an endpoint certificate. For each fixed index pattern, the feasible set obtained by allowing  $\delta \geq 0$  is compact, and the largest feasible margin is attained. In particular, the endpoint question is the positivity test for the optimum of one of finitely many compact degree-two semialgebraic optimization problems.

Under the hypotheses of [proposition 6.70](#), with  $\lambda = \mathfrak{J}_c^{\text{br}}(\mu)$ , feasibility of (6.191)–(6.196) for some positive  $\delta$  implies

$$\alpha_c^{\text{fn}}(\mu) = \alpha_c^\pi = \mathfrak{J}_c^{\text{br}}(\mu)$$

and the exact projected logarithmic tail (6.190) for every stationary law at drift  $\mu$ .

*Proof.* Suppose first that a closed endpoint certificate exists. By the five-exponential reduction in [proposition 6.70](#), it may be chosen with  $N \leq 5$ . Normalize its positive exponential coefficients so that  $\sum_k a_k = 1$ , and let  $I_i$  be its actual maximizing set on face  $i$ . Every coefficient is then strictly positive, every nonleader has a strictly smaller tangential exponent, and both leading oblique sums are strictly negative. Define the finite set of positive numbers

$$\mathcal{R} := \{a_k : 1 \leq k \leq N\} \cup \left\{ -\sum_{k \in I_i} a_k \zeta_k \cdot v_i : i = 1, 2 \right\} \cup \{m_i - \zeta_k \cdot u_i : i = 1, 2, k \notin I_i\}.$$

The first two displayed subcollections are nonempty, and every member of  $\mathcal{R}$  is strictly positive. Therefore

$$\delta := \frac{1}{2} \min \mathcal{R} > 0.$$

For every  $k$ , this choice gives  $a_k \geq 2\delta \geq \delta$ ; for every nonleader it gives  $\zeta_k \cdot u_i \leq m_i - 2\delta \leq m_i - \delta$ ; and for each face it gives  $\sum_{k \in I_i} a_k \zeta_k \cdot v_i \leq -2\delta \leq -\delta$ . Thus (6.192), (6.195), and (6.196) hold. The kernel and target conditions are unchanged from the certificate.

Conversely, assume that the displayed system is feasible with  $\delta > 0$ . The gap conditions show that  $I_i$  is exactly the set of maximizers of  $\zeta_k \cdot u_i$ . The coefficient conditions make all  $a_k$  positive, and (6.196) gives the two strict leading oblique sums. Setting

$$\eta := \sum_{k=1}^N t_k \zeta_k - \lambda c$$

turns (6.193) into  $\eta \in S^\vee$ , so the data form a closed endpoint certificate.

For compactness, fix  $N, I_1, I_2$ . The kernel constraint is

$$\mathbb{B}_\mu = \{\zeta : Q_\mu(\zeta) \leq 0\} = \overline{B}(-\mu, |\mu|),$$

so every exponent satisfies  $|\zeta_k| \leq 2|\mu|$ , and the exponent tuple lies in the compact product  $\mathbb{B}_\mu^N$ . The vectors  $a$  and  $t$  lie in the two closed probability simplexes. Because  $a_k \geq \delta$  and  $\sum_k a_k = 1$ , summing over  $k$  gives

$$0 \leq \delta \leq \frac{1}{N}.$$

The sets  $I_i$  are nonempty. Fix  $k_i \in I_i$ . Then (6.194) gives

$$m_i = \zeta_{k_i} \cdot u_i, \quad |m_i| \leq |\zeta_{k_i}| \leq 2|\mu|.$$

Thus every variable belongs to the compact ambient product

$$\mathbb{B}_\mu^N \times \Delta_{N-1} \times \Delta_{N-1} \times [0, 1/N] \times [-2|\mu|, 2|\mu|]^2.$$

The kernel inequalities, simplex equations and inequalities, target membership in the closed cone  $S^\vee$ , leader equalities, gap inequalities, and boundary inequalities are continuous closed constraints when  $\delta \geq 0$ . The feasible set is therefore a closed subset of this compact product and is compact. The continuous objective  $\delta$  attains its maximum. There are only finitely many choices of  $N, I_1, I_2$ , and the final assertion follows from [proposition 6.70](#).  $\square$

**Definition 6.72** (raywise three-leader rate). Fix  $c \in S_\circ^\vee$ . Let  $\mathfrak{J}_c^{3L}(\mu)$  be the supremum of  $\{0\}$  together with all  $\lambda > 0$  for which there exist

$$\eta \in S^\vee, \quad \zeta_0, \zeta_1, \zeta_2 \in \mathbb{R}^2$$

satisfying

$$(6.197) \quad \zeta_0 = \lambda c + \eta, \quad Q_\mu(\zeta_j) < 0, \quad j = 0, 1, 2,$$

$$(6.198) \quad \zeta_1 \cdot v_1 < 0, \quad \zeta_2 \cdot v_2 < 0,$$

and the strict leading-order inequalities

$$(6.199) \quad \zeta_1 \cdot u_1 > \max\{\zeta_0 \cdot u_1, \zeta_2 \cdot u_1\}, \quad \zeta_2 \cdot u_2 > \max\{\zeta_0 \cdot u_2, \zeta_1 \cdot u_2\}.$$

The superscript 3L refers to the target exponent together with two face-leading exponents.

**Proposition 6.73** (raywise three-leader bound). For every  $c \in S_\circ^\vee$ ,

$$(6.200) \quad \mathfrak{J}_c^{3L}(\mu) \leq \alpha_c^{\text{fin}}(\mu) \leq \mathfrak{J}_c^{12}(\mu) \leq \mathfrak{J}_c^{\text{br}}(\mu).$$

Every strictly feasible value in the definition of  $\mathfrak{J}_c^{3L}(\mu)$  is witnessed by a face-dominating family with exactly three exponent vectors. Moreover, for every  $\rho > 0$ ,

$$(6.201) \quad \mathfrak{J}_c^{3L}(\rho\mu) = \rho \mathfrak{J}_c^{3L}(\mu).$$

*Proof.* Let  $\lambda > 0$  and choose data satisfying (6.197)–(6.199). The family

$$((1, \zeta_1), (1, \zeta_2), (1, \zeta_0))$$

is face-dominating:  $\zeta_1$  is the unique leader on face one and has strictly negative first oblique coefficient, while  $\zeta_2$  is the unique leader on face two and has strictly negative second oblique coefficient. Taking target weights  $(0, 0, 1)$  in (5.44) gives  $\lambda c \in \mathcal{E}_{\text{fin}}(\mu)$ . Hence  $\lambda \leq \alpha_c^{\text{fin}}(\mu)$ , and taking the supremum proves the first inequality in (6.200).

The remaining two inequalities were proved in [proposition 6.64](#). They may also be read directly from the present certificate. Indeed,  $\zeta_i$  is strictly feasible for the one-face dual problem on face  $i$ , so  $\zeta_i \cdot u_i \leq \mathfrak{J}_i^f(\mu)$ . The strict ordering then gives

$$\zeta_0 \cdot u_i < \mathfrak{J}_i^f(\mu), \quad i = 1, 2.$$

Together with  $Q_\mu(\zeta_0) < 0$ , this places  $\zeta_0$  in  $\mathbb{C}_{12}(\mu)$ , and therefore  $\lambda c = \zeta_0 - \eta \in \mathcal{K}_{12}(\mu)$ .

For scaling, multiply  $\lambda, \eta, \zeta_0, \zeta_1, \zeta_2$  by  $\rho$  and use  $Q_{\rho\mu}(\rho\zeta) = \rho^2 Q_\mu(\zeta)$ . All strict sign and ordering inequalities are preserved. Hence

$$\mathfrak{J}_c^{3L}(\rho\mu) \geq \rho \mathfrak{J}_c^{3L}(\mu).$$

Applying the already proved inequality with drift  $\rho\mu$  and scaling factor  $1/\rho$  gives

$$\mathfrak{J}_c^{3L}(\mu) \geq \rho^{-1} \mathfrak{J}_c^{3L}(\rho\mu),$$

which is the reverse inequality after multiplication by  $\rho$ . Thus equality holds in (6.201).  $\square$

**Proposition 6.74** (strict three-leader endpoint criterion). *Assume  $0 < \xi < \pi$ ,  $\alpha < 2$ , and  $\mathcal{E}(\mu) \neq \emptyset$ . Fix  $c \in S_\circ^\vee$ , and put*

$$\lambda := \mathfrak{J}_c^{\text{br}}(\mu).$$

*Suppose that there exist  $\eta \in S^\vee$  and  $\zeta_0, \widehat{\zeta}_1, \widehat{\zeta}_2 \in \mathbb{R}^2$  such that*

$$(6.202) \quad \zeta_0 = \lambda c + \eta, \quad Q_\mu(\zeta_0) \leq 0,$$

$$(6.203) \quad Q_\mu(\widehat{\zeta}_i) \leq 0, \quad \widehat{\zeta}_i \cdot v_i < 0, \quad i = 1, 2,$$

*and*

$$(6.204) \quad \widehat{\zeta}_1 \cdot u_1 > \max\{\zeta_0 \cdot u_1, \widehat{\zeta}_2 \cdot u_1\}, \quad \widehat{\zeta}_2 \cdot u_2 > \max\{\zeta_0 \cdot u_2, \widehat{\zeta}_1 \cdot u_2\}.$$

*Then*

$$(6.205) \quad \alpha_c^{\text{fin}}(\mu) = \alpha_c^\pi = \mathfrak{J}_c^{\text{br}}(\mu)$$

