STATIONARY DISTRIBUTIONS FOR THE NONSEMIMARTINGALE REFLECTED BROWNIAN MOTION WITH DRIFT IN A WEDGE: EXISTENCE, CLASSIFICATION AND ELLIPTIC PDE OBSTRUCTIONS

ZIRAN LIU

Abstract. We consider the reflected Brownian motion with drift in a planar wedge defined by the submartingale problem (Definition 2.1 in [5]). In the convex range $0 < \xi < \pi$ and for $1 \leq \alpha < 2$ (the case when the RBM solution to the submartingale problem is nonsemimatingale), we give the complete stationary-existence phase diagram. At the borderline value $\alpha = 1$, a stationary distribution exists exactly when $n_{\mathscr{L}} \cdot \mu < 0$. In the strict regime $1 < \alpha < 2$, it exists exactly when

$$\mu \notin K_{\mathrm{str}} := \mathrm{cone}\{-v_1, -v_2\}.$$

Thus zero drift and both boundary rays of K_{str} belong to the nonstationary region.

Existence follows from bounded concave truncations of a Foster–Lyapunov function and the Krylov–Bogoliubov theorem. Nonexistence is obtained from bounded one-dimensional supersolutions, a neutral projection argument at $\alpha = 1$, and bounded two-scale localizations of a two-homogeneous Varadhan–Williams gauge in the strict regime.

We also introduce the corresponding finite-measure weak elliptic PDE system and prove the same drift classification for its nonzero finite nonnegative solutions. Every stationary law has zero boundary mass and a strictly positive real-analytic density in the open wedge, satisfying the adjoint stationary equation. Additional compatibility and gauge criteria for the weak oblique elliptic PDE problem are also developed in the later sections.

Contents

Date: June 5, 2026.

2020 *Mathematics Subject Classification.* Primary 60J65; Secondary 60J60, 60K25, 35J25, 35B40.

Key words and phrases. reflected Brownian motion; submartingale problem; stationary distribution; wedge; oblique reflection; Varadhan–Williams harmonic function; Foster–Lyapunov criterion; elliptic PDE; nonsemimartingale reflected diffusion.

1. Introduction and Main Results

Let

$$S = \{(r,\theta) : r \geq 0,\ 0 \leq \theta \leq \xi\} \subset \mathbb{R}^2, \qquad 0 < \xi < \pi,$$

be a convex planar wedge. On its two faces, let v_1 and v_2 be constant oblique reflection directions normalized by $v_i \cdot n_i = 1$. We study the Markov family introduced by Lakner, Liu, and Reed through a submartingale problem with constant drift μ [5]. The reflection angles $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ are encoded by

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

For $1 < \alpha < 2$ the process generally lies outside the semimartingale reflected Brownian-motion framework, so arguments based on boundary local times or a classical basic adjoint relationship are not available. The purpose of this paper is to determine the stationary-existence region for $1 \leq \alpha < 2$ directly from the admissible tests of the defining submartingale problem.

Theorem 1.1 (stationary phase diagram)**.** *Assume $0 < \xi < \pi$.*

(i) *If $\alpha = 1$, let $\mathscr{L} = \operatorname{cone}\{v_1, v_2\}$ and let $n_{\mathscr{L}}$ be the unit normal to this line satisfying $n_{\mathscr{L}} \cdot z > 0$ for $z \in S \setminus \{0\}$. A stationary distribution exists if and only if*

$$n_{\mathscr{L}} \cdot \mu < 0.$$

(ii) *If $1 < \alpha < 2$, a stationary distribution exists if and only if*

$$\mu \notin K_{\mathrm{str}} := \operatorname{cone}\{-v_1, -v_2\}.$$

Thus the whole closed cone K_{str}, including zero drift and its two boundary rays, is a nonexistence region.

The proof is formulated entirely in terms of bounded admissible tests. Bounded one-dimensional profiles give a direct nonexistence criterion. On the critical line $n_{\mathscr{L}} \cdot \mu = 0$, compactly supported neutral projection tests reduce stationarity to a one-dimensional distributional identity. For existence, an exponential Foster–Lyapunov gauge is replaced by bounded concave truncations; the resulting occupation estimates imply tightness, and the C_0-Feller property yields an invariant probability measure. In the strict regime, a two-homogeneous transform of the Varadhan–Williams harmonic function provides a neutral gauge whose bounded two-scale localizations exclude the entire closed cone K_{str}.

The same stationary inequality has a natural weak elliptic formulation. We call a finite nonnegative measure with zero boundary mass a solution of the stationary elliptic system when it satisfies the admissible-test inequality. Testing with both signs yields the neutral identities and the interior adjoint equation without imposing a classical boundary condition on the density.

Theorem 1.2 (elliptic-system phase diagram)**.** *Under the hypotheses of theorem 1.1, the stationary elliptic system has a nonzero finite nonnegative measure solution exactly for the same drifts for which the reflected Brownian motion has a stationary distribution.*

Every stationary distribution charges neither the boundary nor the vertex. Its restriction to S° has a smooth strictly positive density p satisfying

$$\frac{1}{2}\Delta p - \mu \cdot \nabla p = 0.$$

The weak elliptic formulation also yields the interior adjoint equation and a collection of compatibility and supersolution criteria. These consequences follow directly from the weak stationary inequality and use only bounded admissible tests.

The main argument is organized as follows. Section 2 fixes the geometry and the reflected diffusion, and section 3 states the classification. Section 4 derives the stationary inequality and the interior adjoint equation. Sections 5 to 7 establish, respectively, the direct nonexistence criterion, Foster–Lyapunov existence, and nonexistence on the closed reflection cone. Section 8 introduces the weak stationary elliptic system and proves its phase diagram. After section 9, the appendices collect compatibility results, limitations of local correction schemes, and additional elliptic-gauge criteria that are logically independent of the phase-diagram proof. Quantitative moment theory, the closed Laplace-transform domain, and exact projected, radial, and sector logarithmic tails are developed in the companion paper [4].

2. Geometry and basic properties of the reflected diffusion

Throughout the paper,

$$\operatorname{cone}\{w_1, \ldots, w_k\} := \left\{ \sum_{j=1}^{k} a_j w_j : \; a_j \geq 0 \right\}$$

denotes the closed conical hull generated by its arguments. In particular, it is not the compact convex hull. We use this notation consistently because the phase diagram below is organized by conical subsets and their complements.

Let

$$S = \{(r, \theta) : r \geq 0, \; 0 \leq \theta \leq \xi\} \subset \mathbb{R}^2, \qquad 0 < \xi < \pi,$$

be the convex wedge. We write its two unit generators as

$$u_1 = (1, 0), \qquad u_2 = (\cos \xi, \sin \xi),$$

so that $S = \operatorname{cone}\{u_1, u_2\}$. Its two open boundary rays are denoted by ∂S_1 and ∂S_2, excluding the vertex. Let $v_1, v_2 \in \mathbb{R}^2$ be the reflection directions, normalized by

$$v_i \cdot n_i = 1, \qquad i = 1, 2,$$

where n_i is the inward unit normal on ∂S_i. We write

$$D_i = v_i \cdot \nabla, \qquad i = 1, 2.$$

Throughout, $C_b^2(S)$ denotes functions whose values and first two Euclidean derivatives are bounded on S, with derivatives understood on the interior and by continuous extension up to each open face. Every admissible test used below is also constant on a neighborhood of the vertex, so its derivatives extend through the corner. Generator expressions $\mathcal{L}f$ are the bounded open-wedge expressions, with arbitrary bounded values assigned on the boundary when an integral over S is written; the zero-boundary-mass result in proposition 4.2 shows that these boundary choices never affect stationary integrals. For gauges that are introduced before localization and are not inserted directly into the submartingale problem, the following regularity convention is also used: a function W is *facewise C^2 away from the vertex* if it is continuous on S, is C^2 in S°, and its first and second Euclidean derivatives admit continuous limits to compact subsets of each open face. A facewise C^2 gauge is called *locally C^2-extendable away from the vertex* if, for every compact set $K \subset S \setminus \{0\}$, its restriction to K is the restriction of a C^2 function on an open neighborhood of K in $\mathbb{R}^2$. Boundary directional derivatives such as $D_i W$ are interpreted through the open-face limits. Whenever a localized gauge is used as an admissible test, the statement includes this local extendability, or verifies it directly. All functions used as admissible tests below are bounded members of $C_b^2(S)$, typically after localization.

For drift μ, write

$$P_t^{\mu} g(z) := \mathbb{E}_z^{\mu}[g(Z(t))].$$

A probability measure π on S is stationary if $\pi P_t^{\mu} = \pi$ for every $t \geq 0$. The state space S is locally compact and Polish. For $\alpha < 2$, Lakner–Liu–Reed prove the $\widehat{C}(S)$-Feller property [5, Theorem 2.12]; their space $\widehat{C}(S)$ is the usual space $C_0(S)$ of continuous functions vanishing at infinity. This $C_0(S)$-Feller property, together with tightness of occupation measures, is the only semigroup regularity used in the Krylov–Bogoliubov argument below. The superscript μ is suppressed when the drift is fixed.

As in Lakner–Liu–Reed [5], let $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ be the reflection angles and define

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

Throughout the paper we use the following sign convention for these angles. Let t_i be the unit tangent on ∂S_i pointing away from the vertex. The normalization $v_i \cdot n_i = 1$ is then equivalent to

$$v_i = n_i - (\tan\theta_i)t_i, \qquad i = 1, 2.$$

Thus a positive reflection angle means that the tangential component of the reflection direction points toward the vertex. This convention is used in the Varadhan–Williams harmonic calculation in section 7.

The process studied in this paper is specified by the following submartingale problem.

Definition 2.1 (Submartingale problem with drift)**.** For a fixed drift vector $\mu \in \mathbb{R}^2$, a family of probability measures $\{P_z^{\mu} : z \in S\}$ on $C([0,\infty), S)$ is said to solve the submartingale problem with drift if the coordinate process Z satisfies the three conditions of Definition 2.1 in Lakner–Liu–Reed [5]: it starts from z, it makes

$$\left\{ f(Z(t)) - \int_0^t \mu \cdot \nabla f(Z(s))\, ds - \frac{1}{2}\int_0^t \Delta f(Z(s))\, ds : t \geq 0 \right\}$$

a submartingale for every admissible test function $f \in C_b^2(S)$ that is constant near the vertex and satisfies $D_i f \geq 0$ on ∂S_i, and it spends zero occupation time at the vertex.

The following results of Lakner–Liu–Reed [5] will be used.

(i) If $\alpha < 2$, the submartingale problem with drift has a unique solution; this is [5, Theorem 2.3].
(ii) If $\alpha < 2$, the transition semigroup has the $C_0(S)$-Feller property; in the notation of [5], this is the $\widehat{C}(S)$-Feller property of [5, Theorem 2.12].
(iii) For the solution to the submartingale problem, the whole boundary has zero occupation time; this is [5, Lemma 3.3].
(iv) If $0 < \xi < \pi$ and $\alpha \geq 1$, then there exists $b \neq 0$ such that

$$b \cdot z < 0 \quad (z \in S \setminus \{0\}), \qquad b \cdot v_i \geq 0 \quad (i = 1, 2).$$

This follows from the Lakner–Liu–Reed reflection-cone geometry, in particular the statement that $\operatorname{cone}\{v_1, v_2\} \cap S = \{0\}$ in the convex $\alpha \geq 1$ case.
(v) If $0 < \xi < \pi$ and $\alpha = 1$, then $\operatorname{cone}\{v_1, v_2\}$ is a line and $\operatorname{cone}\{v_1, v_2\} \cap S = \{0\}$.

We refer to this vector b as the *geometric separator*. The term *strict nonsemimartingale regime* below refers only to $1 < \alpha < 2$; the boundary case $\alpha = 1$ is treated separately.

The following elementary consequence of the angle convention will be used in the cone calculations.

Lemma 2.2 (strict reflection cone contains the wedge)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Then

$$S = \operatorname{cone}\{u_1, u_2\} \subset \operatorname{cone}\{-v_1, -v_2\}.$$

More explicitly, if the boundary generators have polar angles 0 *and* ξ*, then the ray generated by* $-v_1$ *has angle*

$$\theta_1 - \frac{\pi}{2} < 0,$$

and the ray generated by $-v_2$ *has angle*

$$\xi + \frac{\pi}{2} - \theta_2 > \xi.$$

The counterclockwise angle from $-v_1$ *to* $-v_2$ *is*

$$\pi + \xi - (\theta_1 + \theta_2) = \pi - (\alpha - 1)\xi,$$

which lies in (ξ, π)*. Hence the proper closed cone generated by* $-v_1$ *and* $-v_2$ *contains the whole wedge sector.*

Proof. With the convention $v_i = n_i - (\tan\theta_i)t_i$, the first face has $t_1 = e_r(0)$ and $n_1 = e_\theta(0)$. Thus

$$v_1 = e_\theta(0) - (\tan\theta_1)e_r(0),$$

so v_1 has angle $\pi/2 + \theta_1$ and $-v_1$ has angle $\theta_1 - \pi/2$. Since $\theta_1 < \pi/2$, this angle is strictly below the first boundary ray.

On the second face, $t_2 = e_r(\xi)$ and $n_2 = -e_\theta(\xi)$, hence

$$v_2 = -e_\theta(\xi) - (\tan\theta_2)e_r(\xi).$$

Therefore v_2 has angle $\xi - \pi/2 - \theta_2$, and $-v_2$ has angle $\xi + \pi/2 - \theta_2$. Since $\theta_2 < \pi/2$, this angle is strictly above the second boundary ray.

The counterclockwise aperture from $-v_1$ to $-v_2$ is

$$\big(\xi + \frac{\pi}{2} - \theta_2\big) - \big(\theta_1 - \frac{\pi}{2}\big) = \pi + \xi - (\theta_1 + \theta_2) = \pi - (\alpha - 1)\xi.$$

Because $\alpha > 1$, this aperture is less than π. Because $\theta_1 + \theta_2 < \pi$, it is larger than ξ. Thus $-v_1$ and $-v_2$ form a proper convex cone whose angular interval contains $[0, \xi]$, which is exactly the wedge sector in the convex case. □

Lemma 2.3 (nonempty strict Lyapunov directions)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Let $K_{\mathrm{str}} := \operatorname{cone}\{-v_1, -v_2\}$*. Then* $\mathfrak{A} \neq \varnothing$*. More precisely, if*

$$K_{\mathrm{str}}^\vee := \{a \in \mathbb{R}^2 : a \cdot x \geq 0 \textit{ for every } x \in K_{\mathrm{str}}\},$$

then every $a \in \operatorname{int} K_{\mathrm{str}}^\vee$ *belongs to* $\mathfrak{A}$*.*

Proof. By lemma 2.2, K_{str} is a proper two-dimensional closed cone with aperture

$$\omega_{K_{\mathrm{str}}} = \pi - (\alpha - 1)\xi \in (0, \pi).$$

The dual cone therefore has aperture $\pi - \omega_{K_{\mathrm{str}}} > 0$, so $\operatorname{int} K_{\mathrm{str}}^\vee \neq \varnothing$.

Fix $a \in \operatorname{int} K_{\mathrm{str}}^\vee$. Choose $\varepsilon > 0$ such that

$$B_\varepsilon(a) \subset K_{\mathrm{str}}^\vee.$$

For any $x \in K_{\mathrm{str}} \setminus \{0\}$, the vector

$$a_x := a - \frac{\varepsilon}{2}\frac{x}{|x|}$$

belongs to $K_{\mathrm{str}}^\vee$. Hence

$$0 \leq a_x \cdot x = a \cdot x - \frac{\varepsilon}{2}|x|,$$

and therefore

$$a \cdot x \geq \frac{\varepsilon}{2}|x| > 0, \qquad x \in K_{\mathrm{str}} \setminus \{0\}. \tag{2.1}$$

Since $u_1, u_2, -v_1, -v_2$ are nonzero elements of K_{str}, (2.1) gives

$$a \cdot u_j > 0 \quad (j = 1, 2), \qquad a \cdot (-v_i) > 0 \quad (i = 1, 2).$$

Thus $a \cdot v_i < 0$ for $i = 1, 2$, and therefore $a \in \mathfrak{A}$. Since $\operatorname{int} K_{\mathrm{str}}^{\vee} \neq \varnothing$, it follows that $\mathfrak{A} \neq \varnothing$. □

We use the dual cone notation

$$S^{\vee} := \{\vartheta \in \mathbb{R}^2 : \vartheta \cdot z \geq 0 \text{ for all } z \in S\}, \qquad S^{\vee}_{\circ} := \{\vartheta \in \mathbb{R}^2 : \vartheta \cdot z > 0 \text{ for all } z \in S \setminus \{0\}\}.$$

In the convex regimes $0 < \xi < \pi$, we also fix the following Lyapunov cone notation at the outset.

Definition 2.4 (Lyapunov existence cone)**.** Let

$$u_1 = (1, 0), \qquad u_2 = (\cos\xi, \sin\xi)$$

denote the two generating rays of the wedge. Define

$$\mathfrak{A} := \{a \in \mathbb{R}^2 : a \cdot u_1 > 0,\ a \cdot u_2 > 0,\ a \cdot v_1 \leq 0,\ a \cdot v_2 \leq 0\}$$

and the corresponding drift cone

$$\mathfrak{M}_{\mathrm{Lyap}} := \{\mu \in \mathbb{R}^2 : \exists a \in \mathfrak{A} \text{ such that } a \cdot \mu < 0\}.$$

3. Main results

We write $K_{\mathrm{str}} := \operatorname{cone}\{-v_1, -v_2\}$ in the strict regime.

Theorem 3.1 (stationary phase diagram)**.** *Assume $0 < \xi < \pi$ and $1 \leq \alpha < 2$.*

- (i) *If $\alpha = 1$, let $\mathscr{L} = \operatorname{cone}\{v_1, v_2\}$ and let $n_{\mathscr{L}}$ be the unit normal to $\mathscr{L}$ whose positive half-plane contains $S \setminus \{0\}$. The submartingale problem admits a stationary distribution if and only if $n_{\mathscr{L}} \cdot \mu < 0$.*
- (ii) *If $1 < \alpha < 2$, the submartingale problem admits a stationary distribution if and only if $\mu \notin K_{\mathrm{str}}$.*

Proof. Suppose first that $\alpha = 1$. By proposition 5.10,

$$\mathfrak{M}_{\mathrm{Lyap}} = \{\mu : n_{\mathscr{L}} \cdot \mu < 0\}, \qquad \mathfrak{M}_{\mathrm{sup}} = \{\mu : n_{\mathscr{L}} \cdot \mu > 0\}.$$

Existence on the first half-plane follows from proposition 6.11; nonexistence on the second follows from theorem 5.4. The boundary line is excluded by proposition 5.14.

Now assume $1 < \alpha < 2$. By lemma 2.2, $S \subset K_{\mathrm{str}}$. Hence

$$\operatorname{cone}\{u_1, u_2, -v_1, -v_2\} = K_{\mathrm{str}},$$

and the polar description in proposition 6.10 yields

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus K_{\mathrm{str}}.$$

Thus proposition 6.11 gives existence for $\mu \notin K_{\mathrm{str}}$, whereas theorem 7.12 gives nonexistence for every $\mu \in K_{\mathrm{str}}$. □

Every application of the stationary inequality in the proof of theorem 3.1 uses a bounded admissible test function. In particular, the classification requires neither an unbounded test nor an a priori stationary moment assumption.

Remark 3.2 (logical scope of the auxiliary elliptic criteria). The proof of the phase diagram is logically complete at the end of section 7: the borderline case uses the one-dimensional Lyapunov, supersolution, and neutral-projection arguments, while the strict case uses the Lyapunov complement and the localized Varadhan–Williams gauge. The stationary elliptic system and the compatibility, elliptic-norm, cone-quadratic, and proper-gauge criteria developed in later sections are consequences and alternative obstruction mechanisms; none of them is used as an input to theorem 3.1. This separation also rules out any circular use of the elliptic-system phase diagram in the probabilistic classification.

4. Stationary identities and admissible supersolutions

Throughout this section, $\alpha < 2$. For a fixed drift μ, write

$$\mathcal{L}_\mu = \mu \cdot \nabla + \frac{1}{2}\Delta,$$

and write $\mathcal{L}$ when the drift is fixed by context.

Lemma 4.1 (zero stationary mass from zero occupation times)**.** *Assume $\alpha < 2$, and let π be a stationary distribution for the solution to the submartingale problem with drift μ. Then*

$$\pi(\{0\}) = 0, \qquad \pi(\partial S) = 0. \tag{4.1}$$

Consequently, the stationary integral of any bounded open-wedge expression is independent of the bounded Borel values assigned on ∂S.

Proof. Condition 3 in definition 2.1 gives zero occupation time at the vertex, and [5, Lemma 3.3] gives zero occupation time on the whole boundary. Under the stationary initial law, the following expectation identities are finite because the integrands are bounded. For every $t > 0$,

$$0 = \mathbb{E}_\pi \int_0^t \mathbf{1}_{\{0\}}(Z(s))\, ds = t\,\pi(\{0\}),$$

The zero boundary-occupation identity gives separately

$$0 = \mathbb{E}_\pi \int_0^t \mathbf{1}_{\partial S}(Z(s))\, ds = t\,\pi(\partial S).$$

The last assertion follows because two bounded Borel extensions of the same open-wedge expression differ only on ∂S, a π-null set. □

Proposition 4.2 (stationary inequality and zero boundary mass)**.** *Assume $\alpha < 2$, and let π be a stationary distribution for the solution to the submartingale problem with drift μ. Then for every admissible test function f,*

$$\int_S \mathcal{L} f(x)\, \pi(dx) \le 0. \tag{4.2}$$

Moreover,

$$\pi(\{0\}) = 0, \qquad \pi(\partial S) = 0. \tag{4.3}$$

Proof. The zero-mass statement is lemma 4.1. Fix an admissible test function f. Definition 2.1 implies that

$$M_f(t) := f(Z(t)) - \int_0^t \mathcal{L} f(Z(s))\, ds$$

is a submartingale. Since $f \in C_b^2(S)$, both $f(Z(t))$ and the time integral of the bounded open-wedge generator are integrable after any bounded Borel extension to the boundary. Under a stationary initial law π,

$$\mathbb{E}_\pi[f(Z(t))] = \mathbb{E}_\pi[f(Z(0))],$$

so

$$\mathbb{E}_\pi\Big[\int_0^t \mathcal{L} f(Z(s))\, ds\Big] \le 0.$$

Because $\mathcal{L} f$ is bounded in the open wedge and $\pi(\partial S) = 0$, Fubini's theorem and stationarity give

$$\mathbb{E}_\pi \int_0^t \mathcal{L} f(Z(s))\, ds = \int_0^t \int_S \mathcal{L} f\, d\pi\, ds = t \int_S \mathcal{L} f\, d\pi,$$

where any bounded Borel extension of the open-wedge generator may be used on ∂S. Division by t yields (4.2). □

Corollary 4.3 (stationary equality for neutral boundary tests)**.** *Assume $\alpha < 2$, and let π be a stationary distribution for the drift μ. Suppose $f \in C_b^2(S)$ is constant near the vertex and satisfies*

$$D_i f = 0 \quad \text{on } \partial S_i, \qquad i = 1, 2.$$

Then both f and $-f$ are admissible test functions, the open-wedge generator $\mathcal{L}f$ has a bounded Borel extension to S, and

$$\int_S \mathcal{L}f \, d\pi = 0.$$

The value assigned to $\mathcal{L}f$ on ∂S is immaterial.

Proof. The function $-f$ is also in $C_b^2(S)$ and is constant near the vertex. The boundary equalities imply $D_i f \geq 0$ and $D_i(-f) \geq 0$ on both faces, so both signs are admissible. Since $f \in C_b^2(S)$, the open-wedge generator $\mathcal{L}f$ is bounded and therefore has bounded Borel extensions to S. By proposition 4.2, $\pi(\partial S) = 0$, so the integral is independent of the chosen boundary extension. Applying proposition 4.2 to f and to $-f$ gives opposite inequalities for the same finite integral, hence equality. □

Corollary 4.4 (boundary-null integration of generator bounds)**.** *Assume $\alpha < 2$, and let π be a stationary distribution for the drift μ. Let $f \in C_b^2(S)$, and let G be a bounded Borel function on S. Interpret $\mathcal{L}f$ as any bounded Borel extension to S of the open-wedge generator. If*

$$\mathcal{L}f \geq G \quad \text{on } S^\circ,$$

then

$$\int_S \mathcal{L}f \, d\pi \geq \int_S G \, d\pi.$$

The analogous statement holds for upper bounds and equalities. Consequently, when applying stationary inequalities, generator bounds that are proved in the open wedge and extended continuously to open faces may be integrated without specifying their values at the vertex or on the boundary.

Proof. By proposition 4.2, $\pi(\partial S) = 0$. Hence the inequality $\mathcal{L}f \geq G$, which holds on S°, holds π-almost surely after arbitrary choices of values on ∂S. Both functions are integrable because the chosen extension of $\mathcal{L}f$ is bounded and G is bounded. Integration gives the claim. Changing the extension of $\mathcal{L}f$ on ∂S changes neither side of any stationary integral. □

Corollary 4.5 (pathwise integration of interior generator bounds)**.** *Assume $\alpha < 2$, and fix an initial state $z \in S$. Let $f \in C_b^2(S)$, and let G be a bounded Borel function on S. Interpret $\mathcal{L}f$ as any bounded Borel extension to S of the open-wedge generator. If*

$$\mathcal{L}f \geq G \quad \text{on } S^\circ,$$

then, for every $T < \infty$,

$$\mathbb{E}_z \int_0^T \mathcal{L}f(Z_s) \, ds \geq \mathbb{E}_z \int_0^T G(Z_s) \, ds.$$

The analogous statement holds for upper bounds and equalities. Thus open-wedge generator inequalities for bounded localized tests may be used inside time integrals under any initial state, without specifying special values at the boundary or vertex.

Proof. The solution has zero occupation time on the whole boundary: definition 2.1 gives zero occupation time at the vertex, and [5, Lemma 3.3] gives zero occupation time on ∂S. Hence

$$\int_0^T \mathbf{1}_{\partial S}(Z_s) \, ds = 0 \quad \text{almost surely}$$

for every initial state. The inequality $\mathcal{L}f \geq G$ holds whenever $Z_s \in S^\circ$, and the set of times at which $Z_s \in \partial S$ has Lebesgue measure zero. Since both functions are bounded, integration in time and expectation give the claim. Changing the boundary extension of $\mathcal{L}f$ changes the time integral only on a null set of times. □

Proposition 4.6 (scaling invariance along drift rays). *Assume $\alpha < 2$, so that the Markov family of definition 2.1 with drift is available. For $r > 0$, let*

$$T_r : S \to S, \qquad T_r x = rx.$$

If π is a stationary distribution for drift μ, then $(T_r)_{\#}\pi$ is a stationary distribution for drift μ/r. Equivalently, for every $c > 0$, stationary existence for drift μ is equivalent to stationary existence for drift $c\mu$.

Moreover, if π has interior density p_π, then $(T_r)_{\#}\pi$ has density

$$p_r(y) = r^{-2} p_\pi(y/r), \qquad y \in S^\circ.$$

Consequently, existence and nonexistence are invariant under positive rescaling of a nonzero drift.

Proof. Let Z solve the submartingale problem with drift μ and initial point z, and define

$$Y(t) := rZ(t/r^2), \qquad t \geq 0.$$

For an admissible test function g, set $f(x) = g(rx)$. Then f is admissible: it is constant near the vertex whenever g is, and on ∂S_i,

$$D_i f(x) = r D_i g(rx) \geq 0.$$

The chain rule gives

$$\mathcal{L}_\mu f(x) = \left(\mu \cdot \nabla + \frac{1}{2}\Delta\right) f(x) = r^2 \left(\frac{\mu}{r} \cdot \nabla + \frac{1}{2}\Delta\right) g(rx) = r^2 \mathcal{L}_{\mu/r} g(rx).$$

Let $\mathcal{F}_t^Z$ be the canonical filtration of Z, and put $\mathcal{F}_t^Y := \mathcal{F}_{t/r^2}^Z$. Evaluating the submartingale associated with f at time t/r^2 gives

$$\begin{aligned} f(Z(t/r^2)) - f(Z(0)) &- \int_0^{t/r^2} \mathcal{L}_\mu f(Z(s))\, ds \\ &= g(Y(t)) - g(Y(0)) - \int_0^{t/r^2} r^2 \mathcal{L}_{\mu/r} g(rZ(s))\, ds \\ &= g(Y(t)) - g(Y(0)) - \int_0^t \mathcal{L}_{\mu/r} g(Y(q))\, dq, \end{aligned}$$

where $q = r^2 s$ in the last integral. Thus the required process for Y is an $\mathcal{F}_t^Y$-submartingale. Its initial state is $Y(0) = rz$. For every $T < \infty$,

$$\int_0^T \mathbf{1}_{\{0\}}(Y(t))\, dt = r^2 \int_0^{T/r^2} \mathbf{1}_{\{0\}}(Z(s))\, ds = 0 \qquad \text{almost surely.}$$

The same identity with $\mathbf{1}_{\partial S}$ preserves zero occupation time on the whole boundary. Hence Y solves the Lakner–Liu–Reed submartingale problem with drift μ/r and initial point rz. Uniqueness for $\alpha < 2$ identifies this law with the canonical Markov family at drift μ/r.

If $Z(0) \sim \pi$ and π is stationary for drift μ, then $Z(t/r^2) \sim \pi$ for every t. Therefore $Y(t) = rZ(t/r^2) \sim (T_r)_{\#}\pi$ for every t, so $(T_r)_{\#}\pi$ is stationary for drift μ/r. Replacing r by $1/c$ gives the equivalence between μ and $c\mu$. Finally, for every Borel set $A \subset S^\circ$,

$$(T_r)_{\#}\pi(A) = \pi(r^{-1}A) = \int_A r^{-2} p_\pi(y/r)\, dy,$$

which proves the density transformation. □

Proposition 4.7 (interior adjoint equation and smooth density)**.** *Assume $\alpha < 2$, and let π be a stationary distribution for the solution to the submartingale problem with drift μ. Then*

$$\int_{S^\circ} \mathcal{L}\varphi \, d\pi = 0$$

for every $\varphi \in C_c^2(S^\circ)$. Consequently, the restriction of π to the open wedge S° is absolutely continuous with respect to Lebesgue measure. Its density p_π is C^∞ in S° and satisfies the adjoint stationary equation

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0 \qquad \text{in } S^\circ$$

in the classical sense.

Proof. If $\varphi \in C_c^2(S^\circ)$, then the compact support of φ has positive distance from the vertex and from ∂S. Hence φ and $-\varphi$ are both bounded admissible test functions: they are identically zero in a full neighborhood of the vertex and in a neighborhood of each face, so the oblique boundary inequalities are automatic. In particular, no boundary value of the open-wedge generator enters this argument. Applying proposition 4.2 to both signs and using $\pi(\partial S) = 0$ gives

$$\int_{S^\circ} \mathcal{L}\varphi \, d\pi \le 0, \qquad -\int_{S^\circ} \mathcal{L}\varphi \, d\pi \le 0,$$

and therefore

$$\int_{S^\circ} \mathcal{L}\varphi \, d\pi = 0.$$

This means that the order-zero distribution

$$T := \pi|_{S^\circ}$$

satisfies $\mathcal{L}^*T = 0$ in $\mathcal{D}'(S^\circ)$. We spell out the elliptic regularity step because no regularity of the measure is assumed a priori. Define the distribution

$$Q := e^{-\mu \cdot x}T, \qquad \langle Q, \psi \rangle := \langle T, e^{-\mu \cdot x}\psi \rangle.$$

For $\psi \in C_c^\infty(S^\circ)$, put $\varphi = e^{-\mu \cdot x}\psi$. A direct calculation gives

$$\mathcal{L}\varphi = \left(\mu \cdot \nabla + \frac{1}{2}\Delta\right)(e^{-\mu \cdot x}\psi) = \frac{1}{2}e^{-\mu \cdot x}(\Delta\psi - |\mu|^2\psi).$$

Consequently,

$$\big\langle (\Delta - |\mu|^2)Q, \psi \big\rangle = 2\langle T, \mathcal{L}\varphi \rangle = 0.$$

Thus Q is a distributional solution of the constant-coefficient equation $(\Delta - |\mu|^2)Q = 0$. The operator $P = \Delta - |\mu|^2$ has principal symbol $-|\xi|^2$, which is nonzero for every $\xi \ne 0$; hence P is elliptic. Constant-coefficient elliptic hypoellipticity gives $\operatorname{sing\,supp} Q \subset \operatorname{sing\,supp}(PQ) = \varnothing$, and therefore $Q \in C^\infty(S^\circ)$; see, for example, [3]. Thus Q is represented by a smooth function q on S°. It follows that

$$T = e^{\mu \cdot x}q(x)\, dx,$$

so $\pi|_{S^\circ}$ has the smooth density $p_\pi = e^{\mu \cdot x}q$. Returning to $\mathcal{L}^*T = 0$ gives

$$\mathcal{L}^*p_\pi = \frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0$$

classically in S°, as claimed. □

Corollary 4.8 (strict positivity of the interior stationary density)**.** *Assume $\alpha < 2$. Let π be a stationary distribution for the solution to the submartingale problem with drift μ, and let p_π be the interior density from proposition 4.7. Then*

$$p_\pi(x) > 0 \qquad \text{for every } x \in S^\circ.$$

Proof. By proposition 4.2, $\pi(\{0\}) = 0$ and $\pi(\partial S) = 0$. Hence

$$\int_{S^\circ} p_\pi(x)\, dx = \pi(S^\circ) = 1.$$

In particular, p_π is not identically zero. Because π is a nonnegative measure, its smooth density is nonnegative almost everywhere; continuity therefore gives $p_\pi \geq 0$ at every point of S°. Since p_π satisfies

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0 \qquad \text{in } S^\circ,$$

the strong maximum principle for uniformly elliptic equations with bounded lower-order coefficients applies on every relatively compact connected subdomain of S°; see [2, Chapter 3]. Suppose that $p_\pi(x_0) = 0$ for some $x_0 \in S^\circ$, and set

$$Z_0 := \{x \in S^\circ : p_\pi(x) = 0\}.$$

The set Z_0 is relatively closed by continuity. For every $x \in Z_0$, choose $r_x > 0$ such that $\overline{B}_{2r_x}(x) \Subset S^\circ$. On $B_{2r_x}(x)$, the nonnegative solution p_π attains its minimum zero at the interior point x. The strong maximum principle gives

$$p_\pi \equiv 0 \qquad \text{on } B_{2r_x}(x),$$

so $B_{2r_x}(x) \subset Z_0$. Hence Z_0 is also relatively open. Since S° is connected and $Z_0 \neq \varnothing$, one would have $Z_0 = S^\circ$, which contradicts $\pi(S^\circ) = 1$. Therefore $p_\pi > 0$ throughout S°. □

Proposition 4.9 (ground-state transform and analyticity of the interior density)**.** *Assume $\alpha < 2$. Let π be a stationary distribution for the solution to the submartingale problem with drift μ, and let p_π be the interior density from proposition 4.7. Define*

$$q_\pi(x) := e^{-\mu \cdot x} p_\pi(x), \qquad x \in S^\circ.$$

Then $q_\pi > 0$ in S°, and q_π satisfies

$$\Delta q_\pi = |\mu|^2 q_\pi \qquad \text{in } S^\circ.$$

Consequently both q_π and p_π are real analytic in S°. In particular, when $\mu = 0$, the interior density p_π is a positive harmonic function in the open wedge.

Proof. The positivity of q_π follows from corollary 4.8 and the positivity of the exponential factor. Write

$$p_\pi(x) = e^{\mu \cdot x} q_\pi(x).$$

Then

$$\nabla p_\pi = e^{\mu \cdot x}(\nabla q_\pi + \mu q_\pi),$$

and

$$\Delta p_\pi = e^{\mu \cdot x}\big(\Delta q_\pi + 2\mu \cdot \nabla q_\pi + |\mu|^2 q_\pi\big).$$

Substituting these identities into the interior adjoint equation

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0$$

gives

$$\frac{1}{2}\Delta q_\pi - \frac{1}{2}|\mu|^2 q_\pi = 0,$$

which is exactly $\Delta q_\pi = |\mu|^2 q_\pi$. The operator has analytic constant coefficients, so analytic elliptic regularity implies that every smooth solution is real analytic; see [6]. Hence q_π, and therefore $p_\pi = e^{\mu \cdot x} q_\pi$, is real analytic in S°. □

Remark 4.10 (interior elliptic reductions impose no boundary condition). The conclusions of propositions 4.7 and 4.9 and corollary 4.8 are purely interior statements. They use compactly supported test functions in S° and the fact that any stationary distribution gives no mass to ∂S. No Neumann, oblique, or flux boundary condition for the density is asserted or used in the arguments below. This distinction is necessary in the Lakner–Liu–Reed nonsemimartingale regime, where the process is not represented by boundary local times in the way a semimartingale reflected Brownian motion is.

Proposition 4.11 (separator barrier). *Assume*

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2.$$

Let b be the geometric separator. For any drift $\mu \in \mathbb{R}^2$, choose

$$\gamma > \max\Big\{0, -\frac{2b\cdot\mu}{|b|^2}\Big\}, \qquad \Phi_\gamma(x) = e^{\gamma b\cdot x} - 1. \tag{4.4}$$

Then

$$\Phi_\gamma(0) = 0, \qquad -1 \le \Phi_\gamma \le 0 \text{ on } S, \tag{4.5}$$

$$D_i\Phi_\gamma \ge 0 \quad \text{on } \partial S_i, \qquad \mathcal{L}\Phi_\gamma > 0 \quad \text{on } S^\circ \text{ and by continuous extension to the open faces.} \tag{4.6}$$

Proof. Since $b \cdot x \le 0$ on S, with strict inequality away from the origin, (4.5) follows from the separator property. Also,

$$\nabla\Phi_\gamma = \gamma e^{\gamma b\cdot x} b, \qquad \Delta\Phi_\gamma = \gamma^2 |b|^2 e^{\gamma b\cdot x}.$$

Hence

$$D_i\Phi_\gamma = \gamma e^{\gamma b\cdot x}(b\cdot v_i) \ge 0,$$

and

$$\mathcal{L}\Phi_\gamma = e^{\gamma b\cdot x}\Big(\gamma b\cdot\mu + \frac{1}{2}\gamma^2|b|^2\Big) > 0,$$

by the choice of γ. □

Definition 4.12 (local admissible patching family). Fix a drift $\mu \in \mathbb{R}^2$ and choose γ as in proposition 4.11. A family

$$\{f_\varepsilon : 0 < \varepsilon < \varepsilon_0\} \subset C_b^2(S)$$

is called a *local admissible patching family* for the drift μ if there is a fixed constant $r > 0$ such that, for every sufficiently small $\varepsilon > 0$,

$$f_\varepsilon \equiv \text{const in a neighborhood of } 0, \tag{4.7}$$

$$f_\varepsilon = \Phi_\gamma \qquad \text{on } S \setminus B_{r\varepsilon}, \tag{4.8}$$

$$D_i f_\varepsilon \ge 0 \quad \text{on } \partial S_i, \tag{4.9}$$

$$\mathcal{L} f_\varepsilon \ge 0 \quad \text{on } S^\circ. \tag{4.10}$$

Remark 4.13 (generator inequalities in local patching tests). In definition 4.12, the generator inequality is a pointwise open-wedge inequality, with the open-face values understood through the continuous facewise extensions of the bounded C^2 test. The value assigned at the vertex and on the boundary is irrelevant to stationary integrals by lemma 4.1 and corollary 4.4. Thus the local patching contradiction uses bounded admissible tests only, and no boundary generator term is being introduced.

Theorem 4.14 (patching criterion for nonexistence). *Assume*

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2.$$

Fix a drift vector $\mu \in \mathbb{R}^2$. If there exists a local admissible patching family for μ in the sense of definition 4.12, then the solution to the submartingale problem with drift μ admits no stationary distribution.

Proof. Let π be stationary and let f_ε be a local admissible patching family. By proposition 4.2,

$$0 \ge \int_S \mathcal{L} f_\varepsilon \, d\pi. \tag{4.11}$$

Using definition 4.12 and proposition 4.11,

$$\int_S \mathcal{L} f_\varepsilon \, d\pi \geq \int_{S \setminus B_{r\varepsilon}} \mathcal{L}\Phi_\gamma \, d\pi. \tag{4.12}$$

By proposition 4.11 and the boundary-null integration convention, the open-wedge integrand $\mathcal{L}\Phi_\gamma$ is bounded, nonnegative, and strictly positive on S°. Since $\pi(\partial S) = 0$ and $\pi(\{0\}) = 0$, it is strictly positive π-almost everywhere on $S \setminus \{0\}$. The strict positivity gives a positive integral: up to the π-null boundary,

$$S \setminus \{0\} = \bigcup_{n=1}^{\infty} \left((S \setminus \{0\}) \cap \left\{ \mathcal{L}\Phi_\gamma \geq \frac{1}{n} \right\} \right),$$

and therefore one of the sets in this union has positive π-mass. Hence

$$\int_{S \setminus \{0\}} \mathcal{L}\Phi_\gamma \, d\pi > 0.$$

Moreover, $\mathbf{1}_{S \setminus B_{r\varepsilon}} \uparrow \mathbf{1}_{S \setminus \{0\}}$ as $\varepsilon \downarrow 0$. Put

$$I_\gamma := \int_{S \setminus \{0\}} \mathcal{L}\Phi_\gamma \, d\pi > 0.$$

The monotone convergence theorem gives

$$\lim_{\varepsilon \downarrow 0} \int_{S \setminus B_{r\varepsilon}} \mathcal{L}\Phi_\gamma \, d\pi = I_\gamma.$$

Hence there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$,

$$\int_{S \setminus B_{r\varepsilon}} \mathcal{L}\Phi_\gamma \, d\pi > \frac{I_\gamma}{2} > 0.$$

Together with (4.12), this yields $\int_S \mathcal{L} f_\varepsilon \, d\pi > I_\gamma / 2$, contradicting (4.11). □

Definition 4.15 (vanishing-core admissible contradiction family)**.** Let $H : S \to [0, \infty)$ be continuous and satisfy

$$H(0) = 0, \qquad H(z) > 0 \quad (z \neq 0). \tag{4.13}$$

A family $\{f_\delta : 0 < \delta < \delta_0\} \subset C_b^2(S)$ is called a *vanishing-core admissible contradiction family relative to* H if, for every sufficiently small $\delta > 0$,

$$f_\delta \equiv \text{const} \quad \text{near } 0, \tag{4.14}$$

$$D_i f_\delta \geq 0 \quad \text{on } \partial S_i, \tag{4.15}$$

$$\mathcal{L} f_\delta \geq 0 \quad \text{on } S^\circ, \tag{4.16}$$

$$\mathcal{L} f_\delta(z) > 0 \quad (z \in S^\circ,\ H(z) > \delta). \tag{4.17}$$

Theorem 4.16 (abstract vanishing-core contradiction criterion)**.** *Assume $\alpha < 2$. Let H and $\{f_\delta\}$ be as in the preceding definition. If such a vanishing-core admissible contradiction family exists for a drift μ, then the solution to the Lakner–Liu–Reed submartingale problem with drift μ admits no stationary distribution.*

Proof. Assume, for contradiction, that π is stationary. By proposition 4.2, for every δ,

$$0 \geq \int_S \mathcal{L} f_\delta \, d\pi. \tag{4.18}$$

On the other hand, by the generator conditions in the definition of a vanishing-core admissible contradiction family, the open-wedge integrand $\mathcal{L} f_\delta$ is nonnegative on S° and strictly positive on $S^\circ \cap \{H > \delta\}$. Since $\pi(\partial S) = 0$, and since $H(0) = 0$ makes the vertex irrelevant for $\{H > \delta\}$, the condition $\pi\{H > \delta\} > 0$ implies

$$\pi(A_\delta) > 0, \qquad A_\delta := S^\circ \cap \{H > \delta\}.$$

On A_δ the measurable function $\mathcal{L}f_\delta$ is pointwise positive. Hence

$$A_\delta = \bigcup_{n=1}^{\infty} \left(A_\delta \cap \{\mathcal{L}f_\delta \geq 1/n\}\right),$$

and if $\pi(A_\delta) > 0$, then some set in this increasing union has positive π-mass. Consequently

$$\int_S \mathcal{L}f_\delta \, d\pi > 0, \tag{4.19}$$

which contradicts the stationary inequality for f_δ. Therefore

$$\pi\{H > \delta\} = 0 \qquad (0 < \delta < \delta_0). \tag{4.20}$$

Choose N so large that $1/N < \delta_0$. Since $H(z) > 0$ for every $z \neq 0$,

$$S \setminus \{0\} = \bigcup_{n \geq N} \{H > 1/n\}. \tag{4.21}$$

For every $n \geq N$, the preceding zero-mass conclusion applies with $\delta = 1/n$, and therefore $\pi\{H > 1/n\} = 0$. Thus $\pi(S \setminus \{0\}) = 0$. But proposition 4.2 also gives $\pi(\{0\}) = 0$, contradicting $\pi(S) = 1$. Hence no stationary distribution exists. □

Remark 4.17 (specializations of the criterion). Taking $H(z) = |z|$ and matching the separator barrier Φ_γ outside a shrinking ball gives the patching criterion for nonexistence. Taking $H(z) = c{\cdot}z$ gives the one-dimensional supersolution theorem below. In both cases the admissible supersolution has a generator that is positive outside a core shrinking to the vertex.

5. A direct admissible-supersolution nonexistence cone

This section establishes a direct nonexistence criterion based on bounded functions of one dual coordinate.

Definition 5.1 (direct supersolution cone)**.** Let

$$\mathfrak{B} := \{c \in S_\circ^\vee : c \cdot v_i \geq 0 \text{ for } i = 1, 2\}.$$

Define

$$\mathfrak{M}_{\text{sup}} := \{\mu \in \mathbb{R}^2 : \exists c \in \mathfrak{B} \text{ such that } c \cdot \mu > 0\}.$$

Lemma 5.2 (integrable one-dimensional profiles)**.** *Let $\delta > 0$ and $\rho > 0$. There exists a function $\varphi_\delta \in C^\infty([0,\infty))$ such that*

$$\varphi_\delta(s) = 0 \quad (0 \leq s \leq \delta/2), \qquad \varphi_\delta(s) > 0 \quad (s > \delta),$$
$$\varphi_\delta'(s) \geq -\rho\varphi_\delta(s) \quad (s \geq 0), \qquad \int_0^\infty \varphi_\delta(s)\, ds < \infty.$$

Proof. Choose a nondecreasing function $\psi \in C^\infty([0,\infty))$ such that

$$0 \leq \psi \leq 1, \qquad \psi = 0 \text{ on } [0, \delta/2], \qquad \psi > 0 \text{ on } (\delta, \infty), \qquad \psi = 1 \text{ on } [3\delta/2, \infty).$$

Set

$$\varphi_\delta(s) := \psi(s)e^{-\rho s}.$$

Then φ_δ is smooth, vanishes on $[0, \delta/2]$, is positive for $s > \delta$, and is integrable. Moreover

$$\varphi_\delta'(s) = e^{-\rho s}\psi'(s) - \rho\varphi_\delta(s) \geq -\rho\varphi_\delta(s),$$

because $\psi' \geq 0$. This proves the lemma. □

Lemma 5.3 (one-dimensional bounded supersolution)**.** *Let $c \in S_\circ^\vee$ and suppose*

$$c \cdot v_i \geq 0 \quad (i = 1, 2), \qquad c \cdot \mu > 0.$$

Then for every $\delta > 0$ there exists a function $f_\delta \in C_b^2(S)$ of the form

$$f_\delta(z) = h_\delta(c \cdot z)$$

which is constant in a neighborhood of the vertex, satisfies

$$D_i f_\delta \geq 0 \quad \text{on } \partial S_i, \qquad \mathcal{L} f_\delta \geq 0 \quad \text{on } S^\circ,$$

and such that

$$\mathcal{L} f_\delta(z) > 0 \qquad \text{for } z \in S^\circ \text{ with } c \cdot z > \delta.$$

Proof. Set

$$m := c \cdot \mu > 0, \qquad d := \frac{1}{2}|c|^2.$$

Choose $0 < \rho < m/d$, and let φ_δ be the profile given by lemma 5.2.

Define

$$h_\delta(s) := \int_0^s \varphi_\delta(r)\, dr, \qquad f_\delta(z) := h_\delta(c \cdot z).$$

Then h_δ is bounded, nondecreasing, C^2, and constant on $[0, \delta/2]$. Moreover, $h'_\delta = \varphi_\delta$ and $h''_\delta = \varphi'_\delta$ are bounded, because the profile is smooth, exponentially decaying outside a compact transition region, and constant near the origin. Since $c \in S^\vee_\circ$, the condition $c \cdot z \leq \delta/2$ contains a neighborhood of the vertex in S, so f_δ is constant near the vertex. The composition with the linear map $z \mapsto c \cdot z$ therefore belongs to $C_b^2(S)$.

On ∂S_i,

$$D_i f_\delta(z) = h'_\delta(c \cdot z) c \cdot v_i = \varphi_\delta(c \cdot z) c \cdot v_i \geq 0.$$

Moreover,

$$\mathcal{L} f_\delta(z) = (c \cdot \mu) h'_\delta(c \cdot z) + \frac{1}{2}|c|^2 h''_\delta(c \cdot z) = m\varphi_\delta(s) + d\varphi'_\delta(s),$$

where $s = c \cdot z$. By the differential inequality on φ_δ,

$$\mathcal{L} f_\delta(z) \geq (m - d\rho)\varphi_\delta(s) \geq 0.$$

Because $m - d\rho > 0$ and $\varphi_\delta(s) > 0$ for $s > \delta$, we also get strict positivity whenever $c \cdot z > \delta$. □

Theorem 5.4 (direct nonexistence cone). *Assume $\alpha < 2$. If $\mu \in \mathfrak{M}_{\sup}$, then the solution to the Lakner–Liu–Reed submartingale problem with drift μ admits no stationary distribution.*

Proof. Choose $c \in \mathfrak{B}$ with $c \cdot \mu > 0$. Suppose, for contradiction, that π is a stationary distribution for the drift μ. Fix $\delta > 0$ and let f_δ be given by lemma 5.3. By proposition 4.2,

$$0 \geq \int_S \mathcal{L} f_\delta \, d\pi.$$

On the other hand, lemma 5.3 gives $\mathcal{L} f_\delta \geq 0$ on S° and $\mathcal{L} f_\delta > 0$ on $S^\circ \cap \{c \cdot z > \delta\}$. Since $\pi(\partial S) = 0$, the boundary-null integration convention implies that $\mathcal{L} f_\delta \geq 0$ π-almost everywhere. The preceding display therefore forces $\int_S \mathcal{L} f_\delta \, d\pi = 0$. If

$$A_\delta := S^\circ \cap \{c \cdot z > \delta\}$$

had positive π-mass, then, because $\mathcal{L} f_\delta > 0$ pointwise on A_δ, the decomposition

$$A_\delta = \bigcup_{n=1}^\infty \left(A_\delta \cap \left\{ \mathcal{L} f_\delta \geq \frac{1}{n} \right\} \right)$$

would give $\int_S \mathcal{L} f_\delta \, d\pi > 0$, a contradiction. Hence

$$\pi\big(S^\circ \cap \{c \cdot z > \delta\}\big) = 0.$$

Because $\pi(\partial S) = 0$, this is the same as

$$\pi\{c \cdot z > \delta\} = 0.$$

Since this holds for every $\delta > 0$, applying it with $\delta = 1/n$ and using

$$S \setminus \{0\} = \bigcup_{n \geq 1} \{z \in S : c \cdot z > 1/n\},$$

which follows from $c \in S_\circ^\vee$, gives
$$\pi(S \setminus \{0\}) = 0.$$
This contradicts lemma 4.1, which gives $\pi(\{0\}) = 0$, while π is a probability measure. ∎

Remark 5.5 (geometric interpretation)*.* The cone $\mathfrak{M}_{\mathrm{sup}}$ is the first nonexistence region obtained by a completely explicit admissible supersolution. It is different from the two-dimensional patching problem: here the supersolution is one-dimensional in the coordinate $c \cdot z$, and it works only when a direction c is simultaneously positive on the wedge, nonnegative on both reflection directions, and points outward in the drift sense $c \cdot \mu > 0$.

Proposition 5.6 (maximality of the direct one-dimensional class)**.** *Let $c \in S_\circ^\vee$. Suppose there exists a function $h \in C_b^2([0,\infty))$ such that*
$$h' \geq 0, \qquad h \not\equiv \textit{constant},$$
h is constant in a neighborhood of 0, and
$$f(z) := h(c \cdot z)$$
satisfies
$$D_i f \geq 0 \quad \text{on } \partial S_i, \qquad \mathcal{L} f \geq 0 \quad \text{on } S^\circ.$$
Then
$$c \cdot v_i \geq 0, \qquad i = 1, 2,$$
and
$$c \cdot \mu > 0.$$
Consequently, every nontrivial monotone bounded one-dimensional admissible supersolution of the form $h(c \cdot z)$ arises only from a drift μ in the direct supersolution cone $\mathfrak{M}_{\mathrm{sup}}$.

Proof. Write
$$m := c \cdot \mu, \qquad d := \frac{1}{2}|c|^2 > 0.$$
Since $c \in S_\circ^\vee$, the function $z \mapsto c \cdot z$ maps each open boundary ray ∂S_i onto $(0, \infty)$. Because $h' \geq 0$ and h is not constant, there is $s_0 > 0$ with $h'(s_0) > 0$. For each i choose $z_i \in \partial S_i$ such that $c \cdot z_i = s_0$. Then
$$0 \leq D_i f(z_i) = h'(s_0) c \cdot v_i,$$
so $c \cdot v_i \geq 0$ for $i = 1, 2$.

Next set $p := h'$. Since h is bounded and nondecreasing, $p \geq 0$. Moreover, for every $R > 0$,
$$\int_0^R p(s)\, ds = h(R) - h(0) \leq \sup_{s \geq 0} h(s) - h(0) < \infty,$$
and monotone convergence gives
$$\int_0^\infty p(s)\, ds < \infty.$$
Moreover, since h is constant near 0, there exists $s_1 > 0$ such that $p(s) = 0$ for $0 \leq s \leq s_1$. Choose an interior unit direction $u_* \in S^\circ \cap \mathbb{S}^1$. Since $c \in S_\circ^\vee$, one has $c \cdot u_* > 0$. For every $s > 0$, the point $z_s = (s/(c \cdot u_*))u_*$ lies in S° and satisfies $c \cdot z_s = s$. The open-wedge generator inequality, applied at z_s, gives
$$0 \leq \mathcal{L} f(z_s) = m h'(s) + d h''(s) = m p(s) + d p'(s).$$
Thus
$$p'(s) + \frac{m}{d} p(s) \geq 0.$$
Let $k := m/d$ and define
$$G(s) := e^{ks} p(s).$$

Then

$$G'(s) = e^{ks}\big(p'(s) + kp(s)\big) \geq 0.$$

Since $p = 0$ near 0, $G = 0$ near 0; hence G is nonnegative and nondecreasing.

If $m \leq 0$, then $k \leq 0$. Since h is not constant, p is not identically zero, so there is s_2 with $G(s_2) > 0$. For all $s \geq s_2$,

$$G(s) \geq G(s_2) > 0,$$

and therefore

$$p(s) = e^{-ks}G(s) \geq G(s_2)e^{-ks}.$$

When $k \leq 0$, the right-hand side is not integrable over $[s_2, \infty)$. This contradicts $p \in L^1(0, \infty)$. Hence $m > 0$, i.e. $c \cdot \mu > 0$. □

Remark 5.7 (sharpness within the monotone one-dimensional class). Lemma 5.3 and proposition 5.6 show that $\mathfrak{M}_{\text{sup}}$ is exactly the drift region accessible to bounded monotone supersolutions of the form $h(c \cdot z)$. Thus a drift outside $\mathfrak{M}_{\text{sup}}$ cannot be excluded by a test in this particular monotone bounded-profile class. This conclusion concerns the class, not the dimension of every alternative argument: at $\alpha = 1$ the remaining reflection line is excluded by compactly supported neutral tests of the single coordinate $n_{\mathscr{L}} \cdot z$, whereas in the strict regime the remaining cone is treated by the genuinely two-dimensional Varadhan–Williams gauge; see proposition 5.14 and theorem 7.12.

Proposition 5.8 (nonemptiness of the direct supersolution cone)**.** *Assume* $0 < \xi < \pi$*. Let*

$$V := \text{cone}\{v_1, v_2\}.$$

Then

$$\mathfrak{B} \neq \varnothing \quad \Longleftrightarrow \quad V \cap (-S) = \{0\}. \tag{5.1}$$

Equivalently, a direct admissible supersolution depending on one linear coordinate exists exactly when the cone generated by the reflection directions contains no nonzero vector pointing into the opposite wedge.

Proof. Suppose first that $c \in \mathfrak{B}$. If there were a nonzero vector $w \in V \cap (-S)$, then $-w \in S \setminus \{0\}$. Since $c \in S_\circ^\vee$, we would have

$$c \cdot (-w) > 0,$$

and hence $c \cdot w < 0$. On the other hand, $w \in V = \text{cone}\{v_1, v_2\}$ and $c \cdot v_i \geq 0$ for $i = 1, 2$, so $c \cdot w \geq 0$, a contradiction. Thus $V \cap (-S) = \{0\}$.

Conversely, assume $V \cap (-S) = \{0\}$. We use the finite-dimensional Farkas alternative to obtain the required mixed strict/weak separator. Consider the system

$$c \cdot u_1 \geq 1, \qquad c \cdot u_2 \geq 1, \qquad c \cdot v_1 \geq 0, \qquad c \cdot v_2 \geq 0. \tag{5.2}$$

If this system were infeasible, Farkas' lemma would give nonnegative numbers $\lambda_1, \lambda_2, \beta_1, \beta_2$, with $\lambda_1 + \lambda_2 > 0$, such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \beta_1 v_1 + \beta_2 v_2 = 0.$$

Then

$$\beta_1 v_1 + \beta_2 v_2 = -(\lambda_1 u_1 + \lambda_2 u_2)$$

is a nonzero vector in $V \cap (-S)$, contradicting the hypothesis. Hence (5.2) is feasible. Any feasible c satisfies $c \cdot z > 0$ for every $z \in S \setminus \{0\}$, because $S = \text{cone}\{u_1, u_2\}$ is pointed and both inequalities on the wedge generators are strict after normalization. Thus $c \in S_\circ^\vee$, and the two inequalities $c \cdot v_i \geq 0$ give $c \in \mathfrak{B}$. Therefore $\mathfrak{B} \neq \varnothing$. □

Remark 5.9 (geometric meaning). The Lyapunov existence cone is available under the opposite separation condition $\text{cone}\{v_1, v_2\} \cap S = \{0\}$, which is precisely the convex-regime geometry used to construct the separator b. The direct admissible-supersolution cone instead requires

$\operatorname{cone}\{v_1, v_2\} \cap (-S) = \{0\}$. Thus the two one-dimensional criteria live on opposite geometric sides of the reflection cone.

Proposition 5.10 (borderline geometry when $\alpha = 1$). *Assume*

$$0 < \xi < \pi, \qquad \alpha = 1.$$

Let

$$\mathscr{L} := \operatorname{cone}\{v_1, v_2\}.$$

Then $\mathscr{L}$ is a line through the origin and $\mathscr{L} \cap S = \{0\}$. There is a unique open half-plane bounded by $\mathscr{L}$ that contains $S \setminus \{0\}$. Let $n_{\mathscr{L}}$ be the unit normal to $\mathscr{L}$ pointing into that half-plane, so that

$$n_{\mathscr{L}} \cdot z > 0 \quad (z \in S \setminus \{0\}), \qquad n_{\mathscr{L}} \cdot v_i = 0, \quad i = 1, 2.$$

Then

$$\mathfrak{A} = \{t n_{\mathscr{L}} : t > 0\}, \qquad \mathfrak{B} = \{t n_{\mathscr{L}} : t > 0\}.$$

Consequently,

$$\mathfrak{M}_{\mathrm{Lyap}} = \{\mu : n_{\mathscr{L}} \cdot \mu < 0\}, \qquad \mathfrak{M}_{\mathrm{sup}} = \{\mu : n_{\mathscr{L}} \cdot \mu > 0\},$$

and the set remaining after the two one-dimensional criteria is exactly the reflection line

$$\{\mu : n_{\mathscr{L}} \cdot \mu = 0\} = \mathscr{L}.$$

Proof. The first statement is the borderline part of the Lakner–Liu–Reed geometric classification. Since $\mathscr{L} \cap S = \{0\}$ and S is a closed convex wedge of opening angle less than π, all nonzero points of S lie in one of the two open half-planes bounded by $\mathscr{L}$. This determines $n_{\mathscr{L}}$ uniquely.