*for every stationary law  $\pi$  at drift  $\mu$ , and*

$$(6.206) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z \in S : c \cdot z \geq x\} = -\mathfrak{J}_c^{\text{br}}(\mu).$$

*Thus an exact projected logarithmic tail follows from a feasibility system in three exponent vectors, one cone shift, quadratic kernel inequalities, and strict linear face-ordering inequalities. No global cross-compatibility hypothesis is required.*

*Proof.* Use the three exponent vectors

$$\zeta_0, \quad \widehat{\zeta}_1, \quad \widehat{\zeta}_2$$

with positive coefficients equal to one, target weights  $(1, 0, 0)$ , and the cone shift  $\eta$ . The strict ordering assumptions make  $\widehat{\zeta}_i$  the unique leader on face  $i$ , and (6.203) gives the two strict leading boundary sums. Hence these data form a closed finite-exponential endpoint certificate in the sense of definition 6.69. The conclusion follows from proposition 6.70.  $\square$

**Corollary 6.75** (quadratic-linear three-leader certificate). *Assume  $0 < \xi < \pi$ ,  $\alpha < 2$ , and  $\mathcal{E}(\mu) \neq \emptyset$ . Fix  $c \in S_\circ^\vee$ , and set  $\lambda = \mathfrak{J}_c^{\text{br}}(\mu)$ . Suppose that there exist  $\zeta_0, \widehat{\zeta}_1, \widehat{\zeta}_2 \in \mathbb{R}^2$  satisfying*

$$(6.207) \quad Q_\mu(\zeta_0) \leq 0, \quad (\zeta_0 - \lambda c) \cdot u_i \geq 0, \quad i = 1, 2,$$

$$(6.208) \quad Q_\mu(\widehat{\zeta}_i) \leq 0, \quad \widehat{\zeta}_i \cdot v_i < 0, \quad i = 1, 2,$$

*and the strict ordering inequalities (6.204). Then the frontier equality (6.205) and the exact tail (6.206) hold for every stationary law at drift  $\mu$ .*

*Proof.* For a convex wedge,

$$S^\vee = \{\eta \in \mathbb{R}^2 : \eta \cdot u_i \geq 0, \quad i = 1, 2\}.$$

Thus (6.207) says precisely that  $\eta := \zeta_0 - \lambda c$  belongs to  $S^\vee$ . The hypotheses of proposition 6.74 are therefore satisfied.  $\square$

For  $i = 1, 2$ , define

$$(6.209) \quad m_i^{12} := h_{\mathbb{C}_{12}(\mu)}(u_i), \quad F_i^{12} := \{\vartheta \in \mathbb{C}_{12}(\mu) : \vartheta \cdot u_i = m_i^{12}\}.$$

**Proposition 6.76** (automatic oblique sign on the simultaneous exposed faces). *For the exposed faces in (6.209), one has*

$$(6.210) \quad \min_{\vartheta \in F_i^{12}} \vartheta \cdot v_i \leq 0, \quad i = 1, 2.$$

Thus the sign hypothesis (6.216) is automatic for the clipped kernel disk  $\mathbb{C}_{12}(\mu)$ .

*Proof.* Fix  $i \in \{1, 2\}$ , put  $j = 3 - i$ , and use the orthonormal coordinates

$$\vartheta = au_i + bn_i.$$

Because  $n_i$  is the inward unit normal on face  $i$ , a direct coordinate calculation on either face gives  $u_j = \cos \xi u_i + \sin \xi n_i$ . Together with  $v_i = n_i - (\tan \theta_i)u_i$ , this yields

$$(6.211) \quad \vartheta \cdot u_j = a \cos \xi + b \sin \xi, \quad \vartheta \cdot v_i = b - a \tan \theta_i.$$

Let

$$J_i := \mathfrak{J}_i^f(\mu), \quad m_i := m_i^{12} = h_{\mathbb{C}_{12}(\mu)}(u_i).$$

The origin belongs to both  $\mathbb{B}_\mu$  and every clipping half-space, so  $0 \leq m_i \leq J_i$ .

Consider the compact convex one-face feasible set

$$K_i := \mathbb{B}_\mu \cap \{\vartheta : \vartheta \cdot v_i \leq 0\}.$$

By proposition 6.45,  $\max_{K_i} \vartheta \cdot u_i = J_i$ . Choose  $\widehat{\vartheta}_i \in K_i$  with  $\widehat{\vartheta}_i \cdot u_i = J_i$ . Since  $0 \in K_i$  and  $K_i$  is convex, the segment  $\{t\widehat{\vartheta}_i : 0 \leq t \leq 1\}$  is contained in  $K_i$ . Consequently, for every  $a \in [0, J_i]$ , there is at least one  $b = b(a)$  for which

$$au_i + bn_i \in \mathbb{B}_\mu, \quad b - a \tan \theta_i \leq 0.$$

In particular, this holds at  $a = m_i$ .

The section of the disk at  $a = m_i$  is a nonempty compact interval; write it as

$$\{m_i u_i + bn_i : b_- \leq b \leq b_+\}.$$

The preceding paragraph supplies a point in this interval with  $b \leq m_i \tan \theta_i$ . Hence its lower endpoint satisfies

$$(6.212) \quad b_- \leq m_i \tan \theta_i, \quad (m_i u_i + b_- n_i) \cdot v_i \leq 0.$$

It remains to verify the other clipping inequality. The set  $\mathbb{C}_{12}(\mu)$  is nonempty and compact, so the linear functional  $\vartheta \mapsto \vartheta \cdot u_i$  attains its maximum; hence the exposed face  $F_i^{12}$  is nonempty. Thus the same disk section contains some value  $b_0 \in [b_-, b_+]$  such that

$$m_i \cos \xi + b_0 \sin \xi \leq J_j.$$

Since  $0 < \xi < \pi$ , one has  $\sin \xi > 0$ , and therefore lowering the normal coordinate preserves this inequality:

$$m_i \cos \xi + b_- \sin \xi \leq m_i \cos \xi + b_0 \sin \xi \leq J_j.$$

It follows that

$$p_i := m_i u_i + b_- n_i \in \mathbb{C}_{12}(\mu), \quad p_i \cdot u_i = m_i,$$

so  $p_i \in F_i^{12}$ . Equation (6.212) gives  $p_i \cdot v_i \leq 0$ , proving (6.210).  $\square$

**Lemma 6.77** (quantitative strict-oblique interior perturbation). *Assume  $\vartheta_0 \in \mathcal{E}(\mu)$ , and let  $a \in S_\circ^\vee$  satisfy  $a \cdot v_i < 0$  for  $i = 1, 2$ . Put*

$$\kappa_0 := -Q_\mu(\vartheta_0) > 0, \quad M_0 := |a \cdot (\mu + \vartheta_0)| + \frac{1}{2}|a|^2.$$

Choose

$$(6.213) \quad 0 < \varepsilon_0 < \min \left\{ 1, \frac{\kappa_0}{2(1 + M_0)} \right\}, \quad \vartheta_* := \vartheta_0 + \varepsilon_0 a.$$

Then

$$(6.214) \quad \vartheta_* \in S_\circ^\vee, \quad Q_\mu(\vartheta_*) < -\frac{\kappa_0}{2} < 0, \quad \vartheta_* \cdot v_i < 0 \quad (i = 1, 2).$$

For each  $i$ , put

$$\kappa_* := -Q_\mu(\vartheta_*) > 0, \quad d_i := -\vartheta_* \cdot v_i > 0, \quad L_i := |u_i \cdot (\mu + \vartheta_*)| + \frac{1}{2}|u_i|^2,$$

and choose

$$(6.215) \quad 0 < s_i < \min \left\{ 1, \frac{\kappa_*}{2(1+L_i)}, \frac{d_i}{2(1+|u_i \cdot v_i|)} \right\}.$$

Then

$$Q_\mu(\vartheta_* + s_i u_i) < -\frac{\kappa_*}{2} < 0, \quad (\vartheta_* + s_i u_i) \cdot v_i < -\frac{d_i}{2} < 0.$$

*Proof.* Because  $S_\circ^\vee$  is a convex cone,  $\vartheta_* \in S_\circ^\vee$ . Since  $0 < \varepsilon_0 < 1$ ,

$$\begin{aligned} Q_\mu(\vartheta_*) &= Q_\mu(\vartheta_0) + \varepsilon_0 a \cdot (\mu + \vartheta_0) + \frac{1}{2} \varepsilon_0^2 |a|^2 \\ &\leq -\kappa_0 + \varepsilon_0 M_0 < -\frac{\kappa_0}{2}. \end{aligned}$$

Moreover,

$$\vartheta_* \cdot v_i = \vartheta_0 \cdot v_i + \varepsilon_0 a \cdot v_i < 0,$$

because  $\vartheta_0 \cdot v_i \leq 0$ . For the displaced point, again using  $s_i < 1$ ,

$$Q_\mu(\vartheta_* + s_i u_i) \leq -\kappa_* + s_i L_i < -\frac{\kappa_*}{2},$$

and

$$(\vartheta_* + s_i u_i) \cdot v_i \leq -d_i + s_i |u_i \cdot v_i| < -\frac{d_i}{2}.$$

This proves the lemma.  $\square$

**Proposition 6.78** (exposed-face saturation of the simultaneous-clipping frontier). *Assume*

$$0 < \xi < \pi, \quad 1 < \alpha < 2, \quad \mathcal{E}(\mu) \neq \emptyset.$$

Let  $m_i^{12}$  and  $F_i^{12}$  be defined by (6.209). Suppose that each exposed face contains a point with nonpositive own-face oblique coefficient:

$$(6.216) \quad \min_{\vartheta \in F_i^{12}} \vartheta \cdot v_i \leq 0, \quad i = 1, 2.$$

Then, for every  $c \in S_\circ^\vee$ ,

$$(6.217) \quad \alpha_c^{\text{fn}}(\mu) = \mathfrak{J}_c^{12}(\mu).$$

More precisely, every

$$0 < \lambda < \mathfrak{J}_c^{12}(\mu)$$

belongs to  $\mathcal{E}_{\text{fn}}(\mu)$  and has a face-dominating witness with at most three exponent vectors.