Because $v_1, v_2 \in \mathscr{L}$, one has $n_{\mathscr{L}} \cdot v_i = 0$ for $i = 1, 2$. Hence every positive multiple of $n_{\mathscr{L}}$ belongs both to $\mathfrak{A}$ and to $\mathfrak{B}$.

Conversely, let $a \in \mathfrak{A}$. Since $\mathscr{L} = \operatorname{cone}\{v_1, v_2\}$ is a line, the two reflection directions are opposite generators of the same line. Thus the two inequalities $a \cdot v_1 \le 0$ and $a \cdot v_2 \le 0$ force a to vanish on $\mathscr{L}$. Therefore a is a scalar multiple of one of the two unit normals to $\mathscr{L}$. The condition $a \cdot z > 0$ on $S \setminus \{0\}$ selects precisely the positive multiples of $n_{\mathscr{L}}$. Thus $\mathfrak{A} = \{t n_{\mathscr{L}} : t > 0\}$.

Now let $c \in \mathfrak{B}$. Write $v_2 = -\rho v_1$ with $\rho > 0$. The two inequalities

$$c \cdot v_1 \ge 0, \qquad c \cdot v_2 = -\rho\, c \cdot v_1 \ge 0$$

force $c \cdot v_1 = 0$, and hence c vanishes on the line $\mathscr{L}$. Thus c is a scalar multiple of one of its two unit normals. The condition $c \in S^{\vee}_{\circ}$ selects the positive multiple of $n_{\mathscr{L}}$. Therefore $\mathfrak{B} = \{t n_{\mathscr{L}} : t > 0\}$.

The formulas for $\mathfrak{M}_{\mathrm{Lyap}}$ and $\mathfrak{M}_{\mathrm{sup}}$ now follow directly from their definitions. The boundary set between the two open half-planes is the line $n_{\mathscr{L}} \cdot \mu = 0$, which is exactly $\mathscr{L}$. □

Remark 5.11. The two one-dimensional criteria cover the two open half-planes: the Lyapunov half-plane gives existence and the opposite half-plane gives direct nonexistence. The next proposition rules out the boundary reflection line by a projection argument. The required one-dimensional distributional fact is stated first.

Lemma 5.12 (finite half-line measures with zero second derivative). *Let m be a finite nonnegative Borel measure on $(0, \infty)$. If*

$$\int_{(0,\infty)} \varphi''(s)\, m(ds) = 0 \qquad \text{for every } \varphi \in C_c^2((0, \infty)),$$

then $m = 0$.

Proof. The measure m defines a distribution T_m on $(0, \infty)$, and the displayed identity says that $T_m'' = 0$. We record the elementary distributional integration step. Put $U = T_m'$. Then $U' = 0$. Fix $\rho \in C_c^\infty((0, \infty))$ with $\int \rho = 1$. If $\psi \in C_c^\infty((0, \infty))$ has integral zero, then

$$\chi(s) := \int_0^s \psi(r)\, dr$$

belongs to $C_c^\infty((0,\infty))$: it vanishes near zero because ψ does, and it vanishes for all sufficiently large s because $\int_0^\infty \psi = 0$. Moreover, $\chi' = \psi$. Hence

$$U(\psi) = -U'(\chi) = 0.$$

For arbitrary $\varphi \in C_c^\infty((0,\infty))$, the function $\varphi - (\int \varphi)\rho$ has integral zero. Therefore

$$U(\varphi) = a \int_0^\infty \varphi(s)\, ds, \qquad a := U(\rho),$$

so U is the constant distribution a. If $s \mapsto as$ also denotes the distribution induced by that smooth function, set

$$V := T_m - as.$$

Then $V' = 0$. Retain the test function $\rho \in C_c^\infty((0,\infty))$ with $\int \rho = 1$ and define $b := V(\rho)$. For arbitrary $\varphi \in C_c^\infty((0,\infty))$, put

$$\psi := \varphi - \left(\int_0^\infty \varphi(s)\, ds \right) \rho.$$

Then $\int \psi = 0$, and the compactly supported primitive $\chi(s) = \int_0^s \psi(r)\, dr$ satisfies $\chi' = \psi$. Therefore

$$V(\psi) = -V'(\chi) = 0,$$

and hence

$$V(\varphi) = b \int_0^\infty \varphi(s)\, ds.$$

Consequently

$$T_m(\varphi) = \int_0^\infty (as + b)\varphi(s)\, ds, \qquad \varphi \in C_c^\infty((0,\infty)).$$

Thus m is absolutely continuous with affine density $as + b$ on $(0,\infty)$. Since m is nonnegative, the continuous affine function $as + b$ is nonnegative for Lebesgue-almost every $s > 0$. If it were negative at one point, continuity would make it negative on an interval of positive Lebesgue measure, a contradiction; hence it is nonnegative everywhere on $(0,\infty)$. This forces $a \geq 0$, since otherwise the affine density is negative for all sufficiently large s, and it forces $b \geq 0$, by letting $s \downarrow 0$. If $a > 0$, then $m((0,R))$ grows quadratically in R; if $a = 0$ but $b > 0$, then $m((0,R))$ grows linearly in R. Since m is finite on the whole half-line, both alternatives are impossible. Therefore $a = b = 0$ and $m = 0$. ☐

Proposition 5.13 (neutral linear projection contradiction)**.** *Assume $\alpha < 2$. Let $\ell \in \mathbb{R}^2$ satisfy*

$$\ell \cdot z > 0 \quad (z \in S \setminus \{0\}), \qquad \ell \cdot v_i = 0 \quad (i = 1, 2), \qquad \ell \cdot \mu = 0.$$

Then the Lakner–Liu–Reed submartingale problem with drift μ admits no stationary distribution.

Proof. Suppose, to the contrary, that π is stationary. Let $\varphi \in C_c^2((0,\infty))$, extend φ by zero to a neighborhood of the endpoint, and set

$$f(z) := \varphi(\ell \cdot z), \qquad z \in S.$$

Because ℓ is strictly positive on the compact set $S \cap \mathbb{S}^1$, there is $a_\ell > 0$ with $\ell \cdot z \geq a_\ell |z|$ for $z \in S$. Hence the support of f is contained in a compact subset of $S \setminus \{0\}$. In particular $f \in C_b^2(S)$, is constant on a neighborhood of the vertex, and satisfies

$$D_i f(z) = \varphi'(\ell \cdot z)\, \ell \cdot v_i = 0 \qquad \text{on } \partial S_i.$$

Thus f is a neutral admissible test. By corollary 4.3,

$$0 = \int_S \mathcal{L} f \, d\pi.$$

The open-wedge generator is

$$\mathcal{L}f(z) = (\ell \cdot \mu)\varphi'(\ell \cdot z) + \frac{1}{2}|\ell|^2\varphi''(\ell \cdot z) = \frac{1}{2}|\ell|^2\varphi''(\ell \cdot z),$$

and the boundary-null convention allows this identity to be integrated against π. Therefore

$$\int_S \varphi''(\ell \cdot z)\,\pi(dz) = 0 \qquad \text{for all } \varphi \in C_c^2((0,\infty)).$$

Let m_ℓ be the push-forward of π under $z \mapsto \ell \cdot z$. Since $\ell \cdot z > 0$ on $S \setminus \{0\}$ and $\pi(\{0\}) = 0$, this is a probability measure on $(0,\infty)$. The last display says that its second distributional derivative is zero on the half-line. By lemma 5.12, $m_\ell = 0$, contradicting $m_\ell((0,\infty)) = 1$. □

Proposition 5.14 (critical reflection line at $\alpha = 1$)**.** *Assume*

$$0 < \xi < \pi, \qquad \alpha = 1.$$

Let $\mathscr{L} = \operatorname{cone}\{v_1, v_2\}$ *and let* $n_{\mathscr{L}}$ *be the unit normal from proposition 5.10. If*

$$n_{\mathscr{L}} \cdot \mu = 0,$$

then the solution to the Lakner–Liu–Reed submartingale problem with drift μ *admits no stationary distribution.*

Proof. By proposition 5.10, $n_{\mathscr{L}} \cdot z > 0$ for every $z \in S \setminus \{0\}$. Since $\mathscr{L} = \operatorname{cone}\{v_1, v_2\}$ and $n_{\mathscr{L}} \perp \mathscr{L}$,

$$n_{\mathscr{L}} \cdot v_i = 0, \qquad i = 1, 2.$$

The critical-line hypothesis gives $n_{\mathscr{L}} \cdot \mu = 0$. Therefore proposition 5.13, applied with $\ell = n_{\mathscr{L}}$, rules out a stationary distribution. The proof uses only compactly supported neutral functions of $n_{\mathscr{L}} \cdot z$, not the unbounded normal coordinate itself. □

6. Lyapunov existence theorem and the existence cone

The analytic existence criterion below assumes only $\alpha < 2$ and the sign conditions on a single Lyapunov direction. The subsequent cone identities use $0 < \xi < \pi$ and $1 \le \alpha < 2$.

The Foster–Lyapunov argument uses the following sign convention. If a test function U_n satisfies $D_i U_n \le 0$, then definition 2.1 applies to $-U_n$; equivalently, the corresponding process for U_n is a *super*martingale. As on the nonexistence side, no unbounded Lyapunov function is inserted directly into the submartingale problem; the following two lemmas establish the norm-like and truncation properties used throughout this section.

Lemma 6.1 (positive linear heights are norm-like)**.** *Let* $a \in \mathbb{R}^2$ *satisfy*

$$a \cdot z > 0 \quad (z \in S \setminus \{0\}).$$

Then there is $m_a > 0$ *such that*

$$a \cdot z \ge m_a |z|, \qquad z \in S.$$

Consequently, every sublevel set $\{z \in S : a \cdot z \le r\}$ *is compact. More generally, if a continuous function* $U : S \to [0,\infty)$ *satisfies* $U(z) \to \infty$ *whenever* $a \cdot z \to \infty$*, then* U *is norm-like on* S*.*

Proof. The set $S \cap \mathbb{S}^1$ is compact. The continuous function $u \mapsto a \cdot u$ is strictly positive on this compact set, so

$$m_a := \min_{u \in S \cap \mathbb{S}^1} a \cdot u > 0.$$

For $z \ne 0$, write $z = |z|u$ with $u \in S \cap \mathbb{S}^1$. Then $a \cdot z = |z| a \cdot u \ge m_a |z|$. Hence

$$\{z \in S : a \cdot z \le r\} \subset S \cap \overline{B}_{r/m_a}(0),$$

and this sublevel set is closed in the compact set on the right. Thus it is compact. If $U(z) \to \infty$ whenever $a \cdot z \to \infty$, then for each R there is $M_R < \infty$ such that $\{U \le R\} \subset \{a \cdot z \le M_R\}$. The preceding compactness shows that $\{U \le R\}$ is compact, so U is norm-like. □

Lemma 6.2 (bounded concave truncation profiles)**.** *There exists a sequence $\{\phi_n\}_{n\geq 1}$ of functions in $C^\infty([0,\infty))$ with the following properties. For each n,*

$$\phi_n(0) = 0, \qquad 0 \leq \phi_n(s) \leq s, \qquad 0 \leq \phi_n'(s) \leq 1, \qquad \phi_n''(s) \leq 0, \tag{6.1}$$

$$\phi_n(s) = s \quad (0 \leq s \leq n), \qquad \phi_n(s) = \text{constant} \quad (s \geq n+2), \tag{6.2}$$

and, for every fixed $s \geq 0$,

$$\phi_n(s) \to s, \qquad \phi_n'(s) \to 1. \tag{6.3}$$

Proof. Choose a function $\rho \in C^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\rho' \leq 0$, $\rho(u) = 1$ for $u \leq 0$, and $\rho(u) = 0$ for $u \geq 2$. Define

$$\phi_n(s) := \int_0^s \rho(r-n)\, dr, \qquad s \geq 0.$$

Then $\phi_n \in C^\infty([0,\infty))$, $\phi_n' = \rho(\cdot - n)$, and $\phi_n'' = \rho'(\cdot - n) \leq 0$. The plateau identities follow from the two plateau regions of ρ. Since $0 \leq \phi_n' \leq 1$ and $\phi_n(0) = 0$, one has $0 \leq \phi_n(s) \leq s$. Finally, for each fixed s, once $n > s$, the point s lies in the identity region, so $\phi_n(s) = s$ and $\phi_n'(s) = 1$. ☐

Lemma 6.3 (bounded concave truncations for negative-boundary Lyapunov gauges)**.** *Assume $\alpha < 2$. Let $V : S \to [0,\infty)$ be continuous, facewise C^2 and locally C^2-extendable away from the vertex, constant in a neighborhood of the vertex, and norm-like. Suppose*

$$D_i V \leq 0 \quad \text{on } \partial S_i, \qquad i = 1, 2,$$

through open-face limits. Let ϕ_n be one of the bounded concave truncation profiles from lemma 6.2, and set $V_n = \phi_n(V)$. Then $V_n \in C_b^2(S)$, V_n is constant near the vertex, and

$$D_i V_n \leq 0 \quad \text{on } \partial S_i.$$

Thus $-V_n$ is an admissible test function, and

$$V_n(Z_t) - V_n(Z_0) - \int_0^t \mathcal{L} V_n(Z_s)\, ds$$

is a supermartingale. If, in addition, for some compact $K \subset S$ and constants $\delta, C > 0$,

$$\mathcal{L} V \leq -\delta V + C\mathbf{1}_K \quad \text{on } S^\circ,$$

then, on S°,

$$\mathcal{L} V_n \leq -\delta\, \phi_n'(V) V + C\mathbf{1}_K.$$

The displayed generator inequalities hold in the open wedge; their pathwise time-integral consequences for the bounded truncations follow from corollary 4.5, and their stationary integral consequences follow from corollary 4.4.

Proof. For fixed n, the function V_n is bounded. Since V is constant near the vertex, so is V_n. The possible nonconstant region of V_n is contained in

$$\{V < n+2\} \subset \{V \leq n+2\},$$

and this latter set is compact because V is norm-like. Choose a vertex neighborhood on which V is constant. Outside that neighborhood, the compact set $\{V \leq n+2\}$ is covered by finitely many neighborhoods on which the local C^2-extensions of V are available, and the first and second derivatives of V are bounded on this compact set. The chain rule and the fixed bounded derivatives of ϕ_n therefore give a bounded C^2 extension of V_n on the nonconstant region; the two constant plateaus complete the extension near the vertex and outside $\{V < n+2\}$. Hence $V_n \in C_b^2(S)$. No uniform C_b^2-bound in n is asserted or needed. The boundary sign follows from

$$D_i V_n = \phi_n'(V) D_i V \leq 0.$$

Therefore $-V_n$ is admissible. Applying definition 2.1 to $-V_n$ gives that

$$-V_n(Z_t) + V_n(Z_0) + \int_0^t \mathcal{L}V_n(Z_s)\, ds$$

is a submartingale, equivalently the displayed process with sign reversed is a supermartingale.

Finally, the chain rule in the open wedge gives

$$\mathcal{L}V_n = \phi_n'(V)\mathcal{L}V + \frac{1}{2}\phi_n''(V)|\nabla V|^2 \le \phi_n'(V)\mathcal{L}V,$$

because $\phi_n'' \le 0$. Combining this with the drift inequality for V and $0 \le \phi_n' \le 1$ yields

$$\mathcal{L}V_n \le -\delta\, \phi_n'(V)V + C\mathbf{1}_K.$$

The asserted integration conventions follow from corollaries 4.4 and 4.5. □

Lemma 6.4 (averaged Foster bound from bounded truncations)**.** *Assume $\alpha < 2$, and let $(Z_t)_{t\ge 0}$ be the Markov family started from a fixed state $z \in S$. Let $V : S \to [0, \infty)$ be a norm-like gauge and let $V_n = \phi_n(V)$ be bounded truncations as in lemma 6.3. Suppose that, for some $\delta, C > 0$ and compact $K \subset S$, each V_n satisfies the supermartingale localization*

$$V_n(Z_t) - V_n(Z_0) - \int_0^t \mathcal{L}V_n(Z_s)\, ds \quad \textit{is a supermartingale,}$$

and the open-wedge generator bound

$$\mathcal{L}V_n \le -\delta\, \phi_n'(V)V + C\mathbf{1}_K.$$

For each fixed n, the right-hand side is bounded and Borel. Indeed, the truncation profiles satisfy

$$0 \le \phi_n'(s)s \le n+2, \qquad s \ge 0,$$

because $0 \le \phi_n' \le 1$ and $\phi_n' = 0$ on $[n+2, \infty)$. Then, for every $T > 0$,

$$\frac{1}{T}\int_0^T \mathbb{E}_z[V(Z_s)]\, ds \le \frac{V(z)}{\delta T} + \frac{C}{\delta}.$$

Proof. Set

$$G_n := -\delta\, \phi_n'(V)V + C\mathbf{1}_K.$$

The preceding boundedness observation makes G_n a bounded Borel function. By the pathwise boundary-null convention, corollary 4.5, the open-wedge inequality $\mathcal{L}V_n \le G_n$ may be integrated along the path.

The supermartingale inequality gives

$$\mathbb{E}_z[V_n(Z_T)] - V_n(z) - \mathbb{E}_z \int_0^T \mathcal{L}V_n(Z_s)\, ds \le 0,$$

and hence, since $V_n \ge 0$,

$$-V_n(z) \le \mathbb{E}_z \int_0^T \mathcal{L}V_n(Z_s)\, ds.$$

Combining this with the integrated upper bound by G_n yields

$$-V_n(z) \le -\delta \int_0^T \mathbb{E}_z[\phi_n'(V(Z_s))V(Z_s)]\, ds + C\int_0^T \mathbb{P}_z(Z_s \in K)\, ds.$$

Since the last integral is at most T,

$$\delta \int_0^T \mathbb{E}_z[\phi_n'(V(Z_s))V(Z_s)]\, ds \le V_n(z) + CT.$$

Since $0 \le \phi_n \le \mathrm{id}$, we have $V_n(z) \le V(z)$. Dividing by δT yields

$$\frac{1}{T}\int_0^T \mathbb{E}_z[\phi_n'(V(Z_s))V(Z_s)]\, ds \le \frac{V(z)}{\delta T} + \frac{C}{\delta}.$$

For each fixed sample point and time, $\phi_n'(V(Z_s))V(Z_s) \to V(Z_s)$, because $\phi_n'(r) \to 1$ for every finite $r \geq 0$. Fatou's lemma gives the asserted averaged bound. Every function to which the submartingale problem is applied is bounded; Fatou's lemma is used only after the bounded-test estimates have been obtained. □

Lemma 6.5 (stationary moment bound from bounded truncations). *Assume $\alpha < 2$, and let π be a stationary distribution for the drift μ. Let $V : S \to [0, \infty)$, $V_n = \phi_n(V)$, δ, C, K be as in lemma 6.4, and assume in addition that $-V_n$ is admissible for every n. If*

$$\mathcal{L}V_n \leq -\delta\, \phi_n'(V)V + C\mathbf{1}_K \quad \text{on } S^\circ,$$

then

$$\int_S V\, d\pi \leq \frac{C}{\delta}.$$

Proof. For fixed n, the right-hand side is bounded and Borel by the same estimate

$$0 \leq \phi_n'(s)s \leq n + 2.$$

Since $-V_n$ is admissible, proposition 4.2 gives

$$\int_S \mathcal{L}(-V_n)\, d\pi \leq 0,$$

that is,

$$0 \leq \int_S \mathcal{L}V_n\, d\pi.$$

Using corollary 4.4 to integrate the open-wedge upper bound gives

$$0 \leq \int_S \mathcal{L}V_n\, d\pi \leq -\delta \int_S \phi_n'(V)V\, d\pi + C\pi(K) \leq -\delta \int_S \phi_n'(V)V\, d\pi + C.$$

Thus

$$\int_S \phi_n'(V)V\, d\pi \leq \frac{C}{\delta}.$$

Since $\phi_n'(r)r \to r$ for every finite $r \geq 0$, Fatou's lemma yields the assertion. Thus the stationary inequality is applied only to bounded admissible tests, followed by Fatou's lemma. □

The following two lemmas give the compactness and Krylov–Bogoliubov invariance steps in the standard occupation-measure argument for Feller Markov processes [1].

Lemma 6.6 (norm-like averaged bounds imply tight occupation measures). *Let $(P_t)_{t\geq 0}$ be a Markov semigroup on the locally compact state space S, and fix $z \in S$. Let*

$$\bar{\nu}_T^z := \frac{1}{T}\int_0^T P_s(z, \cdot)\, ds, \qquad T > 0,$$

be the occupation averages. Suppose that $V : S \to [0, \infty)$ is norm-like and that there are constants $A, B < \infty$ such that

$$\int_S V\, d\bar{\nu}_T^z \leq A + \frac{B}{T}, \qquad T > 0.$$

Then the family $\{\bar{\nu}_T^z : T \geq 1\}$ is tight.

Proof. Let $\epsilon > 0$. Since V is norm-like, there exists $R < \infty$ such that

$$K_R := \{y \in S : V(y) \leq R\}$$

is compact and $(A + B)/R < \epsilon$. For every $T \geq 1$, Markov's inequality gives

$$\bar{\nu}_T^z(K_R^c) \leq \frac{1}{R}\int_S V\, d\bar{\nu}_T^z \leq \frac{A + B}{R} < \epsilon.$$

Thus $\{\bar{\nu}_T^z : T \geq 1\}$ is tight. □

Lemma 6.7 (Krylov–Bogoliubov from tight occupation averages)**.** *Let $(P_t)_{t\ge 0}$ be a Feller Markov semigroup on the locally compact Polish space S. Fix $z \in S$, and suppose that the occupation measures*

$$\bar\nu_T^z := \frac{1}{T}\int_0^T P_s(z,\cdot)\,ds$$

are tight. If $\bar\nu_{T_k}^z \Rightarrow \pi$ for some sequence $T_k \to \infty$, then π is stationary.

Proof. Let $g \in C_0(S)$ and fix $t > 0$. Since the semigroup is Feller, $P_t g \in C_0(S)$. Weak convergence gives

$$\int_S P_t g\,d\pi - \int_S g\,d\pi = \lim_{k\to\infty}\left(\int_S P_t g\,d\bar\nu_{T_k}^z - \int_S g\,d\bar\nu_{T_k}^z\right).$$

By the definition of the occupation average and the semigroup property, the expression in parentheses is

$$\begin{aligned}\frac{1}{T_k}\int_0^{T_k}&\big(P_{s+t}g(z) - P_s g(z)\big)\,ds\\ &= \frac{1}{T_k}\left(\int_{T_k}^{T_k+t} P_r g(z)\,dr - \int_0^t P_r g(z)\,dr\right).\end{aligned}$$

Since $|P_r g(z)| \le \|g\|_\infty$, the absolute value is at most $2t\|g\|_\infty/T_k$, which tends to zero. Thus πP_t and π agree on $C_0(S)$. Both are Radon probability measures on the locally compact Polish space S; the Riesz representation theorem therefore gives $\pi P_t = \pi$. Since $t > 0$ was arbitrary, π is stationary. □

Theorem 6.8 (Lyapunov existence criterion)**.** *Assume $\alpha < 2$ so that the Markov family of definition 2.1 with drift is available, unique, and has the Feller property on $C_0(S)$. Suppose there exists $a \in \mathbb{R}^2$ such that*

$$a\cdot z > 0 \quad (z \in S\setminus\{0\}), \tag{6.4}$$

$$a\cdot v_i \le 0, \qquad i = 1,2, \tag{6.5}$$

$$a\cdot\mu < 0. \tag{6.6}$$

Then the solution to the submartingale problem with drift μ admits a stationary distribution.

Proof. We give a direct Foster–Lyapunov proof based on an exponential Lyapunov function. The sign convention requires care: the function constructed below has nonpositive oblique derivatives, so definition 2.1 is applied to its negative.

Choose

$$0 < \beta < -\frac{2a\cdot\mu}{|a|^2}. \tag{6.7}$$

Let $\eta \in C^\infty([0,\infty))$ satisfy

$$\eta(s) = 0 \quad (0 \le s \le \tfrac12), \qquad \eta(s) = 1 \quad (s \ge 1), \qquad \eta' \ge 0. \tag{6.8}$$

Set

$$V(z) := \eta(a\cdot z)e^{\beta a\cdot z}. \tag{6.9}$$

Then $V \ge 0$ and V is constant in a neighborhood of the vertex. Because it is a smooth function of the linear coordinate $a\cdot z$, it is facewise C^2 and locally C^2-extendable away from the vertex. By lemma 6.1, $a\cdot z \to \infty$ whenever $|z| \to \infty$ in S, and the sublevel sets of $a\cdot z$ are compact; hence V is norm-like. Since $a\cdot v_i \le 0$ and $\eta' \ge 0$, on ∂S_i we have

$$D_i V(z) = \big(\eta'(a\cdot z) + \beta\eta(a\cdot z)\big)e^{\beta a\cdot z}\,a\cdot v_i \le 0. \tag{6.10}$$

Thus $-V$ has the admissible boundary sign. Since V is unbounded, we apply the bounded concave truncations of lemma 6.3.

Let ϕ_n be the bounded concave truncation profiles of lemma 6.2, and define

$$V_n := \phi_n(V). \tag{6.11}$$

For each fixed n, the nonconstant part of V_n is contained in the compact set $\{V \le n+2\}$. Therefore $V_n \in C_b^2(S)$ and is constant near the vertex. Moreover,

$$D_i V_n = \phi_n'(V) D_i V \le 0. \tag{6.12}$$

Hence $-V_n$ is admissible. definition 2.1 gives that

$$V_n(Z_t) - V_n(Z_0) - \int_0^t \mathcal{L}V_n(Z_s)\,ds \tag{6.13}$$

is a supermartingale.

A multiplicative drift inequality follows. On the set $\{a \cdot z \ge 1\}$, one has $V = e^{\beta a \cdot z}$, and therefore

$$\mathcal{L}V = \left(\beta a \cdot \mu + \frac{1}{2}\beta^2 |a|^2\right) V. \tag{6.14}$$

By the choice of β, the coefficient

$$\kappa_\beta := \beta a \cdot \mu + \frac{1}{2}\beta^2 |a|^2 \tag{6.15}$$

is strictly negative. Fix, for instance,

$$\delta_\beta := -\frac{1}{2}\kappa_\beta > 0.$$

With

$$K_\beta := \{z \in S : a \cdot z \le 1\}, \tag{6.16}$$

lemma 6.1 shows that K_β is compact. On $S \setminus K_\beta$ the exact identity (6.14) gives

$$\mathcal{L}V = \kappa_\beta V \le -\delta_\beta V.$$

On the compact set K_β, the open-wedge expression $\mathcal{L}V + \delta_\beta V$ is bounded and has continuous open-face extensions; increasing C_β if necessary gives

$$\mathcal{L}V(z) \le -\delta_\beta V(z) + C_\beta \mathbf{1}_{K_\beta}(z), \qquad z \in S^\circ, \tag{6.17}$$

and, by continuous extension, on the open faces. As usual, the value assigned at the vertex is immaterial for later integrals. Applying lemma 6.3 to the drift bound (6.17) gives

$$\mathcal{L}V_n \le -\delta_\beta \phi_n'(V) V + C_\beta \mathbf{1}_{K_\beta}. \tag{6.18}$$

Applying lemma 6.4 with $\delta = \delta_\beta$, $C = C_\beta$, and $K = K_\beta$ gives the uniform averaged bound

$$\frac{1}{T}\int_0^T \mathbb{E}_z[V(Z_s)]\,ds \le \frac{V(z)}{\delta_\beta T} + \frac{C_\beta}{\delta_\beta}. \tag{6.19}$$

Let

$$\bar{\nu}_T^z(\cdot) := \frac{1}{T}\int_0^T P_s(z,\cdot)\,ds. \tag{6.20}$$

Then (6.19) is the averaged moment bound

$$\int_S V\,d\bar{\nu}_T^z \le \frac{V(z)}{\delta_\beta T} + \frac{C_\beta}{\delta_\beta}. \tag{6.21}$$

By lemma 6.1, V is norm-like. Hence lemma 6.6 implies tightness of $\{\bar{\nu}_T^z : T \ge 1\}$. By Prokhorov's theorem, choose a weakly convergent subsequence $\bar{\nu}_{T_k}^z \Rightarrow \pi$ with $T_k \to \infty$. Applying lemma 6.7 to the $C_0(S)$-Feller semigroup shows that π is stationary. This proves the theorem. □

Recall the Lyapunov cone notation from definition 2.4.

Lemma 6.9 (metric-projection separator for a closed convex cone)**.** *Let $C \subset \mathbb{R}^2$ be a nonempty closed convex cone and let $y \notin C$. There is a vector $r \neq 0$ such that*

$$r \cdot x \le 0 \quad (x \in C), \qquad r \cdot y = |r|^2 > 0. \tag{6.22}$$

Equivalently, $a := -r$ satisfies $a \cdot x \ge 0$ for every $x \in C$ and $a \cdot y < 0$.

Proof. Let $d := \operatorname{dist}(y, C)$. Since $0 \in C$, one has $d \le |y|$. Choose $p_n \in C$ with $|y - p_n| \le d + n^{-1}$. Then, for every $n \ge 1$,

$$|p_n| \le |y| + |y - p_n| \le 2|y| + 1.$$

Thus (p_n) lies in the compact set $C \cap \overline{B}_{2|y|+1}(0)$. Passing to a subsequence gives $p_n \to p \in C$, and continuity of the norm gives $|y - p| = d$. Put $r := y - p$. Since $y \notin C$, one has $r \neq 0$. For any $x \in C$ and $t \in [0, 1]$, convexity gives $p + t(x - p) \in C$. The function

$$\phi(t) := |y - p - t(x - p)|^2$$

has a minimum at $t = 0$, and hence

$$0 \le \phi'(0) = -2r \cdot (x - p), \qquad \text{so} \qquad r \cdot (x - p) \le 0.$$

Taking $x = 0$ gives $r \cdot p \ge 0$, while taking $x = 2p \in C$ gives $r \cdot p \le 0$. Thus $r \cdot p = 0$, and the preceding variational inequality reduces to $r \cdot x \le 0$ for every $x \in C$. Finally,

$$r \cdot y = r \cdot (p + r) = |r|^2 > 0.$$

This proves (6.22) and its equivalent formulation. □

Proposition 6.10 (polar description of the Lyapunov existence cone)**.** *Assume $0 < \xi < \pi$ and $1 \le \alpha < 2$. Let*

$$D_0 := \operatorname{cone}\{u_1, u_2, -v_1, -v_2\}.$$

Then

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus D_0.$$

Equivalently, a drift vector μ lies in the Lyapunov stationary-existence region if and only if it is not contained in the closed cone generated by the two wedge rays and the two opposite reflection directions $-v_1, -v_2$.