*Proof.* We first choose a strictly oblique interior exponent. Let  $\vartheta_0 \in \mathcal{E}(\mu)$ , and use lemma 2.3 to choose  $a \in S_\circ^\vee$  with  $a \cdot v_i < 0$  for  $i = 1, 2$ . Apply lemma 6.77 to obtain  $\vartheta_*$  and  $s_i > 0$ . The displaced point  $\vartheta_* + s_i u_i$  is feasible in the one-face dual problem, and

$$\vartheta_* \cdot u_i < (\vartheta_* + s_i u_i) \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu).$$

The hypotheses of lemma 6.65 are now satisfied. Hence  $\vartheta_* \in \text{int } \mathbb{C}_{12}(\mu)$ , and (6.183) gives, in particular,

$$(6.218) \quad \vartheta_* \cdot u_i < m_i^{12}, \quad i = 1, 2.$$

Choose  $p_i \in F_i^{12}$  with  $p_i \cdot v_i \leq 0$ . Fix  $c \in S_\circ^\vee$  and  $0 < \lambda < \lambda' < \mathfrak{J}_c^{12}(\mu)$ . By [lemma 6.60](#), there are  $\zeta \in \mathbb{C}_{12}(\mu)$  and  $\eta' \in S^\vee$  such that

$$\zeta = \lambda'c + \eta'.$$

Choose  $t \in (\lambda/\lambda', 1)$ , and define

$$r_i = tp_i + (1-t)\vartheta_* \quad (i = 1, 2), \quad \zeta_t = t\zeta + (1-t)\vartheta_*.$$

Because  $t < 1$ , strict convexity of the kernel disk, with  $\vartheta_*$  in its interior, places  $r_1, r_2, \zeta_t$  in the open kernel disk. Moreover,

$$(6.219) \quad \zeta_t - \lambda c = t\eta' + (1-t)\vartheta_* + (t\lambda' - \lambda)c \in S^\vee.$$

For each face  $i$ ,

$$\begin{aligned} r_i \cdot u_i &= tm_i^{12} + (1-t)\vartheta_* \cdot u_i, \\ \zeta_t \cdot u_i &= t\zeta \cdot u_i + (1-t)\vartheta_* \cdot u_i \leq r_i \cdot u_i, \end{aligned}$$

and

$$(6.220) \quad r_i \cdot v_i = tp_i \cdot v_i + (1-t)\vartheta_* \cdot v_i < 0.$$

Also  $r_1 \cdot u_1 \geq r_2 \cdot u_1$  and  $r_2 \cdot u_2 \geq r_1 \cdot u_2$ . Equality on face  $i$  occurs exactly when  $p_{3-i} \in F_i^{12}$ . If both equalities occur, then  $p_1, p_2 \in F_1^{12} \cap F_2^{12}$ . The map  $\theta \mapsto (\theta \cdot u_1, \theta \cdot u_2)$  is invertible, so this intersection contains at most one point; hence  $p_1 = p_2$  and  $r_1 = r_2$ . In that case the common corrector has negative oblique coefficient on both faces by (6.220). Therefore all hypotheses of [lemma 5.11\(i\)](#) hold. Choose the corrector coefficients by the explicit formulas (5.50)–(5.51) in the one-tie cases, and denote the resulting strict leading sums by  $b_1, b_2 < 0$ .

The target exponent  $\zeta_t$  has tangential rate no larger than the current maximum on either face. Apply [lemma 5.11\(ii\)](#) with this one additional exponent. For clarity, write

$$m_i^{\text{corr}} := \max\{r_1 \cdot u_i, r_2 \cdot u_i\} = r_i \cdot u_i, \quad i = 1, 2,$$

where the last equality follows from the leader ordering just proved. Thus, if the exponential coefficient of  $\zeta_t$  is any  $\varepsilon \in (0, \varepsilon_*)$ , where

$$\varepsilon_* = \min_{i=1,2} \frac{-b_i}{1 + \mathbf{1}_{\{\zeta_t \cdot u_i = m_i^{\text{corr}}\}}(\zeta_t \cdot v_i)_+},$$

the strict leading sums remain negative. Assign target weight one to  $\zeta_t$  and target weight zero to the correctors. The identity (6.219) is exactly the target-plus-cone shift required in the definition of  $\mathcal{E}_{\text{fin}}(\mu)$ . All exponents satisfy the strict kernel inequality, so the resulting family witnesses

$$\lambda c \in \mathcal{E}_{\text{fin}}(\mu) \quad (0 < \lambda < \mathfrak{J}_c^{12}(\mu))$$

with at most three exponent vectors.

It follows that  $\alpha_c^{\text{fin}}(\mu) \geq \mathfrak{J}_c^{12}(\mu)$ . The opposite inequality is the ray-extent consequence of [proposition 6.64](#). This proves (6.217).  $\square$

**Lemma 6.79** (face-separating perturbation in the borderline geometry). *Assume  $0 < \xi < \pi$  and  $\alpha = 1$ . Define the invertible map*

$$T\vartheta = (\vartheta \cdot u_1, \vartheta \cdot u_2)$$

and let  $d \in \mathbb{R}^2$  be determined by

$$(6.221) \quad Td = (1, -1).$$

Then

$$(6.222) \quad d \cdot v_1 < 0, \quad (-d) \cdot v_2 < 0.$$

Consequently, if  $\vartheta_0 \in S_\circ^\vee$  satisfies

$$Q_\mu(\vartheta_0) < 0, \quad \vartheta_0 \cdot v_i = 0, \quad i = 1, 2,$$

then, for every  $\varepsilon$  satisfying

$$(6.223) \quad 0 < \varepsilon < \varepsilon_0 := \min \left\{ 1, \frac{-Q_\mu(\vartheta_0)}{2(1 + |d \cdot (\mu + \vartheta_0)| + \frac{1}{2}|d|^2)} \right\},$$

$$q_1 = \vartheta_0 + \varepsilon d, \quad q_2 = \vartheta_0 - \varepsilon d$$

satisfy

$$Q_\mu(q_i) < 0, \quad q_i \cdot v_i < 0, \quad i = 1, 2,$$

and

$$q_1 \cdot u_1 > \vartheta_0 \cdot u_1 > q_2 \cdot u_1, \quad q_2 \cdot u_2 > \vartheta_0 \cdot u_2 > q_1 \cdot u_2.$$

*Proof.* In the coordinates fixed in [section 2](#),

$$u_1 = (1, 0), \quad u_2 = (\cos \xi, \sin \xi),$$

so [\(6.221\)](#) gives

$$d = \left( 1, -\frac{1 + \cos \xi}{\sin \xi} \right) = \left( 1, -\cot \frac{\xi}{2} \right).$$

On the first face,

$$v_1 = (-\tan \theta_1, 1),$$

and therefore

$$d \cdot v_1 = -\tan \theta_1 - \cot \frac{\xi}{2}.$$

Since  $\alpha = 1$ , one has  $\xi = \theta_1 + \theta_2$ , and therefore

$$\tan \theta_1 + \cot \frac{\xi}{2} = \frac{\cos((\theta_1 - \theta_2)/2)}{\cos \theta_1 \sin(\xi/2)} > 0.$$

Indeed,  $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ , so  $|\theta_1 - \theta_2| < \pi$ . Thus  $d \cdot v_1 < 0$ . By [lemma 4.1](#), the two reflection vectors are opposite generators of the same line, so  $v_2 = -\rho v_1$  for some  $\rho > 0$ . It follows that

$$(-d) \cdot v_2 = \rho d \cdot v_1 < 0.$$

The tangential inequalities follow directly from  $Td = (1, -1)$ . Put  $\kappa := -Q_\mu(\vartheta_0) > 0$ . For either sign,

$$Q_\mu(\vartheta_0 \pm \varepsilon d) = -\kappa \pm \varepsilon d \cdot (\mu + \vartheta_0) + \frac{1}{2} \varepsilon^2 |d|^2.$$

If  $0 < \varepsilon < \varepsilon_0$  as in [\(6.223\)](#), then

$$Q_\mu(\vartheta_0 \pm \varepsilon d) \leq -\kappa + \varepsilon \left( |d \cdot (\mu + \vartheta_0)| + \frac{1}{2} |d|^2 \right) < -\frac{\kappa}{2} < 0.$$

Finally,

$$q_1 \cdot v_1 = \varepsilon d \cdot v_1 < 0, \quad q_2 \cdot v_2 = \varepsilon (-d) \cdot v_2 < 0,$$

because  $\vartheta_0 \cdot v_i = 0$ . This proves every asserted inequality with an explicit admissible range for  $\varepsilon$ .  $\square$

**Proposition 6.80** (borderline saturation of the simultaneous-clipping frontier). *Assume*

$$0 < \xi < \pi, \quad \alpha = 1, \quad \mathcal{E}(\mu) \neq \emptyset.$$

Then, for every  $c \in S_\circ^\vee$ ,

$$(6.224) \quad \alpha_c^{\text{fin}}(\mu) = \mathfrak{J}_c^{12}(\mu).$$

More precisely, every

$$0 < \lambda < \mathfrak{J}_c^{12}(\mu)$$

belongs to  $\mathcal{E}_{\text{fin}}(\mu)$  and has a face-dominating witness with three exponent vectors.

*Proof.* Choose  $\vartheta_0 \in \mathcal{E}(\mu)$ . By definition,  $\vartheta_0 \in S_\circ^\vee$  and  $\vartheta_0 \cdot v_i \leq 0$  for  $i = 1, 2$ . Since  $v_2 = -\rho v_1$  for some  $\rho > 0$  by [lemma 4.1](#), these two inequalities force

$$\vartheta_0 \cdot v_i = 0, \quad i = 1, 2.$$

Let  $d$  and the explicit threshold  $\varepsilon_0 > 0$  be as in [lemma 6.79](#). Fix

$$0 < \varepsilon < \varepsilon_0$$

and set

$$q_1 := \vartheta_0 + \varepsilon d, \quad q_2 := \vartheta_0 - \varepsilon d.$$

Then the inequalities proved in that lemma give

$$Q_\mu(q_j) < 0 \quad (j = 1, 2), \quad q_1 \cdot v_1 < 0, \quad q_2 \cdot v_2 < 0.$$

Let  $m_i^{12}$  and  $F_i^{12}$  be as in [\(6.209\)](#). By [proposition 6.76](#), choose

$$p_i \in F_i^{12}, \quad p_i \cdot v_i \leq 0, \quad i = 1, 2.$$

Fix  $c \in S_\circ^\vee$  and

$$0 < \lambda < \lambda' < \mathfrak{J}_c^{12}(\mu).$$

By [lemma 6.60](#), there are  $\zeta \in \mathbb{C}_{12}(\mu)$  and  $\eta' \in S^\vee$  such that

$$\zeta = \lambda' c + \eta'.$$

Choose  $t \in (\lambda/\lambda', 1)$  and put

$$r_1 = tp_1 + (1-t)q_1, \quad r_2 = tp_2 + (1-t)q_2, \quad \zeta_t = t\zeta + (1-t)\vartheta_0.$$

Each of these three points lies in the open kernel disk. Moreover,

$$\zeta_t - \lambda c = t\eta' + (1-t)\vartheta_0 + (t\lambda' - \lambda)c \in S^\vee.$$

On the first face,

$$\begin{aligned} r_1 \cdot u_1 &= tm_1^{12} + (1-t)(\vartheta_0 \cdot u_1 + \varepsilon), \\ r_2 \cdot u_1 &\leq tm_1^{12} + (1-t)(\vartheta_0 \cdot u_1 - \varepsilon), \\ \zeta_t \cdot u_1 &\leq tm_1^{12} + (1-t)\vartheta_0 \cdot u_1. \end{aligned}$$

Thus  $r_1$  is the unique tangential leader on face one. On the second face the three tangential exponents satisfy

$$\begin{aligned} r_2 \cdot u_2 &= tm_2^{12} + (1-t)(\vartheta_0 \cdot u_2 + \varepsilon), \\ r_1 \cdot u_2 &\leq tm_2^{12} + (1-t)(\vartheta_0 \cdot u_2 - \varepsilon), \\ \zeta_t \cdot u_2 &\leq tm_2^{12} + (1-t)\vartheta_0 \cdot u_2, \end{aligned}$$

so  $r_2$  is the unique leader on face two. Finally,

$$r_i \cdot v_i = tp_i \cdot v_i + (1-t)q_i \cdot v_i < 0, \quad i = 1, 2.$$

Hence the three exponent vectors  $r_1, r_2, \zeta_t$ , with arbitrary positive exponential coefficients, form a face-dominating family. Give  $\zeta_t$  target weight one and the two correctors target weight zero. The preceding cone-shift identity then proves

$$\lambda c \in \mathcal{E}_{\text{fin}}(\mu).$$

Letting  $\lambda \uparrow \mathfrak{J}_c^{12}(\mu)$ , and using the reverse bound from [proposition 6.64](#), proves [\(6.224\)](#).  $\square$

**Theorem 6.81** (interior of the simultaneous-clipping domain consists of genuine moments).

*Assume*

$$0 < \xi < \pi, \quad 1 \leq \alpha < 2,$$

and let  $\mu$  be a drift for which a stationary distribution exists. Then  $\mathbb{C}_{12}(\mu)$  has nonempty interior and, for every stationary law  $\pi$  at drift  $\mu$ ,

$$(6.225) \quad \text{int } \mathcal{K}_{12}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi).$$

Every point in the left-hand side has a norm-like face-dominating witness with at most four exponential terms.

*Proof.* We first prove that  $\mathbb{C}_{12}(\mu)$  is full-dimensional and fix an interior exponent for the subsequent construction.

Assume first that  $1 < \alpha < 2$ . Stationarity and the strict phase diagram give  $\mathcal{E}(\mu) \neq \emptyset$ ; choose  $\vartheta_0 \in \mathcal{E}(\mu)$ . By lemma 2.3, choose  $a \in S_{\circ}^{\vee}$  with  $a \cdot v_i < 0$  for both faces. Apply lemma 6.77 to obtain  $\vartheta_*$  and the two explicit steps  $s_i > 0$ . The one-face dual representation then gives

$$\vartheta_* \cdot u_i < (\vartheta_* + s_i u_i) \cdot u_i \leq \mathfrak{J}_i^{\text{f}}(\mu).$$

Hence lemma 6.65 applies and gives  $\vartheta_* \in \text{int } \mathbb{C}_{12}(\mu)$ , together with the quantitative support slack  $\vartheta_* \cdot u_i < m_i^{12}$  for both faces.

If  $\alpha = 1$ , choose  $\vartheta_0 \in \mathcal{E}(\mu)$ . The vectors  $q_1, q_2$  from lemma 6.79 remain in the open kernel disk, and  $q_i$  is feasible in the  $i$ -th one-face dual problem while satisfying  $q_i \cdot u_i > \vartheta_0 \cdot u_i$ . Therefore

$$\vartheta_0 \cdot u_i < \mathfrak{J}_i^{\text{f}}(\mu), \quad i = 1, 2,$$

and  $Q_{\mu}(\vartheta_0) < 0$ . Thus lemma 6.65 gives  $\vartheta_0 \in \text{int } \mathbb{C}_{12}(\mu)$ . This establishes nonempty interior in both regimes, with an explicit ball contained in the clipped kernel disk.

Fix now  $\theta \in \text{int } \mathcal{K}_{12}(\mu)$ . By lemma 5.14, there are

$$(6.226) \quad \zeta \in \text{int } \mathbb{C}_{12}(\mu), \quad \eta \in S_{\circ}^{\vee}, \quad \theta + \eta = \zeta.$$

We construct a norm-like finite-exponential witness for this identity.

Suppose first that  $1 < \alpha < 2$ , and use the exponent  $\vartheta_*$  fixed above. By proposition 6.76, choose  $p_i \in F_i^{12}$  with  $p_i \cdot v_i \leq 0$ . For  $i = 1, 2$ , put

$$M_i := m_i^{12}, \quad a_i := \vartheta_* \cdot u_i, \quad z_i := \zeta \cdot u_i.$$

Then  $M_i > a_i$  and  $M_i > z_i$ . Define

$$t_i := \frac{(z_i - a_i)_+}{M_i - a_i} \in [0, 1), \quad t_0 := \max\{t_1, t_2\} < 1, \quad t := \frac{1 + t_0}{2}.$$

Set

$$r_i := t p_i + (1 - t) \vartheta_*, \quad i = 1, 2.$$

The quadratic identity

$$Q_{\mu}(r_i) = t Q_{\mu}(p_i) + (1 - t) Q_{\mu}(\vartheta_*) - \frac{1}{2} t (1 - t) |p_i - \vartheta_*|^2 < 0$$

places  $r_i$  in the open kernel disk. Also

$$r_i \cdot u_i - a_i = t(M_i - a_i) > 0,$$

and, because  $t > t_i$ ,

$$r_i \cdot u_i - z_i = a_i - z_i + t(M_i - a_i) > 0.$$

Thus

$$(6.227) \quad r_i \cdot u_i > \max\{\zeta \cdot u_i, \vartheta_* \cdot u_i\}, \quad i = 1, 2.$$

Furthermore,

$$r_i \cdot v_i = t p_i \cdot v_i + (1 - t) \vartheta_* \cdot v_i < 0.$$

The exposed-face definitions imply  $r_1 \cdot u_1 \geq r_2 \cdot u_1$  and  $r_2 \cdot u_2 \geq r_1 \cdot u_2$ . If both equalities hold, then  $p_1 = p_2$  by invertibility of the two coordinate functionals, and so  $r_1 = r_2$ , with both oblique coefficients negative. Thus lemma 5.11(i) supplies positive corrector coefficients with strict leading sums  $b_1, b_2 < 0$ , using one or two correctors as appropriate.

By (6.227), the two additional exponents  $\zeta$  and  $\vartheta_*$  are strictly below the current maximum on each face. Adding them with arbitrary positive coefficients therefore does not change either leading set; equivalently, lemma 5.11(ii) applies with  $T_1 = T_2 = \emptyset$ . The resulting family is coercive on the wedge because it contains  $\vartheta_* \in S_\circ^\vee$ : indeed,

$$\max_k \zeta_k \cdot u \geq \vartheta_* \cdot u \geq \min_{v \in S \cap \mathbb{S}^1} \vartheta_* \cdot v > 0 \quad (u \in S \cap \mathbb{S}^1).$$

Assign target weight one to  $\zeta$  and zero to all other exponents. Then (6.226) is precisely the identity in proposition 5.13, which yields  $\theta \in \mathcal{D}_{\text{exp}}(\pi)$ .