Proof. First suppose $\mu \in \mathfrak{M}_{\mathrm{Lyap}}$. Then there exists $a \in \mathfrak{A}$ such that $a \cdot \mu < 0$. For each generator of D_0 we have

$$a \cdot u_1 > 0, \qquad a \cdot u_2 > 0, \qquad a \cdot (-v_i) = -a \cdot v_i \ge 0.$$

Thus $a \cdot x \ge 0$ for every $x \in D_0$. If $\mu \in D_0$, this would imply $a \cdot \mu \ge 0$, contradicting $a \cdot \mu < 0$. Hence $\mu \notin D_0$.

Conversely, assume $\mu \notin D_0$. The cone D_0 is finitely generated and therefore closed and convex. Apply lemma 6.9 with $C = D_0$ and $y = \mu$. With $r = \mu - p$ as in that lemma, set $a := -r$. Then

$$a \cdot x \ge 0 \quad (x \in D_0), \qquad a \cdot \mu = -|r|^2 < 0.$$

In particular,

$$a \cdot u_1 \ge 0, \qquad a \cdot u_2 \ge 0, \qquad a \cdot v_i \le 0.$$

The inequalities on u_1, u_2 may be non-strict. The possible non-strict inequalities on the wedge generators are removed by perturbation. In the present convex regime, the geometric separator b exists, and $a_0 := -b$ satisfies

$$a_0 \cdot u_j > 0 \quad (j = 1, 2), \qquad a_0 \cdot v_i \le 0 \quad (i = 1, 2).$$

Put

$$d_\mu := -a \cdot \mu = |r|^2 > 0.$$

If $a_0 \cdot \mu \le 0$, set $\delta_* = 1$; if $a_0 \cdot \mu > 0$, set

$$\delta_* := \min\left\{1, \frac{d_\mu}{2a_0 \cdot \mu}\right\} > 0.$$

For any $0 < \delta < \delta_*$ define

$$a_\delta := a + \delta a_0.$$

Then, for $j = 1, 2$ and $i = 1, 2$,

$$a_\delta \cdot u_j = a \cdot u_j + \delta a_0 \cdot u_j \ge \delta a_0 \cdot u_j > 0,$$

$$a_\delta \cdot v_i = a \cdot v_i + \delta a_0 \cdot v_i \le 0.$$

Moreover, if $a_0 \cdot \mu \le 0$, then $a_\delta \cdot \mu \le -d_\mu < 0$; if $a_0 \cdot \mu > 0$, then

$$a_\delta \cdot \mu = -d_\mu + \delta a_0 \cdot \mu < -\frac{d_\mu}{2} < 0.$$

Thus $a_\delta \in \mathfrak{A}$ and $\mu \in \mathfrak{M}_{\mathrm{Lyap}}$. This proves the identity. □

Proposition 6.11 (Lyapunov existence cone)**.** *Assume $0 < \xi < \pi$ and $1 \le \alpha < 2$. Then $\mathfrak{A}$ is convex and stable under multiplication by positive scalars, $\mathfrak{M}_{\mathrm{Lyap}}$ is an open cone, and for every*

$$\mu \in \mathfrak{M}_{\mathrm{Lyap}} \tag{6.23}$$

the solution to the submartingale problem admits a stationary distribution. Moreover,

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus \mathrm{cone}\{u_1, u_2, -v_1, -v_2\}. \tag{6.24}$$

Proof. The set $\mathfrak{A}$ is an intersection of two open half-spaces and two closed half-spaces, so it is convex and stable under multiplication by positive scalars. The strict inequalities on the wedge generators exclude the zero vector from $\mathfrak{A}$; no argument below uses closure of $\mathfrak{A}$ under multiplication by zero. For each fixed $a \in \mathfrak{A}$, the set $\{\mu : a \cdot \mu < 0\}$ is an open half-space. Hence

$$\mathfrak{M}_{\mathrm{Lyap}} = \bigcup_{a \in \mathfrak{A}} \{\mu \in \mathbb{R}^2 : a \cdot \mu < 0\}$$

is an open cone. The existence claim follows from theorem 6.8. The displayed geometric identity is precisely proposition 6.10. □

Corollary 6.12 (complete borderline classification)**.** *Assume $0 < \xi < \pi$ and $\alpha = 1$. With $n_{\mathscr{L}}$ as in proposition 5.10, the solution to the submartingale problem admits a stationary distribution for $n_{\mathscr{L}} \cdot \mu < 0$, and admits no stationary distribution for $n_{\mathscr{L}} \cdot \mu \ge 0$.*

Proof. If $n_{\mathscr{L}} \cdot \mu < 0$, then $\mu \in \mathfrak{M}_{\mathrm{Lyap}}$ by proposition 5.10, and proposition 6.11 gives existence. If $n_{\mathscr{L}} \cdot \mu > 0$, then $\mu \in \mathfrak{M}_{\mathrm{sup}}$, and theorem 5.4 gives nonexistence. The boundary case $n_{\mathscr{L}} \cdot \mu = 0$ is proposition 5.14. □

Corollary 6.13 (strict Lyapunov complement)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Then

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus \mathrm{cone}\{-v_1, -v_2\}.$$

Consequently, every drift outside the closed reflection cone

$$K_{\mathrm{str}} := \mathrm{cone}\{-v_1, -v_2\}$$

admits a stationary distribution.

Proof. By lemma 2.2,

$$S = \mathrm{cone}\{u_1, u_2\} \subset \mathrm{cone}\{-v_1, -v_2\}.$$

Therefore

$$\mathrm{cone}\{u_1, u_2, -v_1, -v_2\} = \mathrm{cone}\{-v_1, -v_2\}.$$

The conclusion now follows from proposition 6.11. □

The strict geometry identifies the two one-dimensional regions explicitly: the Lyapunov region is the complement of the closed strict cone, whereas the direct one-dimensional admissible-supersolution region is empty.

Proposition 6.14 (strict convex geometry and the collapse of one-dimensional criteria)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Then

$$S \subset \operatorname{cone}\{-v_1, -v_2\}.$$

Consequently,

$$\operatorname{cone}\{u_1, u_2, -v_1, -v_2\} = \operatorname{cone}\{-v_1, -v_2\}$$

and hence

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus \operatorname{cone}\{-v_1, -v_2\}.$$

Moreover,

$$\mathfrak{B} = \varnothing, \qquad \mathfrak{M}_{\mathrm{sup}} = \varnothing.$$

Thus, in the strict case $\alpha > 1$, the two monotone one-coordinate criteria leave exactly the cone

$$\operatorname{cone}\{-v_1, -v_2\}$$

undecided. The nonexistence proof used below on that cone is the genuinely two-dimensional Varadhan–Williams construction; the statement here concerns the reach of the two specified one-coordinate classes and does not assert a general impossibility theorem for every other scalar construction.

Proof. The first inclusion is lemma 2.2. Since $u_1, u_2 \in S$, both wedge generators lie in $\operatorname{cone}\{-v_1, -v_2\}$. Therefore

$$\operatorname{cone}\{u_1, u_2, -v_1, -v_2\} = \operatorname{cone}\{-v_1, -v_2\}.$$

Combining this identity with proposition 6.10 gives the displayed formula for $\mathfrak{M}_{\mathrm{Lyap}}$.

It remains to prove that $\mathfrak{B} = \varnothing$. From the first inclusion we get, by multiplying by -1,

$$-S \subset \operatorname{cone}\{v_1, v_2\}.$$

Since $-S$ contains nonzero vectors,

$$\operatorname{cone}\{v_1, v_2\} \cap (-S) \neq \{0\}.$$

By proposition 5.8, this is equivalent to $\mathfrak{B} = \varnothing$. Hence $\mathfrak{M}_{\mathrm{sup}} = \varnothing$ by definition. The final statement follows from the displayed formula for $\mathfrak{M}_{\mathrm{Lyap}}$. □

Remark 6.15. In the strict regime, no drift in $\operatorname{cone}\{-v_1, -v_2\}$ is detected by this one-dimensional admissible-supersolution criterion.

6.1. **A radial Foster–Lyapunov criterion.** The preceding Lyapunov criterion uses a linear height function. The next criterion gives a distinct sufficient condition. It uses the Euclidean radius and applies in geometries where the drift is uniformly inward in the radial sense and the reflections do not increase the radial coordinate on the boundary.

Let

$$\nu_*(\mu) := \sup_{u \in S \cap \mathbb{S}^1} \mu \cdot u.$$

Let the two boundary rays of the wedge be generated by the unit vectors u_1 and u_2.

Proposition 6.16 (radial Foster–Lyapunov criterion)**.** *Assume $\alpha < 2$. Suppose*

$$\nu_*(\mu) < 0$$

and

$$v_i \cdot u_i \le 0, \qquad i = 1, 2.$$

Then the solution to the Lakner–Liu–Reed submartingale problem with drift μ *admits a stationary distribution. Moreover, for every*

$$0 < \beta < -2\nu_*(\mu),$$

every stationary distribution π *satisfies*

$$\int_S e^{\beta|z|}\,\pi(dz) < \infty.$$

Consequently,

$$\pi\{|z| \ge r\} \le C_\beta e^{-\beta r}, \qquad r \ge 0,$$

for some finite constant C_β.

Proof. Choose $0 < \beta < -2\nu_*(\mu)$. Let $\eta \in C^\infty([0,\infty))$ satisfy

$$\eta(r) = 0 \quad (0 \le r \le 1/2), \qquad \eta(r) = 1 \quad (r \ge 1), \qquad \eta'(r) \ge 0,$$

and choose η so that the function

$$r \longmapsto \eta(r)e^{\beta r}$$

is nondecreasing. Set

$$V_\beta(z) := \eta(|z|)e^{\beta|z|}.$$

The function V_β is constant in a neighborhood of the vertex; the plateau removes the possible nonsmoothness of $|z|$ at 0, so the bounded truncations below are admissible C_b^2 tests. On the boundary ray generated by u_i, the radial derivative gives

$$D_i V_\beta(z) = V_\beta'(|z|)\, v_i \cdot u_i \le 0.$$

For $V_{\beta,n} := \phi_n(V_\beta)$, the chain rule gives

$$D_i V_{\beta,n} = \phi_n'(V_\beta) D_i V_\beta \le 0$$

on each open face. We also verify the global test-function regularity. The nonconstant region of $V_{\beta,n}$ is contained in $\{V_\beta < n+2\}$. Since V_β is norm-like, the closure of this set is compact. Near the vertex the function V_β, and hence its truncation, is constant. On the remaining compact annular part, $|z|$ is smooth, V_β has bounded first and second derivatives, and the fixed profile ϕ_n has bounded first and second derivatives. The chain rule therefore gives bounded derivatives of order at most two. Outside the compact nonconstant region the truncation is again constant. Thus $V_{\beta,n} \in C_b^2(S)$, is constant near the vertex, and $-V_{\beta,n}$ is admissible.

For $|z| = r \ge 1$, writing $u = z/|z|$,

$$\nabla V_\beta = \beta e^{\beta r} u, \qquad \Delta V_\beta = e^{\beta r}\left(\beta^2 + \frac{\beta}{r}\right).$$

Hence

$$\mathcal{L}V_\beta(z) = e^{\beta r}\left(\beta\mu \cdot u + \frac{1}{2}\beta^2 + \frac{\beta}{2r}\right) \le e^{\beta r}\left(\beta\nu_*(\mu) + \frac{1}{2}\beta^2 + \frac{\beta}{2r}\right).$$

Since $\beta\nu_*(\mu) + \frac{1}{2}\beta^2 < 0$, there are $R < \infty$ and $\delta > 0$ such that

$$\mathcal{L}V_\beta(z) \le -\delta V_\beta(z), \qquad |z| \ge R.$$

On the compact set $K_R := S \cap \overline{B}_R$, the open-wedge expression $\mathcal{L}V_\beta + \delta V_\beta$ extends continuously to the open faces and is zero in a neighborhood of the vertex where V_β is constant. Hence

$$C := 1 + \sup_{z \in K_R \cap S^\circ} \big(\mathcal{L}V_\beta(z) + \delta V_\beta(z)\big)_+ < \infty,$$

and

$$\mathcal{L}V_\beta \le -\delta V_\beta + C\mathbf{1}_{K_R} \qquad \text{on } S^\circ.$$

Applying lemma 6.3 to the displayed drift inequality yields, for the bounded truncations $V_{\beta,n}$,

$$\mathcal{L}V_{\beta,n} \le -\delta\phi_n'(V_\beta)V_\beta + C\mathbf{1}_{K_R} \quad \text{on } S^\circ.$$

Then lemma 6.4 gives the uniform averaged bound

$$\frac{1}{T}\int_0^T \mathbb{E}_z[V_\beta(Z_s)]\,ds \le \frac{V_\beta(z)}{\delta T} + \frac{C}{\delta}.$$

Since V_β is norm-like, lemma 6.6 gives tightness of the empirical occupation measures, and the Feller property and lemma 6.7 yield a stationary distribution; see also [1, Ch. 4].

Under any stationary distribution for the drift μ, lemma 6.5 applied to the same bounded truncations gives

$$\int_S V_\beta\,d\pi \le C/\delta < \infty.$$

Since $V_\beta = e^{\beta|z|}$ for $|z| \ge 1$, this implies

$$\int_S e^{\beta|z|}\,\pi(dz) < \infty.$$

The tail bound follows from Markov's inequality. □

Remark 6.17. The radial criterion may overlap with the linear Lyapunov cone. It provides a nonlinear Foster–Lyapunov test that can be checked directly from radial drift and boundary tangential signs, and it gives a concrete stationary-existence criterion when radial information is more accessible than a linear Lyapunov witness.

6.2. **An elliptic Foster–Lyapunov criterion.** The radial criterion above uses the Euclidean norm. The next proposition is its anisotropic analogue. It is the probability-side counterpart of the elliptic-norm admissible supersolution criterion in section C. The signs are reversed: here the quadratic gauge decreases in the drift direction and does not increase under reflection on the boundary.

Proposition 6.18 (elliptic Foster–Lyapunov criterion)**.** *Assume* $\alpha < 2$*. Let* Q *be a symmetric positive definite* 2×2 *matrix and define*

$$\rho_Q(z) = (z^TQz)^{1/2}.$$

Assume

(EL1) $$v_i \cdot Qu_i \le 0, \qquad i = 1, 2,$$

and

(EL2) $$Q\mu \in -S_\circ^\vee.$$

Then the solution to the Lakner–Liu–Reed submartingale problem with drift μ *admits a stationary distribution. Moreover, if*

$$m_Q := \min\{-(Q\mu)\cdot z : z \in S,\ \rho_Q(z) = 1\} > 0, \qquad M_Q := \sup_{z\ne 0} \frac{z^TQ^2z}{z^TQz} < \infty,$$

then for every

$$0 < \beta < \frac{2m_Q}{M_Q},$$

every stationary distribution π *satisfies*

$$\int_S e^{\beta\rho_Q(z)}\,\pi(dz) < \infty.$$

Proof. First note that (EL2) gives $m_Q > 0$, because the set $\{z \in S : \rho_Q(z) = 1\}$ is compact and does not contain the vertex. On $S \setminus \{0\}$,

$$\nabla \rho_Q(z) = \frac{Qz}{\rho_Q(z)}, \qquad |\nabla \rho_Q(z)|^2 = \frac{z^T Q^2 z}{z^T Q z} \le M_Q.$$

Moreover, for $z \neq 0$, put

$$r := \rho_Q(z) > 0, \qquad u := \frac{z}{r}.$$

Then $u \in S$, $\rho_Q(u) = 1$, and

$$\mu \cdot \nabla \rho_Q(z) = \frac{(Q\mu) \cdot z}{\rho_Q(z)} = (Q\mu) \cdot u \le -m_Q. \tag{EL3}$$

Also

$$\Delta \rho_Q(z) = \frac{\operatorname{tr} Q}{\rho_Q(z)} - \frac{z^T Q^2 z}{\rho_Q(z)^3} \le \frac{\operatorname{tr} Q}{\rho_Q(z)}. \tag{EL4}$$

Fix $0 < \beta < 2m_Q/M_Q$ and set

$$\Delta_\beta := m_Q - \frac{1}{2}\beta M_Q > 0.$$

Choose

$$R > \max\left\{1, \frac{\operatorname{tr} Q}{2\Delta_\beta}\right\}.$$

Then

$$-m_Q + \frac{\operatorname{tr} Q}{2R} + \frac{1}{2}\beta M_Q = -\Delta_\beta + \frac{\operatorname{tr} Q}{2R} < 0. \tag{EL5}$$

Let $\eta \in C^\infty([0, \infty))$ satisfy

$$\eta(r) = 0 \quad (0 \le r \le 1/2), \qquad \eta(r) = 1 \quad (r \ge 1), \qquad \eta'(r) \ge 0,$$

and set

$$V_\beta(z) = \eta(\rho_Q(z)) e^{\beta \rho_Q(z)}.$$

Then V_β is constant in a neighborhood of the vertex and is norm-like because $\rho_Q(z) \asymp |z|$. The plateau again removes the nonsmoothness of ρ_Q at the vertex. On the boundary ray generated by u_i, a point has the form $z = ru_i$, and

$$D_i \rho_Q(z) = \frac{r\, v_i \cdot Q u_i}{\rho_Q(r u_i)} \le 0$$

by (EL1). Since $r \mapsto \eta(r) e^{\beta r}$ is nondecreasing, we get

$$D_i V_\beta \le 0.$$

For $V_{\beta,n} := \phi_n(V_\beta)$, one has

$$D_i V_{\beta,n} = \phi_n'(V_\beta) D_i V_\beta \le 0$$

on each open face. Its nonconstant region is contained in the compact set $\{V_\beta \le n + 2\}$, because V_β is norm-like. It is separated from the vertex except for the neighborhood on which V_β is constant. On every compact subset away from the vertex the elliptic norm ρ_Q is smooth, and its first two derivatives are bounded; composing with the fixed smooth functions $r \mapsto \eta(r)e^{\beta r}$ and ϕ_n therefore gives bounded first and second derivatives. The two constant plateaus complete the extension through the vertex and outside the compact nonconstant region. Thus $V_{\beta,n} \in C_b^2(S)$, is constant near the vertex, and $-V_{\beta,n}$ is admissible.

On the region $\rho_Q(z) \ge R$, where $\eta = 1$, we have

$$\mathcal{L} V_\beta = e^{\beta \rho_Q} \left(\beta \mathcal{L} \rho_Q + \frac{1}{2}\beta^2 |\nabla \rho_Q|^2 \right).$$

Using (EL3)–(EL4),

$$\mathcal{L}\rho_Q = \mu \cdot \nabla \rho_Q + \frac{1}{2}\Delta\rho_Q \le -m_Q + \frac{\operatorname{tr} Q}{2\rho_Q(z)}.$$

Therefore (EL5) yields a constant $\delta > 0$ such that

$$\mathcal{L}V_\beta \le -\delta V_\beta \qquad (\rho_Q \ge R).$$

The set $K_R := \{z \in S : \rho_Q(z) \le R\}$ is compact. The open-wedge expression $\mathcal{L}V_\beta + \delta V_\beta$ extends continuously to each open face, and it vanishes near the vertex because V_β is constant there. Consequently

$$C := 1 + \sup_{z \in K_R \cap S^\circ} \big(\mathcal{L}V_\beta(z) + \delta V_\beta(z)\big)_+ < \infty,$$

and

$$\text{(EL6)} \qquad \mathcal{L}V_\beta \le -\delta V_\beta + C\mathbf{1}_{K_R} \qquad \text{on } S^\circ.$$

Applying lemma 6.3 to (EL6) gives, for the bounded truncations $V_{\beta,n}$,

$$\mathcal{L}V_{\beta,n} \le -\delta\phi_n'(V_\beta)V_\beta + C\mathbf{1}_{K_R} \quad \text{on } S^\circ.$$

The bounded-truncation Foster estimate gives the uniform averaged bound

$$\frac{1}{T}\int_0^T \mathbb{E}_z[V_\beta(Z_s)]\,ds \le \frac{V_\beta(z)}{\delta T} + \frac{C}{\delta}.$$

Since V_β is norm-like, the occupation averages are tight, and the Feller property yields a stationary distribution by lemma 6.7.

Finally, under any stationary distribution for the drift μ, applying lemma 6.5 to (EL6) and the same bounded truncations gives

$$\int_S V_\beta\,d\pi \le C/\delta < \infty.$$

Since $V_\beta = e^{\beta\rho_Q}$ outside a compact set, the asserted exponential moment follows. □

Remark 6.19. Proposition 6.18 uses an elliptic gauge: for every positive definite Q, the level sets of ρ_Q are ellipses, not half-spaces. It gives a stationary-existence test inside regions where no monotone one-dimensional Lyapunov direction is available. It is also the sign-reversed probability-side counterpart of the elliptic-norm admissible supersolution criterion in section C.

Proposition 6.20 (two-parameter feasibility form of the elliptic Foster–Lyapunov criterion)**.** *Assume $0 < \xi < \pi$ and $\alpha < 2$. Write the reflection directions and the drift in the wedge basis as*

$$v_i = r_{i1}u_1 + r_{i2}u_2, \qquad i = 1, 2, \qquad \mu = m_1u_1 + m_2u_2.$$

Then the hypotheses of proposition 6.18 are equivalent to the existence of two real parameters

$$x > 0, \qquad \tau \in (-1, 1),$$

such that

$$\text{(EF1)} \qquad r_{11} + r_{12}\tau x \le 0, \qquad r_{22} + r_{21}\frac{\tau}{x} \le 0,$$

and

$$\text{(EF2)} \qquad m_1 + m_2\tau x < 0, \qquad m_2 + m_1\frac{\tau}{x} < 0.$$

Consequently, the elliptic Foster–Lyapunov criterion is a concrete two-scalar feasibility test in wedge coordinates.

Proof. Let Q be symmetric positive definite and set

$$q_{11} := u_1 \cdot Qu_1, \qquad q_{12} := u_1 \cdot Qu_2, \qquad q_{22} := u_2 \cdot Qu_2.$$

Since u_1, u_2 form a basis of $\mathbb{R}^2$, the matrix

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

is positive definite. Multiplying Q by a positive scalar does not change any of the sign conditions in proposition 6.18. Hence we may normalize $q_{11} = 1$. Then there are unique parameters $x > 0$ and $\tau \in (-1, 1)$ such that

$$q_{22} = x^2, \qquad q_{12} = \tau x.$$

Indeed, positive definiteness is exactly $q_{22} > 0$ and $|q_{12}| < \sqrt{q_{22}}$.

The boundary sign condition $v_1 \cdot Qu_1 \leq 0$ becomes

$$r_{11}q_{11} + r_{12}q_{12} \leq 0,$$

which is the first inequality in (EF1). For the second face,

$$v_2 \cdot Qu_2 = (r_{21}u_1 + r_{22}u_2) \cdot Qu_2 = r_{21}q_{12} + r_{22}q_{22} = x^2 \left(r_{22} + r_{21}\frac{\tau}{x} \right).$$

Because $x^2 > 0$, the inequality $v_2 \cdot Qu_2 \leq 0$ is exactly the second inequality in (EF1).

Next, $Q\mu \in -S_\circ^\vee$ means

$$(Q\mu) \cdot u_1 < 0, \qquad (Q\mu) \cdot u_2 < 0.$$

Since $\mu = m_1u_1 + m_2u_2$, these two inequalities are

$$m_1q_{11} + m_2q_{12} < 0, \qquad m_1q_{12} + m_2q_{22} < 0.$$

After substituting $q_{11} = 1$, $q_{12} = \tau x$, and $q_{22} = x^2$, and dividing the second inequality by x^2, we obtain exactly (EF2).

Conversely, given $x > 0$ and $\tau \in (-1, 1)$ satisfying (EF1)–(EF2), define a symmetric positive definite bilinear form in the basis u_1, u_2 by

$$q_{11} = 1, \qquad q_{12} = \tau x, \qquad q_{22} = x^2.$$

This bilinear form is represented by a unique Euclidean symmetric positive definite matrix Q. Reversing the computations above gives $v_i \cdot Qu_i \leq 0$ for $i = 1, 2$ and $Q\mu \in -S_\circ^\vee$. The final assertion follows from proposition 6.18. □

Remark 6.21. The sign pattern in proposition 6.20 is the probability-side analogue of the two-parameter elliptic-norm supersolution criterion in proposition C.7, with all inequalities reversed. Both criteria are therefore expressed by explicit two-parameter feasibility inequalities, one for Foster–Lyapunov existence and the other for admissible-supersolution nonexistence.

Proposition 6.22 (maximality of one-dimensional Foster–Lyapunov functions)**.** *Let $a \in S_\circ^\vee$. Suppose that there exists $h \in C^2([0, \infty))$ such that*

$$h \geq 0, \qquad h' \geq 0, \qquad h \text{ is constant near } 0, \qquad h(s) \to \infty \quad (s \to \infty),$$

and such that, for $U(z) := h(a \cdot z)$,

$$D_iU \leq 0 \quad \text{on } \partial S_i, \qquad i = 1, 2,$$

and there exist constants $R, \delta > 0$ with

$$\mathcal{L}U(z) \leq -\delta \qquad \text{whenever } a \cdot z \geq R.$$

Then

$$a \cdot v_i \leq 0, \qquad i = 1, 2,$$

and

$$a \cdot \mu < 0.$$

Consequently, any stationary-existence proof based on a monotone one-dimensional Foster–Lyapunov function of the form $h(a \cdot z)$ can only produce drifts in $\mathfrak{M}_{\text{Lyap}}$.

Proof. Since $a \in S_\circ^\vee$, we have $a \cdot u_i > 0$ for the two boundary generators u_i. Because h is nondecreasing and unbounded, h' cannot vanish identically; hence there is $s_0 > 0$ such that $h'(s_0) > 0$. For each boundary ray choose

$$z_i := \frac{s_0}{a \cdot u_i} u_i \in \partial S_i.$$

Then $a \cdot z_i = s_0$, and the boundary sign gives

$$0 \geq D_i U(z_i) = h'(s_0) a \cdot v_i.$$

Since $h'(s_0) > 0$, we obtain $a \cdot v_i \leq 0$ for $i = 1, 2$.

It remains to prove $a \cdot \mu < 0$. Put

$$m := a \cdot \mu, \qquad d := \frac{1}{2}|a|^2 > 0, \qquad p := h'.$$

Choose an interior unit direction $u_* \in S^\circ \cap \mathbb{S}^1$. Since $a \in S_\circ^\vee$, one has $a \cdot u_* > 0$. For every $s > 0$, the point $z_s = (s/(a \cdot u_*))u_*$ lies in S° and satisfies $a \cdot z_s = s$. Since the expression for $\mathcal{L}U$ depends on z_s only through $s = a \cdot z_s$, the drift inequality gives, for every $s \geq R$,

$$dp'(s) + mp(s) \leq -\delta.$$

Assume first that $m = 0$. Then

$$p'(s) \leq -\frac{\delta}{d}, \qquad s \geq R,$$

so $p(s)$ becomes negative for large s, contradicting $p = h' \geq 0$.

Assume next that $m > 0$. Multiplying by the integrating factor $e^{ms/d}$ gives

$$\big(e^{ms/d} p(s)\big)' \leq -\frac{\delta}{d} e^{ms/d}.$$

Integrating from R to $t > R$ yields

$$e^{mt/d} p(t) \leq e^{mR/d} p(R) - \frac{\delta}{m}\big(e^{mt/d} - e^{mR/d}\big).$$

Dividing by $e^{mt/d}$ and letting $t \to \infty$ shows that $p(t)$ is eventually bounded above by a negative number, again contradicting $p \geq 0$. Therefore m cannot be nonnegative, and hence $a \cdot \mu = m < 0$. □

Remark 6.23. This proposition is the probability-side analogue of the maximality theorem for the direct one-dimensional admissible-supersolution cone. Together, the two results characterize exactly the drift regions covered by the two monotone one-coordinate classes considered here. They do not, by themselves, determine the dimension of every possible argument on the complement: the borderline reflection line is handled by a neutral one-coordinate identity, while the strict residual cone is handled here by the two-dimensional Varadhan–Williams gauge.

Corollary 6.24 (compatibility of the one-dimensional criteria)**.** *Assume* $0 < \xi < \pi$ *and* $1 \leq \alpha < 2$. *Then*

$$\mathfrak{M}_{\text{sup}} \cap \mathfrak{M}_{\text{Lyap}} = \varnothing. \tag{6.25}$$

Consequently, using proposition 6.10,

$$\mathfrak{M}_{\text{sup}} \subset \operatorname{cone}\{u_1, u_2, -v_1, -v_2\}. \tag{6.26}$$

Proof. If $\mu \in \mathfrak{M}_{\text{Lyap}}$, then proposition 6.11 gives a stationary distribution. If at the same time $\mu \in \mathfrak{M}_{\text{sup}}$, then theorem 5.4 gives nonexistence of a stationary distribution. These two conclusions are incompatible. Hence the intersection is empty. Since proposition 6.10 gives

$$\mathfrak{M}_{\text{Lyap}} = \mathbb{R}^2 \setminus \operatorname{cone}\{u_1, u_2, -v_1, -v_2\},$$

the inclusion (6.26) follows. □

Remark 6.25. This corollary shows that the direct one-dimensional nonexistence cone and the Foster–Lyapunov existence cone are disjoint.