Now let  $\alpha = 1$ . Choose  $\vartheta_0 \in \mathcal{E}(\mu)$  and let  $q_1, q_2$  be the perturbations from lemma 6.79. Choose  $p_i \in F_i^{12}$  with  $p_i \cdot v_i \leq 0$ , and put

$$\begin{aligned} M_i &:= m_i^{12} = p_i \cdot u_i, & D_i &:= |q_i \cdot u_i - M_i|, \\ \Delta_i &:= \min\{M_i - \zeta \cdot u_i, M_i - \vartheta_0 \cdot u_i\} > 0. \end{aligned}$$

Choose  $t \in (0, 1)$  so that

$$(6.228) \quad 1 - t < \min_{i=1,2} \frac{\Delta_i}{2(1 + D_i)}.$$

Set

$$r_1 = tp_1 + (1 - t)q_1, \quad r_2 = tp_2 + (1 - t)q_2.$$

For  $x \in \{\zeta, \vartheta_0\}$ ,

$$r_i \cdot u_i - x \cdot u_i \geq \Delta_i - (1 - t)D_i > \frac{\Delta_i}{2} > 0.$$

Furthermore,

$$(r_1 - r_2) \cdot u_1 = t(p_1 - p_2) \cdot u_1 + (1 - t)(q_1 - q_2) \cdot u_1 > 0,$$

because the first term is nonnegative and the second equals  $2(1 - t)\varepsilon > 0$ . On the second face,

$$(r_2 - r_1) \cdot u_2 = t(p_2 - p_1) \cdot u_2 + (1 - t)(q_2 - q_1) \cdot u_2.$$

The definition of  $F_i^{12}$  gives  $p_2 \cdot u_2 = M_2 \geq p_1 \cdot u_2$ , while lemma 6.79 gives  $(q_2 - q_1) \cdot u_2 = 2\varepsilon$ . Hence

$$(r_2 - r_1) \cdot u_2 \geq 2(1 - t)\varepsilon > 0.$$

Therefore

$$\begin{aligned} r_1 \cdot u_1 &> \max\{r_2 \cdot u_1, \zeta \cdot u_1, \vartheta_0 \cdot u_1\}, \\ r_2 \cdot u_2 &> \max\{r_1 \cdot u_2, \zeta \cdot u_2, \vartheta_0 \cdot u_2\}. \end{aligned}$$

Moreover,

$$Q_\mu(r_i) = tQ_\mu(p_i) + (1 - t)Q_\mu(q_i) - \frac{1}{2}t(1 - t)|p_i - q_i|^2 < 0,$$

and

$$r_i \cdot v_i = tp_i \cdot v_i + (1 - t)q_i \cdot v_i < 0.$$

Thus coefficients one on  $r_1, r_2$  already give strict leading sums, and adding  $\zeta$  and  $\vartheta_0$  does not change the leading sets. The exponent  $\vartheta_0 \in S_\circ^\vee$  makes the family coercive, and assigning target weight one to  $\zeta$  again invokes proposition 5.13 through (6.226).

In either regime the family uses at most two face correctors, the target exponent, and one strictly positive exponent, hence at most four terms. This proves (6.225).  $\square$

*Remark 6.82* (geometry of the simultaneous exposed faces). Each set  $F_i^{12}$  is a compact point or line segment. Its endpoints are intersections of the supporting line  $\vartheta \cdot u_i = m_i^{12}$  with the kernel circle and the two clipping lines. The proof of proposition 6.76 shows more than finite verifiability: the lower endpoint of the disk section in the inward-normal coordinate always lies in  $F_i^{12}$  and has nonpositive own-face oblique coefficient. Thus no additional exposed-face sign hypothesis is needed in the strict-regime saturation theorem.

**Proposition 6.83** (cross-compatible saturation of the simultaneous-clipping frontier). *Assume  $\mathcal{E}(\mu) \neq \emptyset$ . For each face choose a one-face maximizer  $\widehat{\vartheta}_i$  satisfying*

$$(6.229) \quad Q_\mu(\widehat{\vartheta}_i) = 0, \quad \widehat{\vartheta}_i \cdot v_i \leq 0, \quad \widehat{\vartheta}_i \cdot u_i = \mathfrak{J}_i^f(\mu), \quad i = 1, 2,$$

*as supplied by [proposition 6.45](#). Suppose that the two maximizers can be chosen to satisfy the cross-compatibility inequalities*

$$(6.230) \quad \widehat{\vartheta}_1 \cdot u_2 < \mathfrak{J}_2^f(\mu), \quad \widehat{\vartheta}_2 \cdot u_1 < \mathfrak{J}_1^f(\mu).$$

*Then the following assertions hold.*

(i) *If  $\theta \in S_\circ^\vee$  and there are  $\eta \in S^\vee$  and  $\zeta \in \mathbb{R}^2$  such that*

$$(6.231) \quad \theta + \eta = \zeta, \quad Q_\mu(\zeta) < 0, \quad \zeta \cdot u_i < \mathfrak{J}_i^f(\mu), \quad i = 1, 2,$$

*then  $\theta \in \mathcal{E}_{\text{fin}}(\mu)$ . A witnessing face-dominating family can be chosen with at most three exponent vectors.*

(ii) *For every  $c \in S_\circ^\vee$ ,*

$$(6.232) \quad \alpha_c^{\text{fin}}(\mu) = \mathfrak{J}_c^{12}(\mu).$$

*More precisely, every  $\lambda c$  with  $0 < \lambda < \mathfrak{J}_c^{12}(\mu)$  belongs to  $\mathcal{E}_{\text{fin}}(\mu)$  and has a three-exponential witness.*

*Proof.* Choose  $\vartheta_0 \in \mathcal{E}(\mu)$ . It is feasible for each one-face maximization problem, and in fact

$$(6.233) \quad \vartheta_0 \cdot u_i < \mathfrak{J}_i^f(\mu), \quad i = 1, 2.$$

Put  $\kappa_0 := -Q_\mu(\vartheta_0) > 0$ . If  $\vartheta_0 \cdot v_i < 0$ , set  $d_i = u_i$  and  $b_i := -\vartheta_0 \cdot v_i > 0$ , and choose

$$0 < s_i < \min \left\{ 1, \frac{\kappa_0}{2(1 + |d_i \cdot (\mu + \vartheta_0)| + \frac{1}{2}|d_i|^2)}, \frac{b_i}{2(1 + |d_i \cdot v_i|)} \right\}.$$

Then

$$Q_\mu(\vartheta_0 + s_i d_i) < -\frac{\kappa_0}{2}, \quad (\vartheta_0 + s_i d_i) \cdot v_i < -\frac{b_i}{2},$$

and the objective increases by  $s_i$ .

If  $\vartheta_0 \cdot v_i = 0$ , take  $d_i = u_i + (\tan \theta_i)n_i$ , so that  $d_i \cdot v_i = 0$  and  $d_i \cdot u_i = 1$ . Choose

$$0 < s_i < \min \left\{ 1, \frac{\kappa_0}{2(1 + |d_i \cdot (\mu + \vartheta_0)| + \frac{1}{2}|d_i|^2)} \right\}.$$

Then  $Q_\mu(\vartheta_0 + s_i d_i) < -\kappa_0/2$ , the oblique coefficient remains zero, and the objective increases by  $s_i$ . Thus  $\vartheta_0$  cannot maximize the one-face problem, proving (6.233). In particular,  $\mathfrak{J}_i^f(\mu) > 0$ , and the maximizers in (6.229) are nonzero.

We first prove part (i). Define

$$\gamma := \min \left\{ \mathfrak{J}_1^f(\mu) - \max\{\widehat{\vartheta}_2 \cdot u_1, \zeta \cdot u_1\}, \mathfrak{J}_2^f(\mu) - \max\{\widehat{\vartheta}_1 \cdot u_2, \zeta \cdot u_2\} \right\} > 0, \quad \rho := \frac{\gamma}{4}.$$

Choose

$$0 < \delta_i < \min \left\{ \frac{1}{2}, \frac{\rho}{2(1 + |\widehat{\vartheta}_i|)} \right\}, \quad x_i := (1 - \delta_i)\widehat{\vartheta}_i.$$

Since  $Q_\mu(\widehat{\vartheta}_i) = 0$  and  $\widehat{\vartheta}_i \neq 0$ ,

$$Q_\mu(x_i) = -\frac{1}{2}\delta_i(1 - \delta_i)|\widehat{\vartheta}_i|^2 < 0.$$

If  $\widehat{\vartheta}_i \cdot v_i < 0$ , set  $\zeta_i = x_i$ . If  $\widehat{\vartheta}_i \cdot v_i = 0$ , put  $\kappa_i := -Q_\mu(x_i) > 0$  and choose

$$0 < \varepsilon_i < \min \left\{ \frac{\rho}{2(1 + |v_i|)}, \frac{\kappa_i}{2(1 + |v_i \cdot (\mu + x_i)| + \frac{1}{2}|v_i|^2)} \right\}, \quad \zeta_i := x_i - \varepsilon_i v_i.$$

Then

$$Q_\mu(\zeta_i) \leq -\kappa_i + \varepsilon_i \left( |v_i \cdot (\mu + x_i)| + \frac{1}{2}|v_i|^2 \right) < -\frac{\kappa_i}{2} < 0,$$

and  $\zeta_i \cdot v_i = -\varepsilon_i |v_i|^2 < 0$ . In the first case  $x_i \cdot v_i < 0$ . In both cases,

$$(6.234) \quad |\zeta_i - \widehat{\vartheta}_i| < \rho, \quad Q_\mu(\zeta_i) < 0, \quad \zeta_i \cdot v_i < 0.$$