Proposition 6.26 (polar description of the direct nonexistence cone)**.** *Assume* $0 < \xi < \pi$ *and* $\mathfrak{B} \neq \varnothing$*; equivalently, by proposition 5.8,* $\operatorname{cone}\{v_1, v_2\} \cap (-S) = \{0\}$*. Then*

$$\mathfrak{M}_{\sup} = \mathbb{R}^2 \setminus \operatorname{cone}\{-u_1, -u_2, -v_1, -v_2\}. \tag{6.27}$$

Equivalently, under the nonemptiness of $\mathfrak{B}$*, a drift vector belongs to the direct one-dimensional nonexistence region exactly when it is separated by some admissible outward linear functional from the closed cone generated by the opposite wedge rays and the opposite reflection directions.*

Proof. Let

$$C_- := \operatorname{cone}\{-u_1, -u_2, -v_1, -v_2\}.$$

First suppose $\mu \in \mathfrak{M}_{\sup}$. Then there exists $c \in \mathfrak{B}$ such that $c \cdot \mu > 0$. Since $c \in S_\circ^\vee$, we have $c \cdot u_j > 0$ for $j = 1, 2$, and by definition of $\mathfrak{B}$ we have $c \cdot v_i \geq 0$ for $i = 1, 2$. Hence

$$c \cdot x \leq 0 \qquad (x \in C_-).$$

If $\mu \in C_-$, this would imply $c \cdot \mu \leq 0$, contradicting $c \cdot \mu > 0$. Thus $\mu \notin C_-$.

Conversely, assume $\mu \notin C_-$. Apply lemma 6.9 to $C = C_-$ and $y = \mu$. The resulting vector $c \neq 0$ satisfies the stronger normalized relations

$$c \cdot x \leq 0 \quad (x \in C_-), \qquad c \cdot \mu = |c|^2 > 0.$$

The inequalities on the generators of C_- give

$$c \cdot u_j \geq 0 \quad (j = 1, 2), \qquad c \cdot v_i \geq 0 \quad (i = 1, 2).$$

These inequalities may be non-strict on the wedge generators. By the assumed nonemptiness of $\mathfrak{B}$, choose $h \in \mathfrak{B}$. For $\delta > 0$, set

$$c_\delta := c + \delta h.$$

Then for each $j = 1, 2$,

$$c_\delta \cdot u_j = c \cdot u_j + \delta h \cdot u_j > 0,$$

and for each $i = 1, 2$,

$$c_\delta \cdot v_i = c \cdot v_i + \delta h \cdot v_i \geq 0.$$

Thus $c_\delta \in \mathfrak{B}$ for every $\delta > 0$. Put

$$d_\mu := c \cdot \mu = |c|^2 > 0.$$

If $h \cdot \mu \geq 0$, then $c_\delta \cdot \mu \geq d_\mu > 0$ for every $\delta > 0$. If $h \cdot \mu < 0$, choose

$$0 < \delta < \frac{d_\mu}{2|h \cdot \mu|}.$$

Then

$$c_\delta \cdot \mu = d_\mu + \delta h \cdot \mu > \frac{d_\mu}{2} > 0.$$

Hence $\mu \in \mathfrak{M}_{\sup}$. This proves (6.27). ∎

Remark 6.27 (polar descriptions of the drift cones). The Lyapunov existence cone and, when $\mathfrak{B} \neq \varnothing$, the direct nonexistence cone have the parallel polar descriptions

$$\mathfrak{M}_{\text{Lyap}} = \mathbb{R}^2 \setminus \operatorname{cone}\{u_1, u_2, -v_1, -v_2\},$$
$$\mathfrak{M}_{\sup} = \mathbb{R}^2 \setminus \operatorname{cone}\{-u_1, -u_2, -v_1, -v_2\}.$$

These identities describe exactly what the two monotone one-coordinate criteria decide. They do not imply that their complement is full-dimensional or that every further argument must be two-dimensional.

Proposition 6.28 (the region not covered by the two monotone one-coordinate criteria)**.** *Assume* $0 < \xi < \pi$ *and* $1 \le \alpha < 2$, *and define*

$$\mathfrak{R}_{\mathrm{rem}} := \mathbb{R}^2 \setminus (\mathfrak{M}_{\mathrm{Lyap}} \cup \mathfrak{M}_{\mathrm{sup}}). \tag{6.28}$$

Then the following more precise alternatives hold.

(i) *If* $\alpha = 1$, *then*

$$\mathfrak{R}_{\mathrm{rem}} = \{\mu : n_{\mathscr{L}} \cdot \mu = 0\} = \mathscr{L}. \tag{6.29}$$

This one-dimensional remainder is excluded by the neutral projection argument of proposition 5.14.

(ii) *If* $1 < \alpha < 2$, *then*

$$\mathfrak{R}_{\mathrm{rem}} = K_{\mathrm{str}} = \mathrm{cone}\{-v_1, -v_2\}. \tag{6.30}$$

This two-dimensional closed cone is excluded by the Varadhan–Williams argument of theorem 7.12.

(iii) *Whenever* $\mathfrak{B} \neq \varnothing$, *put*

$$D_{\mathrm{Lyap}} := \mathrm{cone}\{u_1, u_2, -v_1, -v_2\}, \qquad D_{\mathrm{sup}} := \mathrm{cone}\{-u_1, -u_2, -v_1, -v_2\}.$$

Then

$$\mathfrak{R}_{\mathrm{rem}} = D_{\mathrm{Lyap}} \cap D_{\mathrm{sup}}. \tag{6.31}$$

Thus $\mathfrak{R}_{\mathrm{rem}}$ *is the set left undecided by the two* monotone *one-coordinate criteria; the argument that completes the classification depends on the regime.*

Proof. If $\alpha = 1$, proposition 5.10 gives

$$\mathfrak{M}_{\mathrm{Lyap}} = \{\mu : n_{\mathscr{L}} \cdot \mu < 0\}, \qquad \mathfrak{M}_{\mathrm{sup}} = \{\mu : n_{\mathscr{L}} \cdot \mu > 0\}.$$

Taking the complement of their union proves (6.29); the final assertion in part (i) is proposition 5.14.

If $1 < \alpha < 2$, proposition 6.14 gives

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus K_{\mathrm{str}}, \qquad \mathfrak{M}_{\mathrm{sup}} = \varnothing.$$

This proves (6.30), and theorem 7.12 supplies the stated nonexistence result.

Finally assume $\mathfrak{B} \neq \varnothing$. By propositions 6.10 and 6.26,

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus D_{\mathrm{Lyap}}, \qquad \mathfrak{M}_{\mathrm{sup}} = \mathbb{R}^2 \setminus D_{\mathrm{sup}}.$$

De Morgan's law now gives

$$\mathfrak{R}_{\mathrm{rem}} = \mathbb{R}^2 \setminus \big((\mathbb{R}^2 \setminus D_{\mathrm{Lyap}}) \cup (\mathbb{R}^2 \setminus D_{\mathrm{sup}})\big) = D_{\mathrm{Lyap}} \cap D_{\mathrm{sup}},$$

which is (6.31). □

Proposition 6.29 (topology of the one-dimensional drift regions)**.** *Under the hypotheses of proposition 6.28, the sets*

$$\mathfrak{M}_{\mathrm{Lyap}}, \qquad \mathfrak{M}_{\mathrm{sup}}$$

are open cones, and $\mathfrak{R}_{\mathrm{rem}}$ *is a closed convex cone. Its dimension depends on the regime: it is the line* $\mathscr{L}$ *when* $\alpha = 1$ *and the two-dimensional cone* K_{str} *when* $1 < \alpha < 2$.

Proof. For each fixed witness $a \in \mathfrak{A}$, the condition $a \cdot \mu < 0$ defines an open half-space; hence $\mathfrak{M}_{\mathrm{Lyap}}$ is open. Likewise, for each fixed $c \in \mathfrak{B}$, the condition $c \cdot \mu > 0$ defines an open half-space, so $\mathfrak{M}_{\mathrm{sup}}$ is open; when $\mathfrak{B} = \varnothing$ it is the empty open set. Both sets are invariant under multiplication by positive scalars.

The explicit alternatives (6.29) and (6.30) show that $\mathfrak{R}_{\mathrm{rem}}$ is respectively a line or a finitely generated closed convex cone. This proves all assertions, including the dimension statement. □

Remark 6.30 (the mechanisms completing the classification). The notation $\mathfrak{R}_{\mathrm{rem}}$ records only the limitation of the two monotone one-coordinate criteria. In the borderline regime the remainder is lower-dimensional and is resolved by a neutral one-coordinate distributional identity. In the strict regime the direct supersolution cone is empty, the remainder is the whole closed cone K_{str}, and the genuinely two-dimensional quadratic Varadhan–Williams gauge supplies the missing nonexistence theorem.

Corollary 6.31 (convex existence slice from the separator)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2.$$

Let b be the geometric separator. If

$$b \cdot \mu > 0, \tag{6.32}$$

then the solution to the submartingale problem with drift μ admits a stationary distribution.

Proof. Set $a := -b$. Then $a \cdot z > 0$ on $S \setminus \{0\}$ and $a \cdot v_i \le 0$ for $i = 1, 2$. If $b \cdot \mu > 0$, then $a \cdot \mu < 0$. Apply theorem 6.8. □

Corollary 6.32 (two special geometric sufficient conditions)**.** *Assume $0 < \xi < \pi$ and $1 \le \alpha < 2$. Each of the following two conditions implies existence of a stationary distribution.*

(a) $\mu \neq 0$, and the closed convex cone

$$\operatorname{cone}\{u_1, u_2, -v_1, -v_2, -\mu\}$$

is a wedge in $\mathbb{R}^2$ with opening angle strictly less than π.

(b) $v_1 = -v_2$, and $u_1, u_2, -\mu$ all lie in one of the two open half-spaces determined by the line $\{\lambda v_1 : \lambda \in \mathbb{R}\}$.

Proof. For part (a), put

$$C := \operatorname{cone}\{u_1, u_2, -v_1, -v_2, -\mu\}.$$

By hypothesis, C is a proper closed two-dimensional cone with aperture strictly smaller than π. Its dual cone $C^\vee$ has nonempty interior. Choose $a \in \operatorname{int} C^\vee$. We first verify that $a \cdot x > 0$ for every nonzero $x \in C$. Since $a \in \operatorname{int} C^\vee$, there is $r_a > 0$ such that

$$B_{r_a}(a) \subset C^\vee.$$

If, contrary to the claim, $a \cdot x = 0$ for some $x \in C \setminus \{0\}$, set $h = -x/|x|$. For every $0 < \varepsilon < r_a$, one has $a + \varepsilon h \in C^\vee$, but

$$(a + \varepsilon h) \cdot x = a \cdot x - \varepsilon |x| = -\varepsilon |x| < 0,$$

contradicting the defining inequality of $C^\vee$ on the point $x \in C$. Applying strict positivity to the five nonzero generators gives

$$a \cdot u_j > 0, \qquad a \cdot v_i < 0, \qquad a \cdot \mu < 0.$$

Thus $a \in \mathfrak{A}$, and proposition 6.11 applies.

For part (b), let $L = \{\lambda v_1 : \lambda \in \mathbb{R}\}$. Choose the unit normal a to L which is positive on the open half-plane containing $u_1, u_2, -\mu$. Then

$$a \cdot u_1 > 0, \qquad a \cdot u_2 > 0, \qquad a \cdot (-\mu) > 0.$$

Because $v_2 = -v_1$, one also has $a \cdot v_1 = a \cdot v_2 = 0$. Hence $a \in \mathfrak{A}$ and $a \cdot \mu < 0$, so proposition 6.11 again applies. □

Remark 6.33. The two sufficient conditions in corollary 6.32 identify two explicit geometric stationary-existence regimes; both are contained in the single Lyapunov cone of proposition 6.11.

7. Strict convex classification by the quadratic Varadhan–Williams gauge

The strict closed-cone nonexistence theorem is based on the classical Varadhan–Williams homogeneous harmonic function [7]. Raising this function to the power that makes it two-homogeneous produces a neutral gauge $W = h^{2/\alpha}$ satisfying $D_i W = 0$, $|\nabla W|^2 \lesssim W$, and $\mathcal{L}_\mu W \geq m_0 > 0$ for $\mu \in K_{\mathrm{str}}$. Bounded two-scale cutoffs then localize the gauge near the vertex and at infinity, so the neutral stationary identity applies without any moment assumption on W.

Lemma 7.1 (Varadhan–Williams harmonic and cone signs)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Let $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ be the Lakner–Liu–Reed reflection angles, measured from the inward normals with positive sign toward the vertex, so that

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

In polar coordinates on S, define

$$q(\theta) := \frac{\cos(\alpha\theta - \theta_1)}{\cos\theta_1}, \qquad h(r, \theta) := r^\alpha q(\theta).$$

Then $q > 0$ on $[0, \xi]$,

$$\Delta h = 0 \quad \text{on } S^\circ, \qquad D_1 h = 0 \quad \text{on } \partial S_1, \qquad D_2 h = 0 \quad \text{on } \partial S_2.$$

Moreover there exist constants $C_1, C_2 > 0$ such that, on $S \setminus \{0\}$,

$$(-v_1) \cdot \nabla h = C_1 r^{\alpha-1} \sin((\alpha - 1)\theta),$$

and

$$(-v_2) \cdot \nabla h = C_2 r^{\alpha-1} \sin((\alpha - 1)(\xi - \theta)).$$

Consequently, for every drift

$$\mu \in \operatorname{cone}\{-v_1, -v_2\},$$

one has

$$\mu \cdot \nabla h \geq 0 \quad \text{on } S \setminus \{0\}.$$

Proof. Let $t_1 = u_1$ and $t_2 = u_2$ be the unit tangents along the two faces pointing away from the vertex, and let n_i be the inward unit normal on ∂S_i. With the angle convention above,

$$v_i = n_i - (\tan\theta_i) t_i, \qquad i = 1, 2.$$

Since $\alpha\xi = \theta_1 + \theta_2$, the angle $\alpha\theta - \theta_1$ ranges from $-\theta_1$ to θ_2. Both endpoints lie in $(-\pi/2, \pi/2)$, hence $q > 0$ on $[0, \xi]$. Also $q'' + \alpha^2 q = 0$, so $\Delta h = 0$ in the wedge interior.

On ∂S_1, one has $n_1 = e_\theta(0)$, $t_1 = e_r(0)$, $q(0) = 1$, and $q'(0) = \alpha \tan\theta_1$. Hence

$$D_1 h = r^{\alpha-1}\{q'(0) - \alpha(\tan\theta_1) q(0)\} = 0.$$

On ∂S_2, one has $n_2 = -e_\theta(\xi)$, $t_2 = e_r(\xi)$, and

$$\frac{q'(\xi)}{q(\xi)} = -\alpha \tan\theta_2.$$

Therefore

$$D_2 h = r^{\alpha-1}\{-q'(\xi) - \alpha(\tan\theta_2) q(\xi)\} = 0.$$

The derivatives in the opposite reflection directions have the following signs. From the displayed formula for h,

$$\nabla h = \frac{\alpha r^{\alpha-1}}{\cos\theta_1}\big[\cos(\alpha\theta - \theta_1) e_r(\theta) - \sin(\alpha\theta - \theta_1) e_\theta(\theta)\big].$$

To make the signs independent of any pictorial convention, we compute the two dot products explicitly. Since

$$-v_1 = (\tan\theta_1) e_r(0) - e_\theta(0),$$

the elementary identities

$$e_r(0)\cdot e_r(\theta)=\cos\theta,\qquad e_r(0)\cdot e_\theta(\theta)=-\sin\theta,$$
$$e_\theta(0)\cdot e_r(\theta)=\sin\theta,\qquad e_\theta(0)\cdot e_\theta(\theta)=\cos\theta$$

give the exact formula

$$(-v_1)\cdot\nabla h=\frac{\alpha r^{\alpha-1}}{\cos^2\theta_1}\sin((\alpha-1)\theta).$$

For the second face,

$$-v_2=e_\theta(\xi)+(\tan\theta_2)e_r(\xi)=\frac{1}{\cos\theta_2}e_\theta(\xi-\theta_2).$$

Also,

$$\cos(\alpha\theta-\theta_1)e_r(\theta)-\sin(\alpha\theta-\theta_1)e_\theta(\theta)=e_r\big((1-\alpha)\theta+\theta_1\big).$$

Using $e_\theta(\varphi)\cdot e_r(\psi)=\sin(\psi-\varphi)$ and $\theta_1+\theta_2=\alpha\xi$, we obtain

$$\begin{aligned}(-v_2)\cdot\nabla h&=\frac{\alpha r^{\alpha-1}}{\cos\theta_1\cos\theta_2}\sin\big((1-\alpha)\theta+\theta_1-\xi+\theta_2\big)\\&=\frac{\alpha r^{\alpha-1}}{\cos\theta_1\cos\theta_2}\sin\big((\alpha-1)(\xi-\theta)\big).\end{aligned}$$

The constants in front are strictly positive because $\theta_i\in(-\pi/2,\pi/2)$. Moreover,

$$0\le(\alpha-1)\theta\le(\alpha-1)\xi<\xi<\pi,$$

and the same bound holds with θ replaced by $\xi-\theta$. Hence both sine factors are nonnegative. The final assertion follows by writing $\mu=a(-v_1)+b(-v_2)$ with $a,b\ge 0$. □

Remark 7.2 (symmetric quadrant check). For the symmetric quadrant model

$$S=\mathbb{R}^2_+,\qquad v_1=-\sigma u_1+u_2,\qquad v_2=u_1-\sigma u_2,\qquad \sigma>1,$$

one has $\xi=\pi/2$ and $\theta_1=\theta_2=\arctan(\sigma)$. Hence

$$\alpha=\frac{4\arctan(\sigma)}{\pi}=\frac{2}{\pi}\left(\pi-2\arctan\frac{1}{\sigma}\right),$$

and the harmonic above becomes

$$h(r,\theta)=r^\alpha\{\cos(\alpha\theta)+\sigma\sin(\alpha\theta)\}.$$

The identities in lemma 7.1 reduce to the explicit formulas used in the symmetric critical-ray computations. This verification fixes the sign convention: positive reflection angle corresponds to a reflection direction tilted toward the vertex from the inward normal.

Lemma 7.3 (quadratic Varadhan–Williams gauge)**.** *Under the assumptions of Lemma 7.1, set*

$$\beta:=\frac{2}{\alpha},\qquad W:=h^\beta.$$

Then W extends continuously to S with $W(0)=0$, is positive on $S\setminus\{0\}$, is C^∞ in S°, is facewise C^2 and locally C^2-extendable away from the vertex, is two-homogeneous, and there are constants $0<c<C<\infty$ such that

$$cr^2\le W(r,\theta)\le Cr^2.$$

Moreover

$$D_iW=0\quad\text{on }\partial S_i,\qquad i=1,2,$$

there is a constant $C_W<\infty$ such that

$$|\nabla W|^2\le C_WW\quad\text{on } S^\circ \text{ and by continuous extension to the open faces,}$$

and there exists a constant $m_0 > 0$, depending only on the wedge and reflection data, such that for every drift $\mu \in \operatorname{cone}\{-v_1, -v_2\}$,

$$\mathcal{L}_\mu W \geq m_0 \quad \text{on } S^\circ \text{ and by continuous extension to the open faces.}$$

Proof. Since h is α-homogeneous, $W = h^{2/\alpha}$ is two-homogeneous. The positivity and continuity of q on $[0, \xi]$ give $W \asymp r^2$ and the continuous extension $W(0) = 0$. Although the intermediate harmonic $h = r^\alpha q(\theta)$ need not be twice differentiable at the vertex for noninteger α, this causes no loss: all differentiations of W below are taken in S° and through open-face limits away from the vertex, while the localized test functions used below are constant near the vertex. Writing $W = r^2 G(\theta)$, with G positive and C^2 on the compact angular interval, shows that W is smooth in the interior and up to each open face away from the vertex. Since $G \in C^2([0, \xi])$, extend G to a C^2 function on a slightly larger angular interval. On every compact annulus away from the origin, the function $r^2 \widetilde{G}(\theta)$, expressed in polar coordinates, gives a C^2 extension to a neighborhood of the annular portion of S. Thus W is locally C^2-extendable away from the vertex. Moreover, ∇W is one-homogeneous with bounded angular coefficient. Hence $|\nabla W|^2 \leq C r^2 \leq C_W W$. The boundary identities follow from

$$D_i W = \beta h^{\beta - 1} D_i h = 0.$$

Because $\Delta h = 0$, the chain rule gives

$$\mathcal{L}_\mu W = \beta h^{\beta-1} \mu \cdot \nabla h + \frac{1}{2} \beta(\beta - 1) h^{\beta-2} |\nabla h|^2.$$

The first term is nonnegative by Lemma 7.1. Since $1 < \alpha < 2$, one has $1 < \beta < 2$, so the second term is strictly positive. More explicitly, the gradient formula in Lemma 7.1 gives

$$|\nabla h(r, \theta)|^2 = \frac{\alpha^2 r^{2\alpha - 2}}{\cos^2 \theta_1},$$

and hence

$$h^{\beta-2} |\nabla h|^2 = \frac{\alpha^2}{\cos^2 \theta_1} q(\theta)^{\beta-2}.$$

Here $\beta - 2 < 0$, but this causes no singularity because q is continuous and strictly positive on the compact interval $[0, \xi]$. Thus the displayed angular function is continuous and strictly positive, and therefore has a positive minimum. One may take, for example,

$$m_0 := \frac{1}{2} \beta(\beta - 1) \frac{\alpha^2}{\cos^2 \theta_1} \min_{0 \leq \theta \leq \xi} q(\theta)^{\beta-2} > 0.$$

This lower bound comes only from the diffusion term, so it is uniform over all $\mu \in \operatorname{cone}\{-v_1, -v_2\}$. □

Remark 7.4 (uniformity over the closed reflection cone). The lower bound in lemma 7.3 is uniform over the whole closed reflection cone $K_{\text{str}} = \operatorname{cone}\{-v_1, -v_2\}$, not merely over compact subsets of normalized drifts. Indeed the drift term

$$\beta h^{\beta-1} \mu \cdot \nabla h$$

is nonnegative throughout the cone, while the strictly positive constant m_0 comes entirely from the diffusion term. Thus no bound on $|\mu|$ and no normalization of the critical rays is used in the closed-cone nonexistence theorem.

Remark 7.5 (vertex regularity and fractional powers). The gauge $W = h^{2/\alpha}$ is two-homogeneous and continuous at the vertex, and the proof does not require differentiability of W at the vertex. In lemma 7.9, the cutoff $F_{\varepsilon,R}$ is identically zero on $[0, \varepsilon^2]$; since $W \asymp |z|^2$, the localized function $F_{\varepsilon,R}(W)$ is constant on a full neighborhood of the vertex. Thus the admissibility and C_b^2 requirements are verified away from the corner and then extended across the corner by this constant plateau.

Lemma 7.6 (two-scale cutoff profiles)**.** *There exists a constant $C_F < \infty$ with the following property. For every $0 < \varepsilon < 1 < R$ satisfying $R^2 > 2\varepsilon^2$, there is a function $F_{\varepsilon,R} \in C^2([0,\infty))$ such that*

$$F_{\varepsilon,R}(s) = 0 \quad (s \le \varepsilon^2), \qquad F_{\varepsilon,R}(s) = s \quad (2\varepsilon^2 \le s \le R^2),$$
$$F_{\varepsilon,R}(s) = \textit{constant} \quad (s \ge 2R^2),$$

and

$$0 \le F'_{\varepsilon,R} \le C_F, \qquad |sF''_{\varepsilon,R}(s)| \le C_F \quad (s \ge 0).$$

Proof. Choose fixed functions $\phi, \psi \in C^2([0,\infty))$ such that $\phi = 0$ on $[0,1]$, $\phi(t) = t$ on $[2,\infty)$, $\phi' \ge 0$, and $\psi(t) = t$ on a neighborhood of $[0,1]$, while ψ is constant on $[2,\infty)$ and $\psi' \ge 0$. Define

$$F_{\varepsilon,R}(s) = \begin{cases} \varepsilon^2\phi(s/\varepsilon^2), & 0 \le s \le R^2, \\ R^2\psi(s/R^2), & s \ge R^2. \end{cases}$$

Because $R^2 > 2\varepsilon^2$, the first branch equals s in a neighborhood of R^2, while the second branch equals s in a neighborhood of R^2. Hence the two pieces glue to a C^2 function. The support and plateau properties follow from the definitions of ϕ and ψ. Finally,

$$sF''_{\varepsilon,R}(s) = (s/\varepsilon^2)\phi''(s/\varepsilon^2)$$

on the inner transition and

$$sF''_{\varepsilon,R}(s) = (s/R^2)\psi''(s/R^2)$$

on the outer transition. The functions $t\phi''(t)$, $t\psi''(t)$, ϕ', and ψ' are bounded, giving a constant independent of ε and R. □

Lemma 7.7 (vanishing quadratic shells)**.** *Let $W : S \to [0,\infty)$ be continuous and satisfy*

$$c|z|^2 \le W(z) \le C|z|^2$$

for some $0 < c < C < \infty$. If π is a probability measure on S with $\pi(\{0\}) = 0$, then for every $\delta > 0$ there exist $0 < \varepsilon < 1 < R$, with $R^2 > 2\varepsilon^2$, such that

$$\pi\{2\varepsilon^2 \le W \le R^2\} > 1 - \delta$$

and

$$\pi\big(\{\varepsilon^2 < W < 2\varepsilon^2\} \cup \{R^2 < W < 2R^2\}\big) < \delta.$$

Proof. The comparison $W \asymp |z|^2$ implies

$$\{W \le 2\varepsilon^2\} \downarrow \{0\} \quad \text{as } \varepsilon \downarrow 0,$$

and

$$\{W \ge R^2\} \downarrow \varnothing \quad \text{as } R \uparrow \infty.$$

By continuity from above for the finite measure π, and by the assumption $\pi(\{0\}) = 0$, we have $\pi\{W \le 2\varepsilon^2\} \to 0$ and $\pi\{W \ge R^2\} \to 0$. Choose ε so small that $\pi\{W \le 2\varepsilon^2\} < \delta/2$, and then choose R so large that $R^2 > 2\varepsilon^2$ and $\pi\{W \ge R^2\} < \delta/2$. The main annulus then has mass greater than $1 - \delta$. The transition shells are contained in

$$\{W \le 2\varepsilon^2\} \cup \{W \ge R^2\},$$

so their mass is less than δ. □

Remark 7.8 (differential inequalities for gauges). For gauges that are not themselves admissible test functions, differential identities and inequalities are understood in the open wedge and by continuous extension to each open face away from the vertex. The stationary argument only applies these gauges after bounded localization. The localized functions are constant near the vertex and belong to $C_b^2(S)$, so their generators are valid bounded test-function generators. Since any stationary distribution has zero boundary mass by proposition 4.2, the values assigned to the displayed gauge inequalities on the open faces are immaterial for the final integrals.

The argument below uses corollary 4.4; the continuous extensions make the boundary-neutral identities unambiguous.

Lemma 7.9 (localized neutral-gauge tests). *Fix a drift μ and let $W : S \to [0,\infty)$ be continuous, facewise C^2, and locally C^2-extendable away from the vertex. Assume that, for some constants $0 < c < C < \infty$, $C_W < \infty$, and $m_0 > 0$,*

$$c|z|^2 \le W(z) \le C|z|^2,$$

$$D_iW = 0 \quad \text{on the open face } \partial S_i, \qquad i = 1, 2,$$

$$|\nabla W|^2 \le C_W W \quad \text{on } S^\circ \text{ and by extension to the open faces,}$$

and

$$\mathcal{L}_\mu W \ge m_0 \quad \text{on } S^\circ \text{ and by extension to the open faces.}$$

For $0 < \varepsilon < 1 < R$ with $R^2 > 2\varepsilon^2$, let $F_{\varepsilon,R}$ be as in lemma 7.6 and set

$$f_{\varepsilon,R} := F_{\varepsilon,R}(W).$$

Then $f_{\varepsilon,R} \in C_b^2(S)$, is constant in a neighborhood of the vertex, is constant outside a compact set, and satisfies

$$D_i f_{\varepsilon,R} = 0 \quad \text{on the open face } \partial S_i, \qquad i = 1, 2.$$

Moreover, with

$$A_{\varepsilon,R} := \{2\varepsilon^2 \le W \le R^2\}, \qquad T_{\varepsilon,R} := \{\varepsilon^2 < W < 2\varepsilon^2\} \cup \{R^2 < W < 2R^2\},$$

which are Borel sets because W is continuous, there is a constant $C_{\text{loc}} < \infty$, depending only on C_F and C_W, such that, on S° and by continuous extension to the open faces,

$$\mathcal{L}_\mu f_{\varepsilon,R} \ge m_0 \mathbf{1}_{A_{\varepsilon,R}} - C_{\text{loc}} \mathbf{1}_{T_{\varepsilon,R}}.$$

The constant C_{loc} is independent of ε and R.

Proof. The comparison $W \asymp |z|^2$ implies that $f_{\varepsilon,R}$ is identically zero near the vertex and constant outside a compact set. On the nonconstant region $W \in [\varepsilon^2, 2R^2]$, the comparison also bounds $|z|$ above and below away from zero; hence this region is compact and disjoint from the vertex. The local C^2-extendability of W, a finite cover of this compact set, and the chain rule for $F_{\varepsilon,R} \in C^2$ give a C^2 extension of the composition on a neighborhood of the nonconstant region. The constant plateaus near the vertex and outside the outer compact set then complete the admissible extension. For each fixed pair (ε, R), the compact support of the nonconstant part and the boundedness of $F'_{\varepsilon,R}$, $F''_{\varepsilon,R}$, and the first two derivatives of W on that compact set give bounded first and second Euclidean derivatives. Hence $f_{\varepsilon,R} \in C_b^2(S)$. No uniform C_b^2-norm in ε or R is asserted or needed. On the open faces,

$$D_i f_{\varepsilon,R} = F'_{\varepsilon,R}(W) D_i W = 0.$$

At the vertex no boundary derivative is needed in the admissibility condition, because $f_{\varepsilon,R}$ is constant in a neighborhood of the vertex.

On the open wedge, and by continuous extension to each open face away from the vertex, the chain rule gives

$$\mathcal{L}_\mu f_{\varepsilon,R} = F'_{\varepsilon,R}(W)\mathcal{L}_\mu W + \frac{1}{2} F''_{\varepsilon,R}(W)|\nabla W|^2.$$

On $A_{\varepsilon,R}$, one has $F'_{\varepsilon,R} = 1$ and $F''_{\varepsilon,R} = 0$, so $\mathcal{L}_\mu f_{\varepsilon,R} \ge m_0$. On the flat regions $W \le \varepsilon^2$ and $W \ge 2R^2$, both derivatives of $F_{\varepsilon,R}$ vanish and the generator is zero. On the transition set $T_{\varepsilon,R}$, the first term is nonnegative because $F'_{\varepsilon,R} \ge 0$ and $\mathcal{L}_\mu W \ge m_0$, while

$$\frac{1}{2} F''_{\varepsilon,R}(W)|\nabla W|^2 \ge -\frac{1}{2}|F''_{\varepsilon,R}(W)| C_W W \ge -\frac{1}{2} C_F C_W.$$

Thus the asserted lower bound holds with $C_{\mathrm{loc}} = C_F C_W/2$. In particular, once the hypotheses $\mathcal{L}_\mu W \geq m_0$ and $|\nabla W|^2 \leq C_W W$ are fixed, the localization error is independent of ε, R, and the size of the drift vector. □

Remark 7.10 (localized cutoff estimates)*.* The conclusion of lemma 7.9 is an integrated-localization estimate rather than a pointwise supersolution statement. In the outer transition shell the second derivative of the cutoff may be negative, so the argument uses the uniform lower bound

$$\mathcal{L}_\mu f_{\varepsilon,R} \geq m_0 \mathbf{1}_{A_{\varepsilon,R}} - C_{\mathrm{loc}} \mathbf{1}_{T_{\varepsilon,R}},$$

together with the fact that the transition-shell mass can be made arbitrarily small under any probability measure with no atom at the vertex. Consequently the neutral-gauge nonexistence criterion requires no moment assumption on the unbounded gauge W.

Proposition 7.11 (quadratic neutral-gauge nonexistence criterion)**.** *Assume $\alpha < 2$, so that the stationary inequality, the zero-vertex-mass conclusion, and the neutral-boundary stationary equality and boundary-null integration convention of proposition 4.2 and corollaries 4.3 and 4.4 are available. Fix a drift μ. Suppose there is a continuous function $W : S \to [0, \infty)$, facewise C^2 and locally C^2-extendable away from the vertex, such that for some constants $0 < c < C < \infty$, $C_W < \infty$, and $m_0 > 0$,*

$$c|z|^2 \leq W(z) \leq C|z|^2, \qquad z \in S,$$
$$D_i W = 0 \quad \text{on } \partial S_i, \qquad i = 1, 2,$$
$$|\nabla W|^2 \leq C_W W \quad \text{on } S^\circ \text{ and by extension to the open faces,}$$

and

$$\mathcal{L}_\mu W \geq m_0 \quad \text{on } S^\circ \text{ and by extension to the open faces.}$$

Then the submartingale problem with drift μ admits no stationary distribution.

Proof. Assume, for contradiction, that π is stationary. For $0 < \varepsilon < 1 < R$ with $R^2 > 2\varepsilon^2$, let $F_{\varepsilon,R}$ be given by lemma 7.6, and set

$$f_{\varepsilon,R} := F_{\varepsilon,R}(W).$$

By lemma 7.9, the function $f_{\varepsilon,R}$ is a bounded neutral admissible test: it belongs to $C_b^2(S)$, is constant near the vertex, and satisfies $D_i f_{\varepsilon,R} = 0$ on both faces. Hence corollary 4.3 and $\pi(\partial S) = 0$ give

$$\int_{S^\circ} \mathcal{L}_\mu f_{\varepsilon,R} \, d\pi = 0.$$

Only the bounded functions $f_{\varepsilon,R}$ enter this identity; no integrability of W under π is assumed.

The same localized-gauge lemma gives, with

$$A_{\varepsilon,R} := \{2\varepsilon^2 \leq W \leq R^2\}, \qquad T_{\varepsilon,R} := \{\varepsilon^2 < W < 2\varepsilon^2\} \cup \{R^2 < W < 2R^2\},$$

the lower bound

$$\mathcal{L}_\mu f_{\varepsilon,R} \geq m_0 \mathbf{1}_{A_{\varepsilon,R}} - C_{\mathrm{loc}} \mathbf{1}_{T_{\varepsilon,R}}$$

on the open wedge, with continuous open-face extensions. By the boundary-null integration convention, corollary 4.4, this bound may be integrated against π. Combining it with stationary equality yields

$$0 \geq m_0 \pi(A_{\varepsilon,R}) - C_{\mathrm{loc}} \pi(T_{\varepsilon,R}).$$

By proposition 4.2, $\pi(\{0\}) = 0$. Applying lemma 7.7 to this W and π, choose ε and R so that, for an arbitrarily small $\delta > 0$,

$$\pi(A_{\varepsilon,R}) > 1 - \delta, \qquad \pi(T_{\varepsilon,R}) < \delta.$$

Taking $\delta < m_0/(m_0 + C_{\mathrm{loc}})$ gives

$$m_0(1 - \delta) - C_{\mathrm{loc}} \delta > 0,$$

contradicting the preceding inequality. Therefore no stationary distribution exists. □

Theorem 7.12 (nonexistence on the closed reflection cone)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

If

$$\mu \in \operatorname{cone}\{-v_1, -v_2\},$$

then the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. Let $W = h^{2/\alpha}$ be the quadratic Varadhan–Williams gauge from lemma 7.3. For every drift $\mu \in \operatorname{cone}\{-v_1, -v_2\}$, that lemma gives

$$W \asymp |z|^2, \qquad D_i W = 0 \quad (i = 1, 2), \qquad |\nabla W|^2 \le C_W W,$$

and

$$\mathcal{L}_\mu W \ge m_0 > 0 \quad \text{on } S^\circ \text{ and by continuous extension to the open faces.}$$

Thus all hypotheses of proposition 7.11 hold. The criterion rules out any stationary distribution for the drift μ. ☐

Corollary 7.13 (strict-regime stationary phase diagram)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Write $K_{\mathrm{str}} := \operatorname{cone}\{-v_1, -v_2\}$. Then the submartingale problem of definition 2.1 with drift μ admits a stationary distribution if and only if

$$\mu \notin K_{\mathrm{str}}.$$

Equivalently, stationary distributions do not exist precisely for drifts in the closed reflection cone K_{str}.

Proof. If $\mu \notin \operatorname{cone}\{-v_1, -v_2\}$, then stationary existence follows from the strict Lyapunov complement, corollary 6.13. If $\mu \in \operatorname{cone}\{-v_1, -v_2\}$, then theorem 7.12 gives nonexistence. ☐

Corollary 7.14 (zero drift and critical rays in the strict regime)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Then the submartingale problem of definition 2.1 admits no stationary distribution for zero drift. It also admits no stationary distribution on either strict critical ray:

$$\mu = t(-v_1), \quad t > 0, \qquad \text{or} \qquad \mu = t(-v_2), \quad t > 0.$$

More generally, every drift in $K_{\mathrm{str}} = \operatorname{cone}\{-v_1, -v_2\}$ is on the nonexistence side of the phase diagram.

Proof. The zero drift and both critical rays are contained in $\operatorname{cone}\{-v_1, -v_2\}$. The conclusion is therefore the corresponding special case of theorem 7.12, or equivalently of corollary 7.13. ☐

8. Structural and elliptic consequences of the stationary inequality

The stationary inequality determines a finite-measure weak elliptic system whose generator inequality is formulated directly on the open wedge. The symbol π is reserved for stationary distributions of the Markov process, whereas Λ denotes a finite nonnegative measure solution of the elliptic system. Every stationary distribution is a probability-measure solution. The comparison below concerns the corresponding drift-existence regions and does not assert that every elliptic-system solution is represented by a Markov stationary law.

Definition 8.1 (stationary elliptic system associated with the submartingale problem)**.** Fix a drift μ. A finite nonnegative Borel measure Λ on S is said to satisfy the stationary elliptic system for the drift μ if the following two conditions hold.

(E1) The measure has no boundary mass:

$$\Lambda(\partial S) = 0.$$

In particular, $\Lambda(\{0\}) = 0$.

(E2) For every admissible $f \in C_b^2(S)$,

$$\int_{S^\circ} \mathcal{L}_\mu f \, d\Lambda \le 0.$$

A solution with $\Lambda(S) = 1$ is called a probability-measure solution. The zero measure is allowed by the definition; all existence and obstruction statements below that concern finite solutions explicitly require the solution to be nonzero. Every stationary distribution for the reflected Brownian motion is a probability-measure solution by proposition 4.2.

Remark 8.2 (boundary values of the generator). The weak inequality is written over S°, where the generator is canonically defined. In view of (E1), it is equivalent to integration over S after assigning arbitrary bounded Borel values to $\mathcal{L}_\mu f$ on ∂S; those values never affect the integral.

Proposition 8.3 (interior and neutral identities). *Let Λ be a finite nonnegative measure solution of the stationary elliptic system for the drift μ. Then*

$$\int_{S^\circ} \mathcal{L}_\mu \varphi \, d\Lambda = 0, \qquad \varphi \in C_c^2(S^\circ), \tag{8.1}$$

and hence

$$\mathcal{L}_\mu^*(\Lambda|_{S^\circ}) = 0 \qquad \text{in } \mathcal{D}'(S^\circ).$$

Moreover, if $f \in C_b^2(S)$ is constant near the vertex and satisfies

$$D_i f = 0 \qquad \text{on } \partial S_i, \quad i = 1, 2,$$

then

$$\int_{S^\circ} \mathcal{L}_\mu f \, d\Lambda = 0. \tag{8.2}$$

Proof. If $\varphi \in C_c^2(S^\circ)$, then both φ and $-\varphi$ are admissible because their supports are separated from the boundary and the vertex. Applying (E2) first to φ and then to $-\varphi$ gives

$$\int_{S^\circ} \mathcal{L}_\mu \varphi \, d\Lambda \le 0, \qquad -\int_{S^\circ} \mathcal{L}_\mu \varphi \, d\Lambda \le 0,$$

and hence (8.1). If $f \in C_b^2(S)$ is neutral, then $D_i f = 0$ on both faces implies $D_i(-f) = 0$, so both f and $-f$ satisfy the admissibility inequalities. Applying (E2) to the two signs gives

$$\int_{S^\circ} \mathcal{L}_\mu f \, d\Lambda \le 0, \qquad -\int_{S^\circ} \mathcal{L}_\mu f \, d\Lambda \le 0,$$

which is (8.2). All integrals in these identities are over the open wedge; by (E1) this is equivalent to any bounded Borel extension of the generator to the boundary. □

Proposition 8.4 (elliptic neutral projection obstruction). *Let Λ be a nonzero finite nonnegative measure solution of the stationary elliptic system of definition 8.1 for the drift μ. Suppose that there exists $\ell \in \mathbb{R}^2$ such that*

$$\ell \cdot z > 0 \quad (z \in S \setminus \{0\}), \qquad \ell \cdot v_i = 0 \quad (i = 1, 2), \qquad \ell \cdot \mu = 0.$$

Then no such Λ exists. Equivalently, the neutral linear projection contradiction of proposition 5.13 is an elliptic-system obstruction and does not require an underlying Markov stationary law.

Proof. Since $0 < \Lambda(S) < \infty$, define

$$\widehat{\Lambda} := \frac{\Lambda}{\Lambda(S)}.$$

Then $\widehat{\Lambda}(\partial S)=0$, and for every admissible f,

$$\int_{S^\circ}\mathcal{L}_\mu f\,d\widehat{\Lambda}=\frac{1}{\Lambda(S)}\int_{S^\circ}\mathcal{L}_\mu f\,d\Lambda\le 0.$$

Thus $\widehat{\Lambda}$ is again an elliptic-system solution. Rename it Λ, so that $\Lambda(S)=1$. Let $\varphi\in C_c^2((0,\infty))$, extend it by zero to a neighborhood of the endpoint, and put

$$f(z)=\varphi(\ell\cdot z).$$

Since ℓ is strictly positive on the compact set $S\cap\mathbb{S}^1$, there is $a_\ell>0$ such that $\ell\cdot z\ge a_\ell|z|$ on S. Hence $f\in C_b^2(S)$, is supported in a compact subset of $S\setminus\{0\}$, and is constant near the vertex. Moreover

$$D_i f=\varphi'(\ell\cdot z)\,\ell\cdot v_i=0\qquad\text{on }\partial S_i.$$

Thus both f and $-f$ are admissible elliptic-system tests. By proposition 8.3,

$$0=\int_{S^\circ}\mathcal{L}_\mu f\,d\Lambda.$$

The open-wedge generator is

$$\mathcal{L}_\mu f(z)=\frac{1}{2}|\ell|^2\varphi''(\ell\cdot z)+(\ell\cdot\mu)\varphi'(\ell\cdot z)=\frac{1}{2}|\ell|^2\varphi''(\ell\cdot z).$$

Let m_ℓ be the push-forward of Λ under $z\mapsto\ell\cdot z$. By (E1) and strict positivity of ℓ on $S\setminus\{0\}$, this is a finite nonnegative measure on $(0,\infty)$ with total mass $\Lambda(S)>0$. The preceding identity gives

$$\int_{(0,\infty)}\varphi''(s)\,m_\ell(ds)=0,\qquad\varphi\in C_c^2((0,\infty)).$$

By lemma 5.12, $m_\ell=0$, contradicting $m_\ell((0,\infty))=\Lambda(S)>0$. □

Proposition 8.5 (elliptic direct admissible-supersolution obstruction)**.** *Let Λ be a nonzero finite nonnegative measure solution of the stationary elliptic system of definition 8.1 for the drift μ. If $\mu\in\mathfrak{M}_{\sup}$, then no such Λ exists.*

Proof. Choose $c\in\mathfrak{B}$ with $c\cdot\mu>0$. Since $0<\Lambda(S)<\infty$, set

$$\widehat{\Lambda}:=\frac{\Lambda}{\Lambda(S)}.$$

The normalization preserves the boundary-null condition, and linearity gives

$$\widehat{\Lambda}(\partial S)=0,\qquad\int_{S^\circ}\mathcal{L}_\mu f\,d\widehat{\Lambda}=\frac{1}{\Lambda(S)}\int_{S^\circ}\mathcal{L}_\mu f\,d\Lambda\le 0$$

for every admissible f. Rename $\widehat{\Lambda}$ as Λ, so that $\Lambda(S)=1$. Fix $\delta>0$, and let

$$f_\delta(z)=h_\delta(c\cdot z)$$

be the bounded one-dimensional profile from lemma 5.3. It belongs to $C_b^2(S)$, is constant near the vertex, and satisfies

$$D_i f_\delta\ge 0\quad\text{on }\partial S_i,\qquad\mathcal{L}_\mu f_\delta\ge 0\quad\text{on }S^\circ,$$

with strict positivity on $S^\circ\cap\{c\cdot z>\delta\}$. Hence f_δ is an admissible elliptic-system test, and (E2) gives

$$0\ge\int_{S^\circ}\mathcal{L}_\mu f_\delta\,d\Lambda.$$

Since (E1) gives $\Lambda(\partial S)=0$ and the generator is nonnegative in the open wedge, the same integral is nonnegative. Thus it is zero. If

$$A_\delta:=S^\circ\cap\{c\cdot z>\delta\}$$

had positive Λ-mass, then the strict positivity of $\mathcal{L}_\mu f_\delta$ on A_δ, together with

$$A_\delta = \bigcup_{n=1}^{\infty} \left(A_\delta \cap \left\{ \mathcal{L}_\mu f_\delta \geq \frac{1}{n} \right\} \right),$$

would force $\int_{S^\circ} \mathcal{L}_\mu f_\delta \, d\Lambda > 0$, a contradiction. Therefore

$$\Lambda\{z \in S : c \cdot z > \delta\} = 0, \qquad \delta > 0,$$

where the boundary has again been removed by (E1). Because $c \in S_\circ^\vee$,

$$S \setminus \{0\} = \bigcup_{n \geq 1} \{z \in S : c \cdot z > 1/n\}.$$

It follows that $\Lambda(S \setminus \{0\}) = 0$. Together with (E1), this gives $\Lambda(S) = 0$, contradicting the normalization. Thus no nonzero finite nonnegative measure solution of the elliptic system exists for $\mu \in \mathfrak{M}_{\text{sup}}$. □

Proposition 8.6 (elliptic strict closed-cone obstruction)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2,$$

and let $K_{\text{str}} = \text{cone}\{-v_1, -v_2\}$*. If* $\mu \in K_{\text{str}}$*, then the stationary elliptic system of definition 8.1 admits no nonzero finite nonnegative measure solution.*

Proof. Suppose that such a solution Λ exists. Since $0 < \Lambda(S) < \infty$, replace Λ by $\widehat{\Lambda} := \Lambda/\Lambda(S)$; the identities $\widehat{\Lambda}(\partial S) = 0$ and

$$\int_{S^\circ} \mathcal{L}_\mu f \, d\widehat{\Lambda} = \frac{1}{\Lambda(S)} \int_{S^\circ} \mathcal{L}_\mu f \, d\Lambda \leq 0$$

show that $\widehat{\Lambda}$ is again an elliptic-system solution. Rename it Λ, so that $\Lambda(S) = 1$.

Let $W = h^{2/\alpha}$ be the quadratic Varadhan–Williams gauge from lemma 7.3. For every $\mu \in K_{\text{str}}$, that lemma gives

$$c|z|^2 \leq W(z) \leq C|z|^2, \qquad D_i W = 0 \ (i = 1, 2), \qquad |\nabla W|^2 \leq C_W W, \qquad \mathcal{L}_\mu W \geq m_0 > 0.$$

We apply the two-scale estimate at the level of the finite measure and write all parameter choices explicitly. Condition (E1) gives $\Lambda(\{0\}) = 0$. Fix

$$0 < \delta < \frac{m_0}{m_0 + C_{\text{loc}}}. \tag{8.3}$$

By lemma 7.7, applied to the probability measure Λ and to $W \asymp |z|^2$, there are $0 < \varepsilon < 1 < R$, with $R^2 > 2\varepsilon^2$, such that

$$\Lambda(A_{\varepsilon,R}) > 1 - \delta, \tag{8.4}$$

$$\Lambda(T_{\varepsilon,R}) < \delta, \tag{8.5}$$

where

$$A_{\varepsilon,R} = \{2\varepsilon^2 \leq W \leq R^2\}, \qquad T_{\varepsilon,R} = \{\varepsilon^2 < W < 2\varepsilon^2\} \cup \{R^2 < W < 2R^2\}.$$

Let

$$f_{\varepsilon,R} = F_{\varepsilon,R}(W).$$

By lemma 7.9, this function belongs to $C_b^2(S)$, is constant near the vertex and outside a compact set, and satisfies $D_i f_{\varepsilon,R} = 0$ on both open faces. Thus both signs are admissible elliptic-system tests, and the neutral identity in proposition 8.3 gives

$$0 = \int_{S^\circ} \mathcal{L}_\mu f_{\varepsilon,R} \, d\Lambda. \tag{8.6}$$

The pointwise open-wedge estimate from the same localization lemma is

$$\mathcal{L}_\mu f_{\varepsilon,R} \geq m_0 \mathbf{1}_{A_{\varepsilon,R}} - C_{\text{loc}} \mathbf{1}_{T_{\varepsilon,R}}.$$

It may be integrated directly over S°; condition (E1) removes all boundary values. Combining it with (8.4)– (8.6) yields

$$0 \geq m_0 \Lambda(A_{\varepsilon,R}) - C_{\mathrm{loc}} \Lambda(T_{\varepsilon,R}) > m_0(1-\delta) - C_{\mathrm{loc}}\delta > 0,$$

where the final inequality is exactly (8.3). This contradiction rules out the normalized solution. Positive rescaling then rules out every nonzero finite nonnegative solution on K_{str}. □

Proposition 8.7 (probabilistic-to-elliptic bridge)**.** *Assume $\alpha < 2$, and let π be a stationary distribution for the reflected Brownian motion with drift μ. Then π is a probability-measure solution of the stationary elliptic system of definition 8.1. Moreover, its interior density is smooth, strictly positive, and real analytic in S°. If $p_\pi = e^{\mu\cdot z} q_\pi$, then*

$$\Delta q_\pi = |\mu|^2 q_\pi \quad \textit{in } S^\circ.$$

Proof. By proposition 4.2,

$$\pi(\partial S) = 0$$

and, for every admissible test function $f \in C_b^2(S)$,

$$\int_{S^\circ} \mathcal{L}_\mu f \, d\pi = \int_S \mathcal{L}_\mu f \, d\pi \leq 0.$$

The equality of the two integrals uses the boundary-null property and an arbitrary bounded Borel extension of the open-wedge generator to ∂S. These two displayed facts are exactly conditions (E1) and (E2) in definition 8.1; since $\pi(S) = 1$, the measure π is a probability-measure solution of that system.

For every $\varphi \in C_c^2(S^\circ)$, both signs are admissible, so proposition 4.7 gives

$$\int_{S^\circ} \mathcal{L}_\mu \varphi \, d\pi = 0, \qquad \pi|_{S^\circ} = p_\pi(z)\, dz, \qquad \frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0.$$

The density is nonnegative and has total mass one in the open wedge; corollary 4.8 therefore yields $p_\pi > 0$ throughout the connected domain S°. Finally, setting $q_\pi = e^{-\mu\cdot z} p_\pi$, the explicit calculation in proposition 4.9 gives

$$\Delta q_\pi = |\mu|^2 q_\pi.$$

Constant-coefficient elliptic analyticity then gives real analyticity of q_π and of $p_\pi = e^{\mu\cdot z} q_\pi$. This proves every assertion of the proposition. □

Proposition 8.8 (interior regularity for elliptic-system solutions)**.** *Let Λ be a nonzero finite nonnegative measure solution of the stationary elliptic system of definition 8.1 for the drift μ. Then there is a unique smooth function p_Λ on S° such that*

$$\Lambda|_{S^\circ} = p_\Lambda(z)\, dz.$$

The density is strictly positive and real analytic and satisfies

$$\frac{1}{2}\Delta p_\Lambda - \mu \cdot \nabla p_\Lambda = 0 \qquad \textit{in } S^\circ.$$

Moreover, with

$$q_\Lambda(z) := e^{-\mu\cdot z} p_\Lambda(z),$$

one has

$$\Delta q_\Lambda = |\mu|^2 q_\Lambda \qquad \textit{in } S^\circ.$$

Proof. By proposition 8.3, the restriction $T := \Lambda|_{S^\circ}$, regarded as a distribution of order zero, satisfies

$$\left(\frac{1}{2}\Delta - \mu \cdot \nabla\right) T = 0 \qquad \text{in } \mathcal{D}'(S^\circ).$$

Multiply the distribution by the smooth positive function $e^{-\mu\cdot z}$ and set

$$Q := e^{-\mu\cdot z} T, \qquad T = e^{\mu\cdot z} Q.$$

The conjugation may be checked directly at the distributional level. If $\psi \in C_c^\infty(S^\circ)$, then

$$\langle (\Delta - |\mu|^2)Q, \psi \rangle = \langle T, e^{-\mu\cdot z}(\Delta - |\mu|^2)\psi \rangle .$$

For $\varphi = e^{-\mu\cdot z}\psi$, a direct calculation gives

$$\left(\frac{1}{2}\Delta + \mu\cdot\nabla\right)\varphi = \frac{1}{2}e^{-\mu\cdot z}(\Delta - |\mu|^2)\psi.$$

The distributional adjoint equation for T therefore implies

$$\langle (\Delta - |\mu|^2)Q, \psi \rangle = 0.$$

Thus

$$(\Delta - |\mu|^2)Q = 0 \qquad \text{in } \mathcal{D}'(S^\circ).$$

The operator $\Delta - |\mu|^2$ has principal symbol $-|\xi|^2$ and is elliptic. Elliptic hypoellipticity [3] first gives $Q \in C^\infty(S^\circ)$, and analytic elliptic regularity [6] then makes this smooth solution real analytic. Thus Q is represented by a real-analytic function q_Λ on S°. Consequently

$$T = e^{\mu\cdot z}q_\Lambda(z)\,dz, \qquad p_\Lambda(z) := e^{\mu\cdot z}q_\Lambda(z)$$

is the unique smooth, real-analytic density of Λ in the open wedge, and it satisfies the asserted adjoint equation. The displayed distributional equation becomes

$$\Delta q_\Lambda = |\mu|^2 q_\Lambda$$

classically.

Since Λ is nonnegative, $p_\Lambda \geq 0$ almost everywhere and hence everywhere by continuity. Condition (E1) and $\Lambda \neq 0$ give $\Lambda(S^\circ) = \Lambda(S) > 0$, so $p_\Lambda \not\equiv 0$. The strong maximum principle [2, Chapter 3] for $\frac{1}{2}\Delta p_\Lambda - \mu\cdot\nabla p_\Lambda = 0$ on the connected domain S° yields $p_\Lambda > 0$ throughout S°. No regularity at the vertex or across the two faces is asserted here. ☐

Theorem 8.9 (stationary elliptic-system phase diagram). *Assume*

$$0 < \xi < \pi, \qquad 1 \leq \alpha < 2.$$

For probability-measure solutions of the stationary elliptic system in definition 8.1, the following drift classification holds. In the strict case, $K_{\mathrm{str}} = \mathrm{cone}\{-v_1, -v_2\}$.

(i) *If $\alpha = 1$, a probability-measure solution exists if and only if*

$$n_{\mathscr{L}}\cdot\mu < 0.$$

(ii) *If $1 < \alpha < 2$, a probability-measure solution exists if and only if*

$$\mu \notin K_{\mathrm{str}}.$$

Equivalently, because the defining conditions in definition 8.1 are homogeneous under multiplication by positive constants, the same drift classification holds for nonzero finite nonnegative measure solutions after normalizing by their total mass.

Proof. The existence implications follow from the probabilistic phase diagram, theorem 3.1, and the bridge proposition 8.7: whenever a stationary distribution exists, it is a probability-measure solution of the stationary elliptic system.

The nonexistence implications also hold for the elliptic system. The direct admissible-supersolution obstruction proposition 8.5 rules out probability-measure solutions whenever $\mu \in \mathfrak{M}_{\mathrm{sup}}$; in the $\alpha = 1$ geometry this is exactly the half-plane $n_{\mathscr{L}}\cdot\mu > 0$. That proof uses only bounded one-dimensional profiles, the boundary-null condition (E1) and the admissible-test inequality (E2). On the reflection line $n_{\mathscr{L}}\cdot\mu = 0$, proposition 8.4, applied with $\ell = n_{\mathscr{L}}$, rules out probability-measure solutions directly from (E1) and the neutral identity in proposition 8.3. The proof uses only compactly supported neutral tests and the finite half-line distribution lemma; it does not require a Markov realization of the measure.

In the strict case, let $K_{\mathrm{str}} = \mathrm{cone}\{-v_1, -v_2\}$. The nonexistence implication for every $\mu \in K_{\mathrm{str}}$, including the two boundary rays and $\mu = 0$, is the elliptic strict closed-cone obstruction of proposition 8.6. That proposition is exactly the two-scale Varadhan–Williams bounded-neutral localization proof rewritten at the level of (E1) and proposition 8.3, so it does not require a Markov realization of the measure. This proves the asserted probability-measure classification.

Finally, if Λ is a nonzero finite nonnegative measure solution of the elliptic system, then $\Lambda(S) \in (0, \infty)$, and $\Lambda/\Lambda(S)$ is a probability-measure solution because the defining conditions (E1) and (E2) are linear in Λ. Conversely, every probability-measure solution is a nonzero finite nonnegative measure solution. The assertion for nonzero finite nonnegative measures is therefore equivalent to the assertion for probability-measure solutions. □

Corollary 8.10 (drift-level equivalence of probabilistic and elliptic stationarity)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2.$$

For a fixed drift μ, the following three assertions are equivalent.

(i) *The Lakner–Liu–Reed reflected Brownian motion with drift μ admits a stationary distribution.*
(ii) *The stationary elliptic system of definition 8.1 admits a probability-measure solution.*
(iii) *The stationary elliptic system of definition 8.1 admits a nonzero finite nonnegative measure solution.*

Equivalently, in the borderline case $\alpha = 1$ these assertions hold exactly when

$$n_{\mathscr{L}} \cdot \mu < 0,$$

and in the strict case $1 < \alpha < 2$ they hold exactly when

$$\mu \notin K_{\mathrm{str}} = \mathrm{cone}\{-v_1, -v_2\}.$$

The equivalence is at the level of drift regions. It does not assert that each abstract elliptic-system solution is represented by a Markov stationary law.

Proof. The implication (i)⇒(ii) is the bridge proposition 8.7. The implication (ii)⇒(iii) is immediate. Conversely, if (iii) holds, normalization gives a probability-measure solution, and theorem 8.9 places μ in the elliptic existence region. That region is identical to the probabilistic existence region in theorem 3.1; hence a stationary distribution exists. The two displayed drift descriptions are the alternatives in theorems 3.1 and 8.9. □

8.1. **Interior elliptic equation.** By propositions 4.2, 4.7 and 4.9 and corollary 4.8, every stationary distribution has zero boundary mass and a strictly positive real-analytic density p_π on S° satisfying

$$\frac{1}{2}\Delta p_\pi - \mu \cdot \nabla p_\pi = 0.$$

The ground-state transform $p_\pi = e^{\mu \cdot z} q_\pi$ gives

$$\Delta q_\pi = |\mu|^2 q_\pi.$$

The oblique reflection enters through the admissible-test inequalities rather than through a classical boundary condition for p_π.

9. Conclusion

We have determined the complete stationary-existence region in the convex cases $\alpha = 1$ and $1 < \alpha < 2$. At $\alpha = 1$, the sign of $n_{\mathscr{L}} \cdot \mu$ gives the classification. In the strict regime, the existence region is the complement of $K_{\mathrm{str}} = \mathrm{cone}\{-v_1, -v_2\}$. The proof combines the stationary inequality with bounded one-dimensional tests, a Foster–Lyapunov compactness argument, and bounded localizations of the Varadhan–Williams gauge.

The same bounded-test identities define a finite-measure weak elliptic system with the same drift phase diagram. Its interior adjoint equation and compatibility consequences provide an elliptic formulation of the classification without introducing boundary fluxes or unbounded test functions. The appendices record supplementary compatibility, impossibility, and gauge criteria. Each of those results is proved in full, but none is used to replace or abbreviate any step of the phase-diagram proof in the main text.