Using  $|u_i| = 1$ ,

$$\begin{aligned} \zeta_1 \cdot u_1 - \zeta_2 \cdot u_1 &\geq \mathfrak{J}_1^f(\mu) - \widehat{\vartheta}_2 \cdot u_1 - 2\rho \geq \frac{\gamma}{2} > 0, \\ \zeta_1 \cdot u_1 - \zeta \cdot u_1 &\geq \mathfrak{J}_1^f(\mu) - \zeta \cdot u_1 - \rho \geq \frac{3\gamma}{4} > 0. \end{aligned}$$

On face two, the same norm estimate is written explicitly as

$$\begin{aligned} \zeta_2 \cdot u_2 - \zeta_1 \cdot u_2 &\geq \mathfrak{J}_2^f(\mu) - \widehat{\vartheta}_1 \cdot u_2 - 2\rho \geq \frac{\gamma}{2} > 0, \\ \zeta_2 \cdot u_2 - \zeta \cdot u_2 &\geq \mathfrak{J}_2^f(\mu) - \zeta \cdot u_2 - \rho \geq \frac{3\gamma}{4} > 0. \end{aligned}$$

Combining the four strict inequalities gives

$$(6.235) \quad \zeta_1 \cdot u_1 > \max\{\zeta_2 \cdot u_1, \zeta \cdot u_1\}, \quad \zeta_2 \cdot u_2 > \max\{\zeta_1 \cdot u_2, \zeta \cdot u_2\}.$$

The family  $(\zeta_1, \zeta_2, \zeta)$ , with any positive exponential coefficients, is therefore face-dominating: the first exponent is the unique leader on face one and the second is the unique leader on face two. Using target weights  $(0, 0, 1)$  in (5.44) gives  $\theta + \eta = \zeta$ , proving  $\theta \in \mathcal{E}_{\text{fin}}(\mu)$ .

For part (ii), the inclusion in proposition 6.64 gives  $\alpha_c^{\text{fin}}(\mu) \leq \mathfrak{J}_c^{12}(\mu)$ . Fix

$$0 < \lambda < \lambda' < \mathfrak{J}_c^{12}(\mu).$$

By lemma 6.60, there are  $\zeta \in \mathbb{C}_{12}(\mu)$  and  $\eta' \in S^\vee$  such that

$$\zeta - \lambda'c = \eta'.$$

Choose  $t \in (\lambda/\lambda', 1)$  and set

$$\zeta_t := t\zeta + (1-t)\vartheta_0.$$

Since  $\vartheta_0$  lies in the open kernel disk and has the strict slack (6.233), while  $\zeta \in \mathbb{C}_{12}(\mu)$ , one has

$$Q_\mu(\zeta_t) < 0, \quad \zeta_t \cdot u_i < \mathfrak{J}_i^f(\mu), \quad i = 1, 2.$$

Moreover,

$$(6.236) \quad \zeta_t - \lambda c = t\eta' + (1-t)\vartheta_0 + (t\lambda' - \lambda)c \in S^\vee.$$

Part (i), with  $\theta = \lambda c$ , now gives  $\lambda c \in \mathcal{E}_{\text{fin}}(\mu)$ . Letting  $\lambda$  increase to  $\mathfrak{J}_c^{12}(\mu)$  proves (6.232) and the stronger subcritical assertion.  $\square$

**Proposition 6.84** (automatic simultaneous-clipping endpoint representation). *Fix  $c \in S_\circ^\vee$  and set*

$$\lambda := \mathfrak{J}_c^{\text{br}}(\mu).$$

*Then*

$$(6.237) \quad \mathfrak{J}_c^{12}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu),$$

*and there exists  $\vartheta \in \mathbb{R}^2$  such that*

$$(6.238) \quad Q_\mu(\vartheta) \leq 0, \quad \vartheta \cdot u_i \leq \mathfrak{J}_i^f(\mu), \quad (\vartheta - \lambda c) \cdot u_i \geq 0, \quad i = 1, 2.$$

*Thus the finite quadratic-linear endpoint system is always feasible; it does not impose an additional geometric restriction on a positive dual ray.*

*Proof.* The equality is proposition 6.67. Since  $\mathcal{K}_{12}(\mu)$  is closed and contains the entire ray segment up to its endpoint,  $\lambda c \in \mathcal{K}_{12}(\mu)$ . Hence there are  $\vartheta \in \mathbb{C}_{12}(\mu)$  and  $\eta \in S^\vee$  with  $\vartheta = \lambda c + \eta$ . Written in the two boundary coordinates, these are exactly the inequalities in (6.238).  $\square$

**Corollary 6.85** (global projected logarithmic tails in the strict regime). *Assume*

$$0 < \xi < \pi, \quad 1 < \alpha < 2,$$

and let  $\mu$  be a drift for which a stationary distribution exists. Then, for every stationary law  $\pi$  at drift  $\mu$  and every  $c \in S^\vee$ ,

$$(6.239) \quad \alpha_c^{\text{fin}}(\mu) = \alpha_c^\pi = \mathfrak{J}_c^{12}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu),$$

and

$$(6.240) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi\{z \in S : c \cdot z \geq x\} = -\mathfrak{J}_c^{\text{br}}(\mu).$$

Moreover, the positive part of the closed transform domain is law-independent:

$$(6.241) \quad D_\pi \cap S^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S^\vee = \mathcal{K}_{12}(\mu) \cap S^\vee.$$

Every strict subcritical point  $\lambda c$ ,  $0 < \lambda < \mathfrak{J}_c^{\text{br}}(\mu)$ , has a face-dominating Lyapunov witness with at most three exponent vectors. No uniqueness assumption on the stationary law is used.

*Proof.* By the strict stationary classification and [proposition 5.4](#), one has  $\mathcal{E}(\mu) \neq \emptyset$ . The exposed-face sign condition in [proposition 6.78](#) is automatic by [proposition 6.76](#). Therefore

$$\alpha_c^{\text{fin}}(\mu) = \mathfrak{J}_c^{12}(\mu),$$

and every strict subcritical ray point has a three-exponential witness. By [proposition 6.67](#),  $\mathfrak{J}_c^{12}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu)$ . The chain and exact-tail conclusion in [theorem 6.62](#) now give (6.239) and (6.240).

The inclusion  $D_\pi \subset \mathcal{K}_{\text{br}}(\mu)$  is [theorem 6.62](#). Both sets are closed, convex, contain the origin, and are stable under subtraction of  $S^\vee$ . Consequently their intersections with every ray generated by  $S^\vee$  are closed intervals beginning at the origin. Equality of the corresponding endpoints therefore gives

$$D_\pi \cap S^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S^\vee.$$

To pass to the boundary of  $S^\vee$ , choose  $\vartheta_* \in \mathcal{E}(\mu)$ . If  $\vartheta \in \mathcal{K}_{\text{br}}(\mu) \cap S^\vee$ , then, for  $n \geq 2$ ,

$$\vartheta_n = (1 - n^{-1})\vartheta + n^{-1}\vartheta_*$$

belongs to  $\mathcal{K}_{\text{br}}(\mu) \cap S^\vee$  by convexity. Hence  $\vartheta_n \in D_\pi$ , and closedness of  $D_\pi$  gives  $\vartheta \in D_\pi$ . This proves

$$D_\pi \cap S^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S^\vee.$$

It remains to compare the two geometric domains on the closed dual cone. The inclusion  $\mathcal{K}_{12}(\mu) \subset \mathcal{K}_{\text{br}}(\mu)$  is (6.180). Conversely, let  $\vartheta \in \mathcal{K}_{\text{br}}(\mu) \cap S^\vee$  and define

$$\vartheta_n = (1 - n^{-1})\vartheta + n^{-1}\vartheta_*.$$

Then  $\vartheta_n \in S^\vee$  and  $\vartheta_n \in \mathcal{K}_{\text{br}}(\mu)$ . Write  $\vartheta_n = \lambda_n c_n$ , where  $\lambda_n = |\vartheta_n| > 0$  and  $c_n = \vartheta_n/|\vartheta_n| \in S^\vee \cap \mathbb{S}^1$ . The raywise equivalence (6.185) gives  $\vartheta_n \in \mathcal{K}_{12}(\mu)$ . Closedness of  $\mathcal{K}_{12}(\mu)$  and  $\vartheta_n \rightarrow \vartheta$  imply  $\vartheta \in \mathcal{K}_{12}(\mu)$ . Hence

$$\mathcal{K}_{\text{br}}(\mu) \cap S^\vee = \mathcal{K}_{12}(\mu) \cap S^\vee,$$

which completes (6.241). □

**Corollary 6.86** (global projected logarithmic tails in the borderline regime). *Assume*

$$0 < \xi < \pi, \quad \alpha = 1,$$

and let  $\mu$  satisfy

$$m := n_{\mathcal{L}} \cdot \mu < 0.$$

Then, for every stationary law  $\pi$  at drift  $\mu$  and every  $c \in S^\vee$ ,

$$(6.242) \quad \alpha_c^{\text{fin}}(\mu) = \alpha_c^\pi = \mathfrak{J}_c^{12}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu),$$

and

$$(6.243) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi \{z \in S : c \cdot z \geq x\} = -\mathfrak{J}_c^{\text{br}}(\mu).$$

Moreover,

$$(6.244) \quad D_\pi \cap S^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S^\vee = \mathcal{K}_{12}(\mu) \cap S^\vee,$$

and every strict subcritical point on a positive dual ray has a three-exponential Lyapunov witness. For  $c = n\mathcal{L}$ , the common rate equals  $-2m$ , consistently with the exact marginal in [proposition 5.22](#).