Appendix A. Compatibility conditions for weak elliptic solutions

This appendix records a compatibility consequence of the weak elliptic formulation. A probability-measure solution of the stationary elliptic system cannot coexist with a local admissible patching family for the same drift. By proposition 8.7, the same conclusion applies to every stationary distribution.

Proposition A.1 (elliptic-system solutions obstruct patching)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2,$$

and fix a drift vector $\mu \in \mathbb{R}^2$. If the stationary elliptic system of definition 8.1 admits a probability-measure solution for the drift μ, then there does not exist any local admissible patching family for that drift. Consequently, the same obstruction holds whenever the reflected Brownian motion admits a stationary distribution.

Proof. Assume, for contradiction, that Λ_0 is a probability-measure solution of the stationary elliptic system and that $\{f_\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ is a local admissible patching family for the same drift. Let $r > 0$ be the fixed constant in definition 4.12. By the elliptic-system inequality (E2),

$$0 \ge \int_{S^\circ} \mathcal{L} f_\varepsilon \, d\Lambda_0. \tag{A.1}$$

On $S^\circ \cap B_{r\varepsilon}$, the patching hypothesis gives $\mathcal{L} f_\varepsilon \ge 0$. On $S^\circ \setminus B_{r\varepsilon}$, one has $f_\varepsilon = \Phi_\gamma$, and therefore $\mathcal{L} f_\varepsilon = \mathcal{L}\Phi_\gamma$. Hence the pointwise open-wedge inequality

$$\mathcal{L} f_\varepsilon \ge \mathbf{1}_{S^\circ \setminus B_{r\varepsilon}} \mathcal{L}\Phi_\gamma \qquad \text{on } S^\circ \tag{A.2}$$

holds. Both sides are bounded Borel functions. Integrating (A.2) against Λ_0 gives

$$\int_{S^\circ} \mathcal{L} f_\varepsilon \, d\Lambda_0 \ge \int_{S^\circ \setminus B_{r\varepsilon}} \mathcal{L}\Phi_\gamma \, d\Lambda_0. \tag{A.3}$$

No boundary value of either generator is involved, because $\Lambda_0(\partial S) = 0$ by (E1).

By proposition 4.11, the function $\mathcal{L}\Phi_\gamma$ is bounded and strictly positive at every point of S°. Since $\Lambda_0(S^\circ) = 1$, its integral is strictly positive. Indeed, with

$$E_n := \{z \in S^\circ : \mathcal{L}\Phi_\gamma(z) \ge n^{-1}\},$$

one has $S^\circ = \bigcup_{n \ge 1} E_n$; hence $\Lambda_0(E_{n_0}) > 0$ for some n_0, and

$$\int_{S^\circ} \mathcal{L}\Phi_\gamma \, d\Lambda_0 \ge n_0^{-1} \Lambda_0(E_{n_0}) > 0. \tag{A.4}$$

Moreover, $\mathbf{1}_{S^\circ \setminus B_{r\varepsilon}} \uparrow \mathbf{1}_{S^\circ}$ as $\varepsilon \downarrow 0$. Monotone convergence applied to the nonnegative function $\mathcal{L}\Phi_\gamma$ and (A.4) therefore yields an $\varepsilon_1 > 0$ such that

$$\int_{S^\circ \setminus B_{r\varepsilon}} \mathcal{L}\Phi_\gamma \, d\Lambda_0 > 0, \qquad 0 < \varepsilon < \varepsilon_1.$$

For such ε, (A.3) contradicts (A.1). Thus no local admissible patching family can coexist with a probability-measure solution. If a stationary distribution exists, proposition 8.7 makes it such a solution, proving the final assertion. □

Corollary A.2 (patching is impossible on the existence cone)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2.$$

If $\mu \in \mathfrak{M}_{\mathrm{Lyap}}$, then no local admissible patching family can exist for that drift. In particular, if $b \cdot \mu > 0$, then no local admissible patching family can exist for that drift.

Proof. Combine proposition 6.11 and corollary 6.31 with proposition A.1. □

Remark A.3 (elliptic interpretation). The local admissible patching family is a supersolution family for the linear elliptic operator

$$\mathcal{L} = \mu \cdot \nabla + \frac{1}{2}\Delta$$

with oblique inequalities on a singular wedge domain. Thus every stationary existence theorem yields an obstruction theorem for this elliptic patching problem on the same drift region.

Appendix B. Limitations of local admissible corrections

We identify two natural correction mechanisms that cannot satisfy the simultaneous admissibility and generator inequalities required by the patching criterion: scalar flattening of the separator barrier and compactly supported separated-variable corrections built from a positive angular mode.

B.1. Impossibility of scalar flattening.

Assume

$$0 < \xi < \pi, \qquad 1 \le \alpha < 2,$$

and fix the separator barrier Φ_γ from proposition 4.11. Define

$$c_\gamma := \gamma b \cdot \mu + \frac{1}{2}\gamma^2 |b|^2, \qquad d_\gamma := \frac{1}{2}\gamma^2 |b|^2.$$

The choice of γ in proposition 4.11 gives $c_\gamma > 0$, and the definition gives $d_\gamma > 0$.

Proposition B.1 (impossibility of scalar flattening)**.** *Let $g \in C^2([-1,0])$ and define*

$$f(x) = g(\Phi_\gamma(x)).$$

If

$$\mathcal{L} f \ge 0 \quad \text{on } S^\circ,$$

then for every $s \in (-1,0)$,

$$c_\gamma g'(s) + d_\gamma (1+s) g''(s) \ge 0.$$

Equivalently, with

$$a_\gamma := \frac{c_\gamma}{d_\gamma}, \qquad H(s) := (1+s)^{a_\gamma} g'(s),$$

then $a_\gamma > 0$ and

$$H'(s) \ge 0 \quad \text{for all } s \in (-1,0).$$

In particular, if g is constant on some interval $[s_0, 0]$ with $s_0 \in (-1,0)$, then

$$g'(s) \le 0 \quad \text{for all } s \in [-1, s_0].$$

Therefore no function of the form $f = g(\Phi_\gamma)$ can simultaneously be constant near the vertex, equal to Φ_γ outside a compact set, and satisfy $\mathcal{L} f \ge 0$ on S°.

Proof. We have

$$\nabla \Phi_\gamma = \gamma(1+\Phi_\gamma) b, \qquad \Delta \Phi_\gamma = \gamma^2 |b|^2 (1+\Phi_\gamma),$$

thus

$$\mathcal{L}\Phi_\gamma = (1+\Phi_\gamma) c_\gamma, \qquad |\nabla \Phi_\gamma|^2 = 2 d_\gamma (1+\Phi_\gamma)^2.$$

Applying the chain rule to $f = g(\Phi_\gamma)$ yields

$$\mathcal{L}f = (1+\Phi_\gamma)\Big(c_\gamma g'(\Phi_\gamma) + d_\gamma(1+\Phi_\gamma)g''(\Phi_\gamma)\Big).$$

Since every value in $(-1,0)$ is attained by Φ_γ along an interior ray, the assumption $\mathcal{L}f \geq 0$ and the displayed chain-rule formula imply

$$c_\gamma g'(s) + d_\gamma(1+s)g''(s) \geq 0, \qquad -1 < s < 0.$$

Writing $a_\gamma = c_\gamma/d_\gamma$, we have

$$H'(s) = (1+s)^{a_\gamma - 1}\big((1+s)g''(s) + a_\gamma g'(s)\big) \geq 0, \qquad -1 < s < 0,$$

so H is nondecreasing on $(-1,0)$. If g is constant on $[s_0,0]$, then $H(s_0)=0$, and monotonicity gives $H(s) \leq 0$ for $-1 < s \leq s_0$. Since $(1+s)^{a_\gamma} > 0$, this gives $g'(s) \leq 0$ on $(-1,s_0]$, and the endpoint value at $s=-1$ follows by continuity when it is needed.

If f were constant near the vertex, then g would be constant on some $[s_0,0]$. If $f = \Phi_\gamma$ outside a compact set, then along any fixed interior ray we must have $g(s) = s$ on an interval $[-1, s_r]$, hence $g'(s) = 1$ there. This contradicts the preceding inequality $g' \leq 0$ near -1. □

B.2. **Impossibility of positive-mode compactly supported separated-variable correctors.**

Lemma B.2 (positive angular mode)**.** *Assume*

$$0 < \xi < \pi, \qquad m \in \Big(\frac{\pi}{\xi}, \frac{2\pi}{\xi}\Big).$$

Let ψ solve

$$\psi''(\theta) + m^2\psi(\theta) = 1, \qquad \psi(0) = \psi(\xi) = 0.$$

Then

$$\psi(\theta) > 0 \quad (0 < \theta < \xi), \qquad \psi'(0) > 0, \qquad \psi'(\xi) < 0.$$

Proof. The explicit solution is

$$\psi(\theta) = \frac{1-\cos(m\theta)}{m^2} + \frac{\cos(m\xi)-1}{m^2\sin(m\xi)}\sin(m\theta).$$

Writing $\delta = m\xi/2 \in (\pi/2, \pi)$, the preceding formula can be rewritten as

$$\psi(\theta) = \frac{\cos\delta - \cos(m(\theta - \xi/2))}{m^2\cos\delta}.$$

For $0 < \theta < \xi$ we have $|m(\theta - \xi/2)| < \delta$; since cos is even and strictly decreasing on $[0,\pi]$, this gives $\cos(m(\theta-\xi/2)) > \cos\delta$. Both the numerator and the denominator in the displayed expression are therefore negative, and hence $\psi(\theta) > 0$. Moreover,

$$\psi'(\theta) = \frac{\sin(m(\theta-\xi/2))}{m\cos\delta},$$

so $\psi'(0) > 0$ and $\psi'(\xi) < 0$, because $\cos\delta < 0$. □

Proposition B.3 (impossibility of positive-mode compactly supported separated-variable correctors)**.** *Let ψ be the positive angular mode from lemma B.2. Let*

$$\eta \in C_c^2((c_3,d_3)), \qquad \eta \geq 0, \qquad \eta \not\equiv 0,$$

and define

$$\chi(r,\theta) = \eta(r)r^m\psi(\theta).$$

Then

$$\Delta\chi \geq 0 \quad \textit{cannot hold on the whole shell } S \cap (B_{d_3} \setminus B_{c_3}).$$

Consequently, no compactly supported separated-variable corrector of this positive angular-mode form can serve as the local correction required in theorem 4.14.

Proof. A direct polar-coordinate computation yields

$$\Delta\chi = r^{m-2}\Big(\eta(r) + \psi(\theta)[r^2\eta''(r) + (2m+1)r\eta'(r)]\Big).$$

Let $\theta_* \in (0,\xi)$ be a point at which ψ attains its maximum, and put

$$\Psi_* := \psi(\theta_*) = \max_{[0,\xi]} \psi.$$

The maximum point lies in the open interval because $\psi > 0$ on $(0,\xi)$ and $\psi(0) = \psi(\xi) = 0$. If $\Delta\chi \geq 0$ everywhere, then evaluating the displayed formula for $\Delta\chi$ at $\theta = \theta_*$ gives

$$\eta(r) + \Psi_*[r^2\eta''(r) + (2m+1)r\eta'(r)] \geq 0 \quad \text{for all } r.$$

Set $r = e^s$ and $h(s) = \eta(e^s)$. Then

$$r\eta'(r) = h'(s), \qquad r^2\eta''(r) = h''(s) - h'(s),$$

so the preceding inequality becomes

$$h(s) + \Psi_*\big(h''(s) + 2mh'(s)\big) \geq 0.$$

Now define

$$y(s) := e^{ms}h(s).$$

Then

$$\Psi_* y''(s) + (1 - \Psi_* m^2)y(s) \geq 0.$$

At θ_*, one has $\psi''(\theta_*) \leq 0$. Hence the differential equation for ψ gives

$$1 = m^2\psi(\theta_*) + \psi''(\theta_*) \leq m^2\Psi_*.$$

Hence $\Psi_* m^2 - 1 \geq 0$, and the preceding displayed inequality implies

$$y''(s) \geq 0.$$

Thus y is convex. But $y = e^{ms}\eta(e^s)$ is nonnegative, nonzero, and compactly supported. Choose s_0 with $y(s_0) > 0$. Since y vanishes for all sufficiently negative and sufficiently positive s, choose $a < s_0 < b$ with $y(a) = y(b) = 0$. Convexity gives

$$y(s_0) \leq \frac{b - s_0}{b - a}y(a) + \frac{s_0 - a}{b - a}y(b) = 0,$$

a contradiction. □

Appendix C. Additional elliptic-gauge criteria

This appendix develops supplementary sufficient conditions for nonexistence based on bounded monotone compositions or compact localizations of proper gauges. The elliptic-norm criterion uses $\rho_Q(z) = (z^TQz)^{1/2}$; the cone-quadratic criterion uses boundary-distance coordinates; and the final criterion treats a general proper gauge. The abstract elliptic-norm and proper-gauge results require $\alpha < 2$. Coordinate formulas and cone-quadratic results assume $0 < \xi < \pi$, while statements concerning the closed reflection cone assume $1 < \alpha < 2$. In every case the function inserted into the stationary inequality is bounded, belongs to $C_b^2(S)$, and is constant near the vertex.

Here Q denotes a symmetric positive definite matrix, ρ_Q its elliptic norm, q a cone-quadratic gauge, and H a general proper gauge. Abstract finite nonnegative measure solutions of the stationary elliptic system are denoted by Λ.

Let Q be a symmetric positive definite 2×2 matrix and define

$$\rho_Q(z) := (z^TQz)^{1/2}.$$

Let u_1, u_2 denote the two unit generators of the boundary rays of the wedge. The following two elementary conditions are used:

$$v_i \cdot Qu_i \geq 0, \qquad i = 1, 2, \tag{EN1}$$

and

$$Q\mu \in S_\circ^\vee. \tag{EN2}$$

Condition (EN1) is the oblique boundary sign condition for the elliptic norm ρ_Q. Condition (EN2) says that the drift points outward in the Q-dual gauge.

Lemma C.1 (a bounded one-dimensional profile with controlled logarithmic derivative)**.** *Let $\gamma > 0$ and $\delta > 0$. There exists $h_\delta \in C_b^2([0,\infty))$ such that*

$$h_\delta = 0 \quad \text{on } [0, \delta/2], \qquad h'_\delta(r) > 0 \quad (r > \delta),$$
$$h'_\delta(r) \geq 0, \qquad h''_\delta(r) \geq -\gamma h'_\delta(r) \quad \text{for all } r \geq 0.$$

Proof. Choose $\zeta_\delta \in C^\infty([0,\infty))$ such that $0 \leq \zeta_\delta \leq 1$, $\zeta_\delta = 0$ on $[0, \delta/2]$, $\zeta_\delta > 0$ on (δ, ∞), $\zeta'_\delta = 0$ outside a compact subset of $(\delta/2, \delta)$, and $\zeta'_\delta \geq 0$. Define

$$h_\delta(r) = \int_0^r \zeta_\delta(s) e^{-\gamma s}\, ds.$$

Then h_δ is bounded, nondecreasing, and constant near zero. Moreover

$$h'_\delta(r) = \zeta_\delta(r) e^{-\gamma r}, \qquad h''_\delta(r) = \big(\zeta'_\delta(r) - \gamma\zeta_\delta(r)\big) e^{-\gamma r} \geq -\gamma h'_\delta(r).$$

This proves the lemma. □

Proposition C.2 (elliptic-norm admissible supersolution criterion)**.** *Assume $\alpha < 2$. Assume there exists a symmetric positive definite matrix Q satisfying (EN1) and (EN2). Then the solution to the Lakner–Liu–Reed submartingale problem with drift μ admits no stationary distribution.*

Proof. We construct a vanishing-core admissible contradiction family and then apply theorem 4.16.

The following elementary differential estimates hold for ρ_Q. On $S \setminus \{0\}$,

$$\nabla \rho_Q(z) = \frac{Qz}{\rho_Q(z)}, \qquad |\nabla \rho_Q(z)|^2 \leq M_Q$$

for some finite constant M_Q. Also

$$\Delta \rho_Q(z) = \frac{\operatorname{tr} Q}{\rho_Q(z)} - \frac{z^T Q^2 z}{\rho_Q(z)^3}.$$

In two dimensions the numerator is nonnegative, because after diagonalizing $Q = \operatorname{diag}(q_1, q_2)$ one has

$$(\operatorname{tr} Q)(z^T Q z) - z^T Q^2 z = q_1 q_2 |z|^2 \geq 0.$$

Therefore

$$\Delta \rho_Q \geq 0.$$

By (EN2), the function $z \mapsto (Q\mu) \cdot z$ is strictly positive on $S \setminus \{0\}$. Hence, on the compact set $\{z \in S : \rho_Q(z) = 1\}$,

$$m_Q := \min_{\rho_Q(z)=1,\ z \in S} \mu \cdot \nabla \rho_Q(z) = \min_{\rho_Q(z)=1,\ z \in S} (Q\mu) \cdot z > 0.$$

Fix $z \in S \setminus \{0\}$ and put

$$r := \rho_Q(z) > 0, \qquad u := \frac{z}{r}.$$

Then $u \in S$, $\rho_Q(u) = 1$, and

$$\mu \cdot \nabla \rho_Q(z) = \frac{(Q\mu) \cdot z}{\rho_Q(z)} = (Q\mu) \cdot u \geq m_Q.$$

Consequently,

$$\mathcal{L}\rho_Q = \mu \cdot \nabla \rho_Q + \frac{1}{2}\Delta \rho_Q \geq m_Q.$$

Since $M_Q > 0$, fix explicitly any

$$0 < \gamma < \frac{2m_Q}{M_Q}.$$

Equivalently,

$$m_Q - \frac{\gamma M_Q}{2} > 0.$$

For each $\delta > 0$, let h_δ be given by lemma C.1, and set

$$f_\delta(z) = h_\delta(\rho_Q(z)).$$

Since h_δ is constant on $[0, \delta/2]$ and $\rho_Q(z) \asymp |z|$ on S, the composition f_δ is constant in a neighborhood of the vertex. Away from that neighborhood ρ_Q is smooth and $\nabla \rho_Q$ is bounded. More explicitly, on the nonconstant region one has $\rho_Q(z) \geq \delta/2$, hence $|z| \geq c_\delta > 0$; the possible singularity of $D^2\rho_Q$ at the vertex is therefore absent. A direct differentiation gives, for $z \neq 0$,

$$D^2\rho_Q(z) = \frac{Q}{\rho_Q(z)} - \frac{(Qz)(Qz)^T}{\rho_Q(z)^3}.$$

Let $\lambda_{\min}, \lambda_{\max} > 0$ be the extreme eigenvalues of Q. Since $\sqrt{\lambda_{\min}}|z| \leq \rho_Q(z) \leq \sqrt{\lambda_{\max}}|z|$,

$$\|D^2\rho_Q(z)\| \leq \frac{\lambda_{\max}}{\sqrt{\lambda_{\min}}|z|} + \frac{\lambda_{\max}^2|z|^2}{\lambda_{\min}^{3/2}|z|^3} = \frac{C_Q}{|z|},$$

where $C_Q := \lambda_{\max}/\sqrt{\lambda_{\min}} + \lambda_{\max}^2/\lambda_{\min}^{3/2}$. On the nonconstant region $\rho_Q(z) \geq \delta/2$, hence $|z| \geq \delta/(2\sqrt{\lambda_{\max}})$, and the preceding bound is uniform there. The profile h_δ has bounded first and second derivatives. Therefore the chain-rule expressions

$$\nabla f_\delta = h'_\delta(\rho_Q)\nabla\rho_Q, \qquad D^2 f_\delta = h''_\delta(\rho_Q)\nabla\rho_Q \otimes \nabla\rho_Q + h'_\delta(\rho_Q)D^2\rho_Q$$

are bounded on the nonconstant region, and hence $f_\delta \in C_b^2(S)$.

On the boundary side ∂S_i, a point has the form $z = ru_i$. Hence

$$D_i\rho_Q(z) = \frac{v_i \cdot Qz}{\rho_Q(z)} = \frac{r\, v_i \cdot Qu_i}{\rho_Q(ru_i)} \geq 0$$

by (EN1). Since $h'_\delta \geq 0$,

$$D_i f_\delta = h'_\delta(\rho_Q)D_i\rho_Q \geq 0.$$

Thus f_δ is admissible.

Finally, on $S \setminus \{0\}$,

$$\mathcal{L}f_\delta = h'_\delta(\rho_Q)\mathcal{L}\rho_Q + \frac{1}{2}h''_\delta(\rho_Q)|\nabla\rho_Q|^2.$$

Using lemma C.1,

$$\mathcal{L}f_\delta \geq h'_\delta(\rho_Q)\left(m_Q - \frac{\gamma M_Q}{2}\right) \geq 0.$$

Moreover this quantity is strictly positive whenever $\rho_Q(z) > \delta$, because then $h'_\delta(\rho_Q(z)) > 0$. Hence $\{f_\delta\}_{\delta>0}$ is a vanishing-core admissible contradiction family with core function $H = \rho_Q$. By theorem 4.16, no stationary distribution exists. □

Proposition C.3 (one-parameter feasibility test for the elliptic-norm criterion)**.** *Let $\mu \neq 0$. Fix*

$$e_\mu := \frac{\mu}{|\mu|}$$

and let n_μ be either unit vector perpendicular to e_μ. For each $w \in S_\circ^\vee$ satisfying $w \cdot \mu > 0$, write

$$w_e := w \cdot e_\mu, \qquad w_n := w \cdot n_\mu, \qquad \alpha_w := \frac{w_e}{|\mu|} > 0, \qquad \beta_w := \frac{w_n}{|\mu|}.$$

For $t \in \mathbb{R}$, define the symmetric matrix $Q_t = Q_t(w)$ by its matrix in the orthonormal basis (e_μ, n_μ):

$$[Q_t]_{(e_\mu,n_\mu)} = \begin{pmatrix} \alpha_w & \beta_w \\ \beta_w & t \end{pmatrix}.$$

Then $Q_t\mu = w$, and Q_t is positive definite if and only if

$$t > t_0(w) := \frac{\beta_w^2}{\alpha_w}.$$

Moreover every symmetric positive definite matrix Q satisfying $Q\mu = w$ is equal to $Q_t(w)$ for a unique $t > t_0(w)$.

For any vector x, write

$$x_e := x \cdot e_\mu, \qquad x_n := x \cdot n_\mu.$$

Set

$$A_i(w) := v_{i,e}\big(\alpha_w u_{i,e} + \beta_w u_{i,n}\big) + \beta_w v_{i,n} u_{i,e}, \qquad B_i := v_{i,n} u_{i,n}, \qquad i = 1, 2.$$

Then

$$v_i \cdot Q_t u_i = A_i(w) + tB_i.$$

Consequently, the matrix hypotheses (EN1)–(EN2) hold for some symmetric positive definite matrix Q if and only if there exist $w \in S_\circ^\vee$ with $w \cdot \mu > 0$ and a number $t > t_0(w)$ such that

$$A_i(w) + tB_i \geq 0, \qquad i = 1, 2.$$

Equivalently, after fixing w, the remaining matrix feasibility problem is the intersection of the open half-line $(t_0(w), \infty)$ with two affine half-line constraints in the single scalar variable t. If in addition $\alpha < 2$, any feasible choice yields stationary nonexistence by proposition C.2.

Proof. The identity $Q_t\mu = w$ follows from the first column of $[Q_t]_{(e_\mu,n_\mu)}$, since $\mu = |\mu| e_\mu$. The positive-definiteness criterion is the elementary criterion for a 2×2 symmetric matrix:

$$\alpha_w > 0, \qquad \alpha_w t - \beta_w^2 > 0.$$

Now let Q be any symmetric matrix with $Q\mu = w$. In the basis (e_μ, n_μ), the relation $Q\mu = w$ fixes the first column of Q, and symmetry fixes the first row. Hence only the lower-right entry remains free, which gives exactly the family $Q_t(w)$. Positive definiteness imposes $t > t_0(w)$.

Finally, for arbitrary vectors x, y, direct multiplication gives

$$x \cdot Q_t y = x_e(\alpha_w y_e + \beta_w y_n) + x_n(\beta_w y_e + t y_n).$$

Taking $x = v_i$ and $y = u_i$ yields

$$v_i \cdot Q_t u_i = v_{i,e}(\alpha_w u_{i,e} + \beta_w u_{i,n}) + \beta_w v_{i,n} u_{i,e} + t v_{i,n} u_{i,n} = A_i(w) + tB_i.$$

Thus, for fixed w, the two oblique boundary sign conditions $v_i \cdot Q_t u_i \geq 0$ are precisely the two affine inequalities displayed above. Conversely, if a positive definite Q satisfies (EN1)–(EN2), then $w := Q\mu \in S_\circ^\vee$ and $w \cdot \mu = \mu \cdot Q\mu > 0$, so $Q = Q_t(w)$ for the corresponding unique $t > t_0(w)$. This proves the matrix feasibility assertion; the stationary-nonexistence conclusion under $\alpha < 2$ is then exactly proposition C.2. □

Remark C.4. Proposition C.3 turns the matrix part of the elliptic-norm criterion into a concrete one-dimensional feasibility test once an outward dual vector $w = Q\mu$ is chosen. The search is no longer over all positive definite matrices, but over $w \in S_\circ^\vee$ and one scalar parameter t. This is the form in which the criterion can be analyzed inside the closed reflection cone.

Remark C.5. The criterion in proposition C.2 is two-dimensional. The admissible test functions are not functions of a single linear coordinate, and the level sets are ellipses determined by the positive definite matrix Q. Thus this proposition gives a concrete matrix search problem for nonexistence inside regions where one-dimensional supersolutions are unavailable.

Proposition C.6 (finite-dimensional form of the elliptic-norm criterion)**.** *Assume* $0 < \xi < \pi$. *Let*

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

be symmetric. The matrix hypotheses (EN1)–(EN2) *in proposition C.2 are equivalent to the feasibility of the following finite-dimensional system:*

$$q_{11} > 0, \qquad q_{11}q_{22} - q_{12}^2 > 0, \tag{ENF0}$$

$$v_i \cdot Qu_i \geq 0, \qquad i = 1, 2, \tag{ENF1}$$

and

$$(Q\mu) \cdot u_j > 0, \qquad j = 1, 2. \tag{ENF2}$$

In particular, the matrix part of the elliptic-norm criterion is a finite-dimensional semialgebraic feasibility problem in the three real variables (q_{11}, q_{12}, q_{22})*. If, in addition,* $\alpha < 2$*, feasibility of this system implies stationary nonexistence by proposition C.2.*

Proof. A real symmetric two-by-two matrix is positive definite if and only if (ENF0) holds. The boundary condition (ENF1) is exactly (EN1). Since

$$S = \operatorname{cone}\{u_1, u_2\},$$

the condition $Q\mu \in S_\circ^\vee$ is equivalent to strict positivity on the two generating rays:

$$(Q\mu) \cdot u_1 > 0, \qquad (Q\mu) \cdot u_2 > 0.$$

This is precisely (ENF2). All constraints are polynomial equalities or inequalities in (q_{11}, q_{12}, q_{22}), with (ENF1) and (ENF2) linear and (ENF0) the positive-definiteness condition. This proves the claim. □

Proposition C.7 (two-parameter feasibility form of the elliptic-norm criterion)**.** *Assume* $0 < \xi < \pi$ *and* $\alpha < 2$*. Then the two wedge generators* u_1, u_2 *form a basis of* $\mathbb{R}^2$*. Write*

$$v_i = r_{i1}u_1 + r_{i2}u_2, \qquad \mu = m_1 u_1 + m_2 u_2.$$

Then the elliptic-norm hypotheses (EN1)–(EN2) *are equivalent to the existence of two real parameters*

$$x > 0, \qquad \tau \in (-1, 1),$$

such that

$$r_{11} + r_{12}\tau x \geq 0, \qquad r_{22} + r_{21}\frac{\tau}{x} \geq 0, \tag{EN3}$$

and

$$m_1 + m_2\tau x > 0, \qquad m_2 + m_1\frac{\tau}{x} > 0. \tag{EN4}$$

Consequently, whenever such (x, τ) *exists, the solution to the Lakner–Liu–Reed submartingale problem with drift* μ *admits no stationary distribution.*

Proof. Let Q be symmetric positive definite. Relative to the basis u_1, u_2, set

$$q_{11} := u_1 \cdot Qu_1, \qquad q_{22} := u_2 \cdot Qu_2, \qquad q_{12} := u_1 \cdot Qu_2.$$

The matrix

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

is positive definite, hence $q_{11} > 0$, $q_{22} > 0$, and $q_{12}^2 < q_{11}q_{22}$. Since multiplying Q by a positive scalar does not change any of the signs in (EN1)–(EN2), normalize $q_{11} = 1$. Then write

$$q_{22} = x^2, \qquad q_{12} = \tau x, \qquad x > 0, \quad \tau \in (-1, 1).$$

Now compute the four relevant signs. First,

$$v_1 \cdot Qu_1 = (r_{11}u_1 + r_{12}u_2) \cdot Qu_1 = r_{11}q_{11} + r_{12}q_{12} = r_{11} + r_{12}\tau x,$$

and

$$v_2 \cdot Qu_2 = (r_{21}u_1 + r_{22}u_2) \cdot Qu_2 = r_{21}q_{12} + r_{22}q_{22} = x^2\left(r_{22} + r_{21}\frac{\tau}{x}\right).$$

Thus (EN1) is exactly (EN3). The two drift-dual coordinates are computed independently as

$$(Q\mu) \cdot u_1 = \mu \cdot Qu_1 = m_1q_{11} + m_2q_{12} = m_1 + m_2\tau x,$$

and

$$(Q\mu) \cdot u_2 = \mu \cdot Qu_2 = m_1q_{12} + m_2q_{22} = x^2\left(m_2 + m_1\frac{\tau}{x}\right).$$

Since $S_\circ^\vee = \{\theta : \theta \cdot u_1 > 0, \theta \cdot u_2 > 0\}$, condition (EN2) is exactly (EN4).