*Proof.* Choose  $\beta \in (0, -2m)$  and put

$$\vartheta_0 = \beta n\mathcal{L}.$$

Then  $\vartheta_0 \in S_\circ^\vee$ ,  $\vartheta_0 \cdot v_i = 0$ , and

$$Q_\mu(\vartheta_0) = \beta m + \frac{1}{2}\beta^2 < 0.$$

Thus  $\mathcal{E}(\mu) \neq \emptyset$ , and [proposition 6.80](#) gives

$$\alpha_c^{\text{fin}}(\mu) = \mathfrak{J}_c^{12}(\mu).$$

The raywise clipping equality in [proposition 6.67](#) and the outer bounds in [theorem 6.62](#) prove [\(6.242\)](#) and [\(6.243\)](#).

We now prove the positive-domain identity. The broken-path theorem gives  $D_\pi \subset \mathcal{K}_{\text{br}}(\mu)$ . Equality of the ray endpoints in [\(6.242\)](#) implies

$$D_\pi \cap S_\circ^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S_\circ^\vee.$$

Let  $\vartheta \in \mathcal{K}_{\text{br}}(\mu) \cap S^\vee$  and put

$$\vartheta_n = (1 - n^{-1})\vartheta + n^{-1}\vartheta_0.$$

Convexity gives  $\vartheta_n \in \mathcal{K}_{\text{br}}(\mu)$ , and strict positivity of  $\vartheta_0$  on  $S \setminus \{0\}$  gives  $\vartheta_n \in S_\circ^\vee$ . Hence  $\vartheta_n \in D_\pi$ , and closedness of  $D_\pi$  gives  $\vartheta \in D_\pi$ . Thus

$$D_\pi \cap S^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S^\vee.$$

For the equality with  $\mathcal{K}_{12}(\mu)$ , the inclusion  $\mathcal{K}_{12}(\mu) \subset \mathcal{K}_{\text{br}}(\mu)$  is already known. Conversely, for the same  $\vartheta_n$ , write  $\vartheta_n = \lambda_n c_n$  with  $c_n \in S_\circ^\vee$ . The raywise equivalence [\(6.185\)](#) implies  $\vartheta_n \in \mathcal{K}_{12}(\mu)$ . Letting  $n \rightarrow \infty$  and using closedness gives  $\vartheta \in \mathcal{K}_{12}(\mu)$ . This proves [\(6.244\)](#). Finally, for  $c = n\mathcal{L}$ , [corollary 5.23](#) gives  $\alpha_c^\pi = -2m$ , and the claimed consistency follows from [\(6.242\)](#).  $\square$

**Theorem 6.87** (global projected logarithmic tails in the classified regimes). *Assume*

$$0 < \xi < \pi, \quad 1 \leq \alpha < 2,$$

and let  $\mu$  be a drift for which a stationary distribution exists. Then, for every stationary law  $\pi$  at drift  $\mu$  and every  $c \in S_\circ^\vee$ ,

$$\begin{aligned} \alpha_c^{\text{fin}}(\mu) &= \alpha_c^\pi = \mathfrak{J}_c^{12}(\mu) = \mathfrak{J}_c^{\text{br}}(\mu), \\ \lim_{x \rightarrow \infty} \frac{1}{x} \log \pi \{z \in S : c \cdot z \geq x\} &= -\mathfrak{J}_c^{\text{br}}(\mu), \end{aligned}$$

and

$$D_\pi \cap S^\vee = \mathcal{K}_{12}(\mu) \cap S^\vee = \mathcal{K}_{\text{br}}(\mu) \cap S^\vee.$$

Every strictly subcritical point  $\lambda c$ ,  $0 < \lambda < \mathfrak{J}_c^{\text{br}}(\mu)$ , has a face-dominating Lyapunov witness with at most three exponential terms. The conclusions are independent of uniqueness of the stationary law.

*Proof.* For  $1 < \alpha < 2$ , this is [corollary 6.85](#). For  $\alpha = 1$ , it is [corollary 6.86](#).  $\square$

**Theorem 6.88** (full transform domain and exact radial and sector logarithmic tails). *Assume*

$$0 < \xi < \pi, \quad 1 \leq \alpha < 2,$$

and let  $\mu$  be a drift for which a stationary distribution exists. Then  $\mu \neq 0$ , and for every stationary law  $\pi$  at drift  $\mu$ ,

$$(6.245) \quad D_\pi = \mathcal{K}_{12}(\mu) = \mathcal{K}_{\text{br}}(\mu).$$

In particular, the entire closed exponential-moment domain, not only its positive dual part, is independent of the choice of stationary law. Moreover,

$$(6.246) \quad \text{int } \mathcal{D}_{\text{exp}}(\pi) = \text{int } \mathcal{K}_{12}(\mu), \quad \text{int } \mathcal{K}_{12}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi) \subset \mathcal{K}_{12}(\mu).$$

No assertion is made here about the finiteness of moments at every point of  $\partial \mathcal{K}_{12}(\mu)$ .

For a nonempty compact angular interval  $C \subset S \cap \mathbb{S}^1$  containing more than one point, define

$$(6.247) \quad \mathfrak{J}_C(\mu) := \min_{u \in C} h_{\mathbb{C}_{12}(\mu)}(u).$$

Then

$$(6.248) \quad \alpha_C^\pi = \mathfrak{J}_C(\mu)$$

and

$$(6.249) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log \pi \left\{ z \neq 0 : |z| \geq r, \frac{z}{|z|} \in C \right\} = -\mathfrak{J}_C(\mu).$$

For  $C = S \cap \mathbb{S}^1$ , this gives the radial identities

$$(6.250) \quad \alpha_{\text{rad}}^\pi = \min_{u \in S \cap \mathbb{S}^1} h_{\mathbb{C}_{12}(\mu)}(u), \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log \pi \{|z| \geq r\} = -\alpha_{\text{rad}}^\pi.$$

Together with [theorem 6.87](#), these statements determine the logarithmic tails for every positive projection, the radial tail, and every nondegenerate closed angular sector in all classified stationary regimes.

*Proof.* The classification theorem gives  $\mu \neq 0$ : in the borderline case  $n_{\mathcal{L}} \cdot \mu < 0$ , while in the strict case  $0 \in K_{\text{str}}$  belongs to the nonexistence cone. By [theorem 6.81](#),

$$\text{int } \mathcal{K}_{12}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi).$$

Since  $\mathcal{K}_{12}(\mu)$  is closed, convex, and has nonempty interior, it is the closure of its interior. Taking closures therefore gives  $\mathcal{K}_{12}(\mu) \subset D_\pi$ . The broken-path outer bound gives  $D_\pi \subset \mathcal{K}_{\text{br}}(\mu)$ , and [corollary 6.68](#) identifies the two geometric domains. This proves (6.245). Since  $\text{int } \mathcal{K}_{12}(\mu) \subset \mathcal{D}_{\text{exp}}(\pi)$  and  $\mathcal{D}_{\text{exp}}(\pi) \subset D_\pi = \mathcal{K}_{12}(\mu)$ , taking interiors gives (6.246).

Fix a compact nondegenerate interval  $C$ , and write  $J = \mathfrak{J}_C(\mu)$ . The construction in [theorem 6.81](#) places a vector of  $S^\vee$  in  $\mathbb{C}_{12}(\mu)$ ; compactness of  $C$  therefore gives  $J > 0$ . Let  $0 \leq q < J$ .

For  $u \in S$ , the support function of the recession cone is

$$h_{-S^\vee}(u) = \sup_{\eta \in S^\vee} (-\eta \cdot u) = 0,$$

because  $u \in (S^\vee)^\vee = S$ , and the value zero is attained at  $\eta = 0$ . Hence

$$(6.251) \quad h_{\mathcal{K}_{12}(\mu)}(u) = h_{\mathbb{C}_{12}(\mu) - S^\vee}(u) = h_{\mathbb{C}_{12}(\mu)}(u), \quad u \in S.$$

For every  $u \in C$ , compactness of  $\mathbb{C}_{12}(\mu)$  supplies  $\bar{\theta}_u \in \mathbb{C}_{12}(\mu)$  attaining the last support value. Since that value is at least  $J > q$ ,  $\bar{\theta}_u \cdot u > q$ .

Choose one fixed  $\theta_0 \in \text{int } \mathcal{K}_{12}(\mu)$ , and choose  $r_0 > 0$  with  $B_{r_0}(\theta_0) \subset \mathcal{K}_{12}(\mu)$ . For every  $\varepsilon \in (0, 1)$ , convexity gives

$$B_{\varepsilon r_0}((1 - \varepsilon)\bar{\theta}_u + \varepsilon\theta_0) = (1 - \varepsilon)\bar{\theta}_u + \varepsilon B_{r_0}(\theta_0) \subset \mathcal{K}_{12}(\mu).$$

Thus the convex combination is an interior point. Put

$$\Delta_u := \bar{\theta}_u \cdot u - q > 0, \quad B_u := |(\theta_0 - \bar{\theta}_u) \cdot u|,$$

and choose

$$0 < \varepsilon < \min \left\{ 1, \frac{\Delta_u}{2(1+B_u)} \right\}.$$

Define

$$\theta_u := (1 - \varepsilon)\bar{\theta}_u + \varepsilon\theta_0.$$

Then

$$\theta_u \cdot u - q = \Delta_u + \varepsilon(\theta_0 - \bar{\theta}_u) \cdot u \geq \Delta_u - \varepsilon B_u > \frac{\Delta_u}{2} > 0.$$

Then  $\theta_u \in \text{int } \mathcal{K}_{12}(\mu)$ ,  $\theta_u \cdot u > q$ , and  $\theta_u \in \mathcal{D}_{\text{exp}}(\pi)$  by [theorem 6.81](#).

Continuity of  $v \mapsto \theta_u \cdot v$  gives a relative open neighborhood  $V_u$  of  $u$  in  $C$  on which  $\theta_u \cdot v > q$ . Compactness of  $C$  gives a finite subcover  $V_{u_1}, \dots, V_{u_m}$ . Write  $\theta_j = \theta_{u_j}$ . If  $z \neq 0$  and  $z/|z| \in C$ , choose  $j$  with  $z/|z| \in V_{u_j}$ . Then

$$\theta_j \cdot z = |z|\theta_j \cdot \frac{z}{|z|} > q|z|,$$

and therefore, pointwise on the cone over  $C$ ,

$$e^{q|z|} \leq \max_{1 \leq j \leq m} e^{\theta_j \cdot z} \leq \sum_{j=1}^m e^{\theta_j \cdot z}.$$

Each term in the finite sum is  $\pi$ -integrable. Hence the restricted  $q$ -radial moment is finite. Markov's inequality gives

$$\Pi_C(r) \leq e^{-qr} \int_{\{z \neq 0: z/|z| \in C\}} e^{q|z|} \pi(dz),$$

and therefore

$$\liminf_{r \rightarrow \infty} -\frac{1}{r} \log \Pi_C(r) \geq q.$$

Letting  $q \uparrow J$  yields

$$\liminf_{r \rightarrow \infty} -\frac{1}{r} \log \Pi_C(r) \geq J.$$

Choose  $u_* \in C$  attaining the minimum in [\(6.247\)](#). Since  $C$  has more than one point, there are interior directions  $u_n \in S^\circ \cap \text{relint } C$  with  $u_n \rightarrow u_*$ . For each  $n$ , choose a relative open angular neighborhood  $U_n$  of  $u_n$  whose closure is contained in  $C$ . Since  $U_n \subset C$ , one has  $\Pi_C(r) \geq \Pi_{U_n}(r)$ . Applying the one-face broken-path lower bound separately for the two faces gives

$$\limsup_{r \rightarrow \infty} -\frac{1}{r} \log \Pi_C(r) \leq \limsup_{r \rightarrow \infty} -\frac{1}{r} \log \Pi_{U_n}(r) \leq \min_{i=1,2} h_{\mathbb{C}_i(\mu)}(u_n).$$

By [lemma 6.52](#), one of the sets  $\mathbb{C}_i(\mu)$  is the full kernel disk, while the other equals  $\mathbb{C}_{12}(\mu)$ . Hence the right-hand side is  $h_{\mathbb{C}_{12}(\mu)}(u_n)$ . Letting  $n \rightarrow \infty$  gives the reverse inequality with rate  $J$ . This proves [\(6.249\)](#); the abscissa equality follows from [proposition 6.4](#). Taking the full angular interval gives [\(6.250\)](#).  $\square$

**Corollary 6.89** (analyticity on the exact transform interior). *Under the hypotheses of [theorem 6.88](#), the Laplace transform*

$$M_\pi(\zeta) := \int_S e^{\zeta \cdot z} \pi(dz)$$

*is holomorphic on the tube domain*

$$\{\zeta \in \mathbb{C}^2 : \text{Re } \zeta \in \text{int } \mathcal{K}_{12}(\mu)\}.$$

*For every compact set  $K \Subset \text{int } \mathcal{K}_{12}(\mu)$  and every integer  $m \geq 0$ ,*

$$\sup_{\theta \in K} \int_S |z|^m e^{\theta \cdot z} \pi(dz) < \infty.$$

*Proof.* By [theorem 6.88](#), every point of  $\text{int } \mathcal{K}_{12}(\mu)$  is an exponential-moment vector. Fix such a point  $\theta_0$ . Choose  $\varepsilon > 0$  so small that

$$\theta_0 + \varepsilon(\sigma_1, \sigma_2) \in \text{int } \mathcal{K}_{12}(\mu), \quad \sigma_1, \sigma_2 \in \{-1, 1\}.$$

If  $\|\text{Re } \zeta - \theta_0\|_\infty < \varepsilon/2$ , then for every  $z = (z_1, z_2) \in S$ ,

$$(\text{Re } \zeta - \theta_0) \cdot z \leq \frac{\varepsilon}{2}(|z_1| + |z_2|).$$

For every integer  $m \geq 0$ , polynomial growth is absorbed by the remaining half of the exponential margin: there is  $C_{m,\varepsilon} < \infty$  such that

$$|z|^m e^{\frac{\varepsilon}{2}(|z_1|+|z_2|)} \leq C_{m,\varepsilon} e^{\varepsilon(|z_1|+|z_2|)}.$$

Choose  $\sigma_j \in \{-1, 1\}$  with  $\sigma_j z_j = |z_j|$ . Then

$$e^{\theta_0 \cdot z + \varepsilon(|z_1|+|z_2|)} = e^{[\theta_0 + \varepsilon(\sigma_1, \sigma_2)] \cdot z} \leq \sum_{\tau_1, \tau_2 \in \{-1, 1\}} e^{[\theta_0 + \varepsilon(\tau_1, \tau_2)] \cdot z}.$$

Consequently every complex derivative of order at most  $m$  is dominated, on this neighborhood of  $\theta_0$ , by a fixed  $\pi$ -integrable sum of the four corner exponentials. Differentiation under the integral sign is therefore justified there. A finite covering of a compact set  $K \Subset \text{int } \mathcal{K}_{12}(\mu)$  gives the stated locally uniform bounds and holomorphy on the full tube domain.  $\square$

*Remark 6.90* (scope of the logarithmic theory). The preceding theorems complete the logarithmic-order problem in the classified stationary regimes: they identify the closure of the Laplace-transform domain and determine the exact exponential rates of all strictly positive projections, the radial tail, and every nondegenerate closed angular sector. They do not determine pointwise integrability at every critical transform vector, subexponential prefactors, or pointwise density asymptotics. Those questions require information beyond the convex geometry of the closed transform domain.

## 7. DISCUSSION AND FURTHER DIRECTIONS

The results above complete the logarithmic-scale stationary theory in the classified range  $1 \leq \alpha < 2$ . For every stationary law,

$$D_\pi = \mathbb{C}_{12}(\mu) - S^\vee,$$

the interior of this closed set consists of genuine exponential-moment vectors, and its support function gives the exact logarithmic rates for all strictly positive projections, the radial tail, and every nondegenerate closed angular sector. The proof also shows that these quantities are determined by the drift and reflection geometry alone, without assuming uniqueness of the stationary law. The remaining questions concern finer asymptotic resolution rather than a gap in the closed-domain or logarithmic-tail theorem.

**7.1. Boundary moments and transform singularities.** The present theorem identifies  $\overline{\mathcal{D}_{\text{exp}}(\pi)}$  and  $\text{int } \mathcal{D}_{\text{exp}}(\pi)$ , but it does not decide, point by point, whether a vector on  $\partial D_\pi$  belongs to  $\mathcal{D}_{\text{exp}}(\pi)$ . A complete boundary classification should distinguish exposed smooth points, endpoints of a clipping arc, intersections with the dual-cone boundary, and possible tangential degeneracies. This requires control of the Laplace transform as an actual analytic object near the frontier, not only knowledge of its convergence domain.

**7.2. Sharp one-term asymptotics.** The next asymptotic level is to determine whether, for a given  $c \in S_\circ^\vee$ ,

$$\pi\{c \cdot Z \geq x\} \sim C_c x^{\kappa_c} e^{-\alpha_c x},$$

and to obtain analogous fixed-sector and fixed-ray formulas. In semimartingale models, transform and boundary-value methods lead to pole, branch-point, and saddle-point regimes; see, for

example, [1, 4, 5]. For the present process, an analogous analysis would have to identify the correct transform or renewal structure compatible with the submartingale formulation and with one-face localization.

**7.3. Raywise density and reflecting-face boundary layers.** The stationary density is strictly positive and real analytic in the open wedge, but the logarithmic results do not determine its pointwise asymptotics along a fixed interior ray or in a shrinking neighborhood of a reflecting face. A natural objective is a uniform expansion that interpolates between interior rays, face-dominated directions, and the transition angles at which the active clipping mechanism changes. Such a result would also clarify how the one-face optimal path is encoded in the stationary density despite the absence of a global boundary-local-time representation.

**7.4. The vertex, uniqueness, and long-time convergence.** The present arguments deliberately avoid any boundary condition for the stationary density at the vertex. A finer local analysis could seek vertex asymptotics, determine whether the weak stationary identities select a unique normalized solution, and relate the stationary tail geometry to recurrence rates and convergence to equilibrium. These questions are logically separate from the logarithmic tail theorem proved here and may require additional potential-theoretic or coupling input.

**7.5. Extensions of the model.** Further directions include projections on  $\partial S^\vee$ , one-ray angular limits, nonidentity covariance matrices, nonconvex wedges, and parameter ranges outside  $1 \leq \alpha < 2$ . The closed domain  $\mathbb{C}_{12}(\mu) - S^\vee$  and the associated optimal paths provide a reference geometry against which such extensions can be formulated and tested.

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