Conversely, given (x, τ) satisfying (EN3)–(EN4), define a positive definite bilinear form in the basis u_1, u_2 by the Gram matrix

$$\begin{pmatrix} 1 & \tau x \\ \tau x & x^2 \end{pmatrix}.$$

Its determinant is

$$x^2(1 - \tau^2) > 0,$$

so the form is positive definite and is represented by a unique Euclidean symmetric positive definite matrix Q. For this matrix,

$$\begin{aligned} v_1 \cdot Qu_1 &= r_{11} + r_{12}\tau x, \\ v_2 \cdot Qu_2 &= x^2\left(r_{22} + r_{21}\frac{\tau}{x}\right), \\ (Q\mu) \cdot u_1 &= m_1 + m_2\tau x, \\ (Q\mu) \cdot u_2 &= x^2\left(m_2 + m_1\frac{\tau}{x}\right). \end{aligned}$$

The first two quantities are nonnegative by (EN3); the last two are strictly positive by (EN4). Hence Q satisfies (EN1)–(EN2), and proposition C.2 gives nonexistence. □

Remark C.8. Proposition C.7 turns the elliptic-norm criterion into a concrete two-scalar feasibility problem. This is a convenient normal form for analyzing the reflection cone: x changes the relative stretching of the two wedge generators, while τ records the cross-term of the quadratic gauge. The result is purely linear algebraic and uses no probabilistic input beyond the already-proved elliptic-norm criterion.

Corollary C.9 (a concrete isotropic elliptic-norm sufficient condition)**.** *Assume $\alpha < 2$ and*

$$\mu \in S_\circ^\vee \quad \text{and} \quad v_i \cdot u_i \geq 0, \qquad i = 1, 2.$$

Then the solution to the Lakner–Liu–Reed submartingale problem with drift μ admits no stationary distribution.

Proof. Take $Q = I$. Then Q is positive definite, $Q\mu = \mu \in S_\circ^\vee$, and $v_i \cdot Qu_i = v_i \cdot u_i \geq 0$. Thus proposition C.2 applies. □

Remark C.10. Proposition C.6 is well suited to the reflection-cone problem because it converts the two-dimensional supersolution search into an explicit three-variable feasibility problem. Unlike the one-dimensional direct cone, the conditions involve the full matrix Q, so they can hold even when no admissible supersolution depending on a single linear coordinate exists.

C.1. **Elliptic matrix cones and consistency.** The elliptic Foster–Lyapunov criterion and the elliptic admissible-supersolution criterion are two elliptic criteria with opposite signs. Their drift regions are described explicitly and are consistent with the one-dimensional phase diagram developed earlier.

Define the elliptic Foster–Lyapunov cone by

$$\mathfrak{M}_{\mathrm{ell}}^{-} := \left\{\mu \in \mathbb{R}^2 : \exists Q = Q^T > 0 \text{ such that } v_i \cdot Qu_i \le 0 \ (i=1,2),\ Q\mu \in -S_\circ^\vee\right\},$$

and the elliptic admissible-supersolution cone by

$$\mathfrak{M}_{\mathrm{ell}}^{+} := \left\{\mu \in \mathbb{R}^2 : \exists Q = Q^T > 0 \text{ such that } v_i \cdot Qu_i \ge 0 \ (i=1,2),\ Q\mu \in S_\circ^\vee\right\}.$$

Here the superscript is only a sign mnemonic: the plus cone corresponds to an outward elliptic gauge and yields nonexistence; the minus cone corresponds to an inward elliptic gauge and yields existence.

Proposition C.11 (elliptic matrix cones are open and consistent). *Assume* $0 < \xi < \pi$ *and* $\alpha < 2$. *The sets* $\mathfrak{M}_{\mathrm{ell}}^{-}$ *and* $\mathfrak{M}_{\mathrm{ell}}^{+}$ *are open cones. Moreover:*

(i) If $\mu \in \mathfrak{M}_{\mathrm{ell}}^{-}$, *then a stationary distribution exists.*
(ii) If $\mu \in \mathfrak{M}_{\mathrm{ell}}^{+}$, *then no stationary distribution exists.*
(iii) The elliptic nonexistence cone does not meet the Lyapunov existence cone:

$$\mathfrak{M}_{\mathrm{ell}}^{+} \cap \mathfrak{M}_{\mathrm{Lyap}} = \varnothing.$$

(iv) The elliptic Foster cone does not meet the direct one-dimensional supersolution cone:

$$\mathfrak{M}_{\mathrm{ell}}^{-} \cap \mathfrak{M}_{\mathrm{sup}} = \varnothing.$$

(v) In the strict case $1 < \alpha < 2$,

$$\mathfrak{M}_{\mathrm{ell}}^{+} \subset \operatorname{cone}\{-v_1, -v_2\}.$$

Thus every drift covered by the elliptic-norm nonexistence criterion in the strict case lies in the two-dimensional reflection cone.

Proof. We first prove openness and conic invariance. Fix a symmetric positive definite matrix Q. The condition $Q\mu \in S_\circ^\vee$ is an open condition on μ, because $S_\circ^\vee$ is open relative to $\mathbb{R}^2$. The same is true for $Q\mu \in -S_\circ^\vee$. Therefore, for each fixed admissible Q, the corresponding set of drifts is open. Taking the union over all admissible matrices Q shows that both $\mathfrak{M}_{\mathrm{ell}}^{-}$ and $\mathfrak{M}_{\mathrm{ell}}^{+}$ are open. If μ belongs to either cone and $r > 0$, then the same matrix Q works for $r\mu$, since $Q(r\mu) = rQ\mu$. Hence both sets are cones.

Part (i) is exactly proposition 6.18; part (ii) is exactly proposition C.2. If $\mu \in \mathfrak{M}_{\mathrm{ell}}^{+} \cap \mathfrak{M}_{\mathrm{Lyap}}$, then (ii) gives nonexistence of a stationary distribution while the Lyapunov theorem gives existence. This is impossible. Hence (iii) holds. The proof of (iv) is the same: $\mathfrak{M}_{\mathrm{ell}}^{-}$ gives existence, whereas $\mathfrak{M}_{\mathrm{sup}}$ gives nonexistence.

For (v), recall from proposition 6.14 that in the strict case

$$\mathfrak{M}_{\mathrm{Lyap}} = \mathbb{R}^2 \setminus \operatorname{cone}\{-v_1, -v_2\}.$$

By (iii), $\mathfrak{M}_{\mathrm{ell}}^{+}$ is disjoint from $\mathfrak{M}_{\mathrm{Lyap}}$. Therefore it must be contained in $K_{\mathrm{str}} = \operatorname{cone}\{-v_1, -v_2\}$. □

Remark C.12. The last assertion locates the elliptic-norm criterion within the phase diagram. It does not extend into the stationary-existence region and, in the strict regime, applies only inside the cone not covered by the one-dimensional Lyapunov and direct supersolution criteria. It is therefore intrinsically two-dimensional.

Proposition C.13 (hyperbolic feasibility form of the elliptic criteria). *Assume* $0 < \xi < \pi$ *and* $\alpha < 2$. *Write*

$$v_i = r_{i1}u_1 + r_{i2}u_2, \qquad \mu = m_1u_1 + m_2u_2.$$

Define

$$\mathscr{H}_{\mathrm{hyp}} := \{(0,0)\} \cup \{(y,z) \in \mathbb{R}^2 : 0 < yz < 1\}.$$

Then the elliptic admissible-supersolution criterion of proposition C.7 is equivalent to the existence of $(y,z) \in \mathscr{H}_{\rm hyp}$ *such that*

$$r_{11} + r_{12}y \ge 0, \qquad r_{22} + r_{21}z \ge 0, \tag{H+1}$$

$$m_1 + m_2 y > 0, \qquad m_2 + m_1 z > 0. \tag{H+2}$$

The elliptic Foster–Lyapunov criterion of proposition 6.20 is, with all four signs reversed, equivalent to the existence of $(y,z) \in \mathscr{H}_{\rm hyp}$ *such that*

$$r_{11} + r_{12}y \le 0, \qquad r_{22} + r_{21}z \le 0, \tag{H-1}$$

$$m_1 + m_2 y < 0, \qquad m_2 + m_1 z < 0. \tag{H-2}$$

Thus both elliptic criteria reduce to linear inequalities in two variables, constrained only by the elementary hyperbolic condition $(y,z) \in \mathscr{H}_{\rm hyp}$.

Proof. We first treat the supersolution criterion. In proposition C.7, set

$$y := \tau x, \qquad z := \frac{\tau}{x}.$$

If $\tau = 0$, then $(y,z) = (0,0)$. If $\tau \neq 0$, then

$$yz = \tau^2 \in (0,1),$$

so $(y,z) \in \mathscr{H}_{\rm hyp}$. The inequalities (EN3)–(EN4) become exactly (H+1)–(H+2).

Conversely, suppose $(y,z) \in \mathscr{H}_{\rm hyp}$ satisfies (H+1)–(H+2). If $(y,z) = (0,0)$, take $\tau = 0$ and any $x > 0$. If $0 < yz < 1$, then y and z have the same sign. Define

$$\tau := \operatorname{sgn}(y)\sqrt{yz} \in (-1,1), \qquad x := \frac{y}{\tau} > 0.$$

Then

$$\tau x = y, \qquad \frac{\tau}{x} = z.$$

Substituting these into (H+1)–(H+2) gives (EN3)–(EN4). Hence proposition C.7 applies.

For the Foster–Lyapunov criterion, begin with a feasible pair (x,τ) in proposition 6.20 and use the same change of variables

$$y = \tau x, \qquad z = \frac{\tau}{x}.$$

The positive-definiteness restrictions again give either $(y,z) = (0,0)$ or $0 < yz < 1$, while the four inequalities (EF1)–(EF2) become exactly (H-1)–(H-2). Conversely, given $(y,z) \in \mathscr{H}_{\rm hyp}$ satisfying the latter inequalities, take $\tau = 0$ and arbitrary $x > 0$ when $(y,z) = (0,0)$; when $0 < yz < 1$, set

$$\tau = \operatorname{sgn}(y)\sqrt{yz}, \qquad x = \frac{y}{\tau}.$$

Then $x > 0$, $|\tau| < 1$, $\tau x = y$, and $\tau/x = z$. Substitution recovers (EF1)–(EF2), proving the claimed equivalence for the Foster–Lyapunov criterion as well. □

Remark C.14. The variables y and z are the normalized off-diagonal coefficients in the two boundary directions. The condition $0 < yz < 1$, together with the isolated point $(0,0)$, is equivalent to positive definiteness after normalization. The feasible set is therefore the intersection of four half-planes with the hyperbolic region $\mathscr{H}_{\rm hyp}$.

C.2. **A symmetric strict model: limits of the elliptic-norm criteria.** The hyperbolic feasibility form identifies the drifts covered by the elliptic-norm criterion. The following symmetric example shows that a substantial part of the closed reflection cone lies outside this gauge class.

Proposition C.15 (symmetric strict quadrant verification)**.** *Let*

$$S = \mathbb{R}^2_+, \qquad u_1 = e_1, \qquad u_2 = e_2,$$

and assume

$$v_1 = -\sigma u_1 + u_2, \qquad v_2 = u_1 - \sigma u_2, \qquad \sigma > 1.$$

Write

$$\mu = m_1 u_1 + m_2 u_2.$$

Then the elliptic admissible-supersolution cone is empty:

$$\mathfrak{M}^+_{\text{ell}} = \varnothing.$$

Moreover, the elliptic Foster–Lyapunov cone coincides with the linear Lyapunov cone:

$$\mathfrak{M}^-_{\text{ell}} = \mathfrak{M}_{\text{Lyap}} = \{\sigma m_1 + m_2 < 0\} \cup \{m_1 + \sigma m_2 < 0\}.$$

Consequently, for this symmetric strict model the elliptic-norm criteria do not resolve the closed reflection cone

$$\operatorname{cone}\{\sigma u_1 - u_2, -u_1 + \sigma u_2\}.$$

Proof. In the notation of proposition C.13, the coefficients are

$$r_{11} = -\sigma, \quad r_{12} = 1, \quad r_{21} = 1, \quad r_{22} = -\sigma.$$

For the elliptic admissible-supersolution criterion, (H+1) becomes

$$-\sigma + y \geq 0, \qquad -\sigma + z \geq 0.$$

Thus any feasible point would have

$$y \geq \sigma, \qquad z \geq \sigma.$$

Since $\sigma > 1$, this implies $yz \geq \sigma^2 > 1$, which is incompatible with $(y, z) \in \mathscr{H}_{\text{hyp}}$. Hence $\mathfrak{M}^+_{\text{ell}} = \varnothing$.

For the elliptic Foster criterion, the boundary inequalities are

$$y \leq \sigma, \qquad z \leq \sigma,$$

and the drift inequalities are

$$m_1 + m_2 y < 0, \qquad m_2 + m_1 z < 0.$$

We claim that such $(y, z) \in \mathscr{H}_{\text{hyp}}$ exists if and only if

$$\sigma m_1 + m_2 < 0 \quad \text{or} \quad m_1 + \sigma m_2 < 0.$$

First suppose $\sigma m_1 + m_2 < 0$. Take $z = \sigma$. Then

$$m_2 + m_1 z = m_2 + \sigma m_1 < 0.$$

Also

$$m_1 + \frac{m_2}{\sigma} = \frac{\sigma m_1 + m_2}{\sigma} < 0.$$

Put

$$d := -\left(m_1 + \frac{m_2}{\sigma}\right) > 0.$$

If $m_2 = 0$, choose $y = 1/(2\sigma)$. If $m_2 \neq 0$, choose

$$0 < \varepsilon < \min\left\{\frac{1}{2\sigma}, \frac{d}{2|m_2|}\right\}, \qquad y := \frac{1}{\sigma} - \varepsilon.$$

Then $0 < y < 1/\sigma$, and

$$m_1 + m_2 y = -d - m_2 \varepsilon \leq -d + |m_2|\varepsilon < -\frac{d}{2} < 0.$$

Thus $yz = \sigma y < 1$, so $(y, z) \in \mathscr{H}_{\text{hyp}}$, and all Foster inequalities hold. Now suppose $m_1 + \sigma m_2 < 0$. Take $y = \sigma$. Then

$$m_1 + m_2 y = m_1 + \sigma m_2 < 0,$$

and

$$m_2 + \frac{m_1}{\sigma} = \frac{m_1 + \sigma m_2}{\sigma} < 0.$$

Put
$$\widetilde{d} := -\Big(m_2 + \frac{m_1}{\sigma}\Big) > 0.$$
If $m_1 = 0$, choose $z = 1/(2\sigma)$. If $m_1 \neq 0$, choose
$$0 < \widetilde{\varepsilon} < \min\left\{\frac{1}{2\sigma}, \frac{\widetilde{d}}{2|m_1|}\right\}, \qquad z := \frac{1}{\sigma} - \widetilde{\varepsilon}.$$
Then $0 < z < 1/\sigma$, and
$$m_2 + m_1 z = -\widetilde{d} - m_1\widetilde{\varepsilon} \le -\widetilde{d} + |m_1|\widetilde{\varepsilon} < -\frac{\widetilde{d}}{2} < 0.$$
Thus $yz = \sigma z < 1$, so this choice also belongs to $\mathscr{H}_{\rm hyp}$ and satisfies all Foster inequalities.

Conversely, suppose that a Foster-feasible $(y, z) \in \mathscr{H}_{\rm hyp}$ exists, and assume for contradiction that
$$\sigma m_1 + m_2 \ge 0, \qquad m_1 + \sigma m_2 \ge 0.$$
If $m_1, m_2 > 0$, the two drift inequalities force
$$y < -\frac{m_1}{m_2} < 0, \qquad z < -\frac{m_2}{m_1} < 0,$$
so $yz > 1$, contradicting $(y, z) \in \mathscr{H}_{\rm hyp}$. If $m_1 < 0 < m_2$, then $\sigma m_1 + m_2 \ge 0$ gives
$$\frac{m_2}{-m_1} \ge \sigma.$$
The second drift inequality gives
$$z > \frac{m_2}{-m_1} \ge \sigma,$$
contradicting the boundary inequality $z \le \sigma$. If $m_2 < 0 < m_1$, then $m_1 + \sigma m_2 \ge 0$ gives
$$\frac{m_1}{-m_2} \ge \sigma,$$
whereas the first drift inequality gives
$$y > \frac{m_1}{-m_2} \ge \sigma,$$
contradicting the boundary inequality $y \le \sigma$. If $m_1 = 0$, the two assumed inequalities give $m_2 \ge 0$. When $m_2 > 0$, the second drift inequality is
$$m_2 + m_1 z = m_2 < 0,$$
which is impossible; when $m_2 = 0$, both strict drift inequalities read $0 < 0$. If $m_2 = 0$, the assumed inequalities give $m_1 \ge 0$. When $m_1 > 0$, the first drift inequality is
$$m_1 + m_2 y = m_1 < 0,$$
which is impossible; the case $m_1 = 0$ was just treated. Finally, $m_1, m_2 < 0$ makes both $\sigma m_1 + m_2$ and $m_1 + \sigma m_2$ strictly negative. Hence at least one of the two strict inequalities must hold.

Finally, in this model
$$\operatorname{cone}\{-v_1, -v_2\} = \operatorname{cone}\{\sigma u_1 - u_2, -u_1 + \sigma u_2\},$$
and membership in this cone is exactly
$$\sigma m_1 + m_2 \ge 0, \qquad m_1 + \sigma m_2 \ge 0.$$
Thus its complement is exactly the set displayed above, which is $\mathfrak{M}_{\rm Lyap}$ by the strict-regime geometry. This proves the proposition. □

Remark C.16. The elliptic matrix criteria introduced above are two-dimensional tools, but they do not by themselves close the symmetric closed reflection cone. In particular, the central drift direction $\mu \parallel u_1 + u_2$ lies inside the closed reflection cone and is not covered by either elliptic criterion. The Varadhan–Williams gauge is therefore needed to reach the whole closed cone.

C.3. **A general cone-quadratic admissible-supersolution criterion.** The symmetric criterion below is a special case of a general cone-adapted quadratic criterion. In Cartesian coordinates write

$$s_1(x,y) := y, \qquad s_2(x,y) := \sin\xi\, x - \cos\xi\, y.$$

Then

$$S = \{s_1 \geq 0,\ s_2 \geq 0\},$$

and ∇s_i is the inward unit normal on the corresponding face. The normalization $v_i \cdot n_i = 1$ therefore gives

$$D_1 s_1 = 1, \qquad D_2 s_2 = 1.$$

Proposition C.17 (a general cone-quadratic supersolution criterion)**.** *Assume* $0 < \xi < \pi$ *and* $\alpha < 2$*, and fix the drift* μ *in the generator* $\mathcal{L} = \mathcal{L}_\mu$*. Let*

$$q(z) = A s_1(z)^2 + B s_1(z) s_2(z) + C s_2(z)^2$$

with real constants A, B, C*, and assume that*

$$q(z) > 0 \qquad (z \in S \setminus \{0\}).$$

Set

$$\kappa_{12} := D_1 s_2, \qquad \kappa_{21} := D_2 s_1,$$

and

$$\mu_j := \mu \cdot \nabla s_j, \qquad j = 1, 2.$$

Assume further that

$$\text{(CQ1a)} \qquad B + 2C\kappa_{12} \geq 0, \qquad B + 2A\kappa_{21} \geq 0,$$

$$\text{(CQ1b)} \qquad 2A\mu_1 + B\mu_2 > 0, \qquad B\mu_1 + 2C\mu_2 > 0,$$

and

$$\text{(CQ1c)} \qquad \Delta q \geq 0.$$

Then the Lakner–Liu–Reed submartingale problem with drift μ *admits no stationary distribution.*

Proof. The boundary signs are as follows. On ∂S_1, where $s_1 = 0$, one has

$$D_1 q = s_2\big(B D_1 s_1 + 2C D_1 s_2\big) = s_2(B + 2C\kappa_{12}) \geq 0.$$

On ∂S_2, where $s_2 = 0$, direct differentiation gives

$$D_2 q = s_1\big(B D_2 s_2 + 2A D_2 s_1\big) = s_1(B + 2A\kappa_{21}) \geq 0.$$

Next compute the drift part. Since

$$\partial_{s_1} q = 2A s_1 + B s_2, \qquad \partial_{s_2} q = B s_1 + 2C s_2,$$

we have

$$\mu \cdot \nabla q = (2A\mu_1 + B\mu_2) s_1 + (B\mu_1 + 2C\mu_2) s_2.$$

By the strict inequalities in the assumptions this is strictly positive on $S \setminus \{0\}$. Together with $\Delta q \geq 0$, this gives

$$\mathcal{L} q = \mu \cdot \nabla q + \frac{1}{2}\Delta q > 0 \qquad \text{on } S \setminus \{0\}.$$

To choose a bounded increasing profile while preserving the generator sign, we use the following quantitative form of the preceding positivity. Let

$$\Sigma_q := \{u \in S : q(u) = 1\}.$$

Since q is continuous, homogeneous of degree two, and strictly positive on $S \setminus \{0\}$, the set Σ_q is compact. The strict drift inequalities imply

$$d_q := \min_{u \in \Sigma_q} \mu \cdot \nabla q(u) > 0.$$

Moreover

$$G_q := \max_{u \in \Sigma_q} |\nabla q(u)|^2 < \infty,$$

and $G_q > 0$ by Euler's identity $u \cdot \nabla q(u) = 2q(u) = 2$ on Σ_q. For $z \in S \setminus \{0\}$, write $r_q = q(z)^{1/2}$ and $u = z/r_q \in \Sigma_q$. Since ∇q is homogeneous of degree one,

$$\mu \cdot \nabla q(z) = r_q \, \mu \cdot \nabla q(u) \geq r_q d_q, \qquad |\nabla q(z)|^2 = r_q^2 |\nabla q(u)|^2 \leq r_q^2 G_q.$$

Using also $\Delta q \geq 0$, we obtain

$$\mathcal{L}q(z) \geq r_q d_q \geq \frac{d_q}{G_q} \frac{|\nabla q(z)|^2}{\sqrt{1+q(z)}} \qquad (z \in S \setminus \{0\}),$$

because $r_q/\sqrt{1+r_q^2} \leq 1$. Thus the required ratio bound holds with $a_0 = d_q/G_q$. Euler's identity also shows that $\nabla q(z) \neq 0$ for every $z \in S \setminus \{0\}$, since $z \cdot \nabla q(z) = 2q(z) > 0$.

Fix $0 < a < 4a_0$. For $\delta > 0$, choose a smooth nondecreasing cutoff ψ_δ with $\psi_\delta = 0$ on $[0, \delta/2]$ and $\psi_\delta = 1$ on $[\delta, \infty)$. Define

$$h_\delta'(s) = \psi_\delta(s) e^{-a\sqrt{1+s}}, \qquad h_\delta(s) = \int_0^s h_\delta'(r)\, dr.$$

Then $h_\delta \in C_b^2([0,\infty))$, h_δ is constant near zero, $h_\delta' \geq 0$, and

$$h_\delta''(s) \geq -\frac{a}{2\sqrt{1+s}} h_\delta'(s).$$

Set

$$f_\delta(z) := h_\delta(q(z)).$$

Since q is positive and two-homogeneous on $S \setminus \{0\}$, there are constants $0 < c_q \leq C_q < \infty$ and $C_1, C_2 < \infty$ such that

$$c_q|z|^2 \leq q(z) \leq C_q |z|^2, \qquad |\nabla q(z)| \leq C_1 |z|, \qquad \|D^2 q(z)\| \leq C_2, \qquad z \in S.$$

The plateau of h_δ on $[0, \delta/2]$ makes f_δ constant on the neighborhood $\{q < \delta/2\}$ of the vertex. On the compact transition set $\{\delta/2 \leq q \leq \delta\}$, all terms in the chain rule are bounded. For $q(z) \geq \delta$, the cutoff satisfies $\psi_\delta = 1$, and hence

$$h_\delta'(q(z)) = e^{-a\sqrt{1+q(z)}}, \qquad |h_\delta''(q(z))| = \frac{a}{2\sqrt{1+q(z)}} e^{-a\sqrt{1+q(z)}}.$$

Therefore, on $\{q \geq \delta\}$,

$$|\nabla f_\delta(z)| \leq C_1 |z| e^{-a\sqrt{1+c_q|z|^2}},$$

$$\|D^2 f_\delta(z)\| \leq \frac{aC_1^2|z|^2}{2\sqrt{1+c_q|z|^2}} e^{-a\sqrt{1+c_q|z|^2}} + C_2 e^{-a\sqrt{1+c_q|z|^2}}.$$

The two right-hand sides are bounded on $[0,\infty)$ and tend to zero as $|z| \to \infty$. Combining these bounds with the compact transition set and the vertex plateau proves $f_\delta \in C_b^2(S)$. The boundary sign follows from $D_i q \geq 0$:

$$D_i f_\delta = h_\delta'(q) D_i q \geq 0.$$

For the generator,

$$\mathcal{L} f_\delta = h_\delta'(q)\mathcal{L}q + \frac{1}{2} h_\delta''(q) |\nabla q|^2 \geq h_\delta'(q) \left(\mathcal{L}q - \frac{a}{4\sqrt{1+q}} |\nabla q|^2 \right).$$

The bracket is strictly positive on $S \setminus \{0\}$: by the preceding ratio bound and the choice $a < 4a_0$,

$$\mathcal{L}q - \frac{a}{4\sqrt{1+q}}|\nabla q|^2 \geq \left(a_0 - \frac{a}{4}\right)\frac{|\nabla q|^2}{\sqrt{1+q}} > 0.$$

Therefore $\mathcal{L}f_\delta \geq 0$, and the inequality is strict whenever $q > \delta$, because then $h'_\delta(q) > 0$. Thus $\{f_\delta\}$ is a vanishing-core admissible contradiction family, with core function q, and theorem 4.16 rules out stationary distributions. □

Remark C.18. The conditions in proposition C.17 reduce the verification of a two-dimensional cone-quadratic test to finitely many scalar inequalities in the boundary-distance coordinates. The symmetric strict criterion in section C.4 is obtained by taking $s_1 = x$, $s_2 = y$, and a suitable asymmetric or symmetric choice of (A, B, C).

Proposition C.19 (two-parameter form of the cone-quadratic criterion)**.** *Assume $0 < \xi < \pi$ and $\alpha < 2$. Keep the notation of proposition C.17. A cone-quadratic gauge satisfying the hypotheses of proposition C.17 exists if and only if there exist*

$$x > 0, \qquad \tau > -2,$$

such that

$$\tau + 2x\kappa_{12} \geq 0, \qquad \tau x + 2\kappa_{21} \geq 0, \tag{CQ2a}$$

$$2\mu_1 + \tau x\mu_2 > 0, \qquad \tau\mu_1 + 2x\mu_2 > 0, \tag{CQ2b}$$

and

$$1 + x^2 - \tau x \cos\xi \geq 0. \tag{CQ2c}$$

In particular, whenever such (x, τ) exists, the submartingale problem with drift μ admits no stationary distribution.

Proof. First suppose that (x, τ) satisfies the displayed inequalities, and set

$$A = 1, \qquad C = x^2, \qquad B = \tau x.$$

Then

$$q = s_1^2 + \tau x s_1 s_2 + x^2 s_2^2.$$

We verify exactly when this quadratic form is strictly positive on $\{s_1 \geq 0, s_2 \geq 0\} \setminus \{0\}$. If $s_1 = 0$, then $q = x^2 s_2^2 > 0$. If $s_1 > 0$, put $t = s_2/s_1 \geq 0$. Then

$$\frac{q}{s_1^2} = 1 + \tau x t + x^2 t^2.$$

For $\tau \geq 0$, this quantity is at least one. For $-2 < \tau < 0$, completing the square gives

$$1 + \tau x t + x^2 t^2 = x^2\left(t + \frac{\tau}{2x}\right)^2 + 1 - \frac{\tau^2}{4} \geq 1 - \frac{\tau^2}{4} > 0.$$

Conversely, if $\tau = -2$, then $q = 0$ at $(s_1, s_2) = (x, 1)$. If $\tau < -2$, the polynomial $1 + \tau x t + x^2 t^2$ has the two positive roots

$$t_\pm = \frac{-\tau \pm \sqrt{\tau^2 - 4}}{2x} > 0,$$

so the form vanishes at nonzero points of the quadrant. Thus strict positivity on the cone is equivalent to $\tau > -2$.

The boundary sign conditions in proposition C.17 become

$$B + 2C\kappa_{12} = x(\tau + 2x\kappa_{12}) \geq 0,$$

and

$$B + 2A\kappa_{21} = \tau x + 2\kappa_{21} \geq 0.$$

The two drift inequalities become

$$2A\mu_1 + B\mu_2 = 2\mu_1 + \tau x\mu_2 > 0,$$

and

$$B\mu_1 + 2C\mu_2 = x(\tau\mu_1 + 2x\mu_2) > 0.$$

Finally, since $|\nabla s_1| = |\nabla s_2| = 1$ and $\nabla s_1 \cdot \nabla s_2 = -\cos\xi$,

$$\frac{1}{2}\Delta q = A + B\nabla s_1 \cdot \nabla s_2 + C = 1 + x^2 - \tau x\cos\xi,$$

which is nonnegative by assumption. Hence proposition C.17 applies.

Conversely, suppose a cone-quadratic gauge

$$q = As_1^2 + Bs_1s_2 + Cs_2^2$$

satisfies the hypotheses of proposition C.17. Positivity of q on the two boundary rays gives $A > 0$ and $C > 0$. Multiplying q by a positive constant does not change any of the signs in the criterion, so normalize $A = 1$. Write

$$x = \sqrt{C} > 0, \qquad \tau = \frac{B}{x}.$$

The preceding equivalence, applied to $q = s_1^2 + \tau x s_1 s_2 + x^2 s_2^2$, gives $\tau > -2$. Substituting $A = 1$, $B = \tau x$, and $C = x^2$ into the boundary, drift, and Laplacian inequalities of proposition C.17 gives exactly the three displayed inequalities in the statement. □

Remark C.20. Thus cone-quadratic feasibility is a two-parameter semialgebraic problem. Compared with the elliptic-norm criteria, the parameter τ is allowed to range over $(-2, \infty)$, rather than the positive-definite interval $(-2, 2)$ after normalization. This additional freedom allows cone-adapted quadratics to cover drift regions not reached by elliptic norms.

Corollary C.21 (right or obtuse wedges with positive inward normal drift)**.** *Assume*

$$\alpha < 2, \qquad \frac{\pi}{2} \le \xi < \pi.$$

Keep the notation of proposition C.17, and assume that the drift has strictly positive inward normal components,

$$\mu_1 = \mu \cdot \nabla s_1 > 0, \qquad \mu_2 = \mu \cdot \nabla s_2 > 0.$$

Then the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. We apply proposition C.19 with $x = 1$. Choose

$$\tau \ge \max\{0, -2\kappa_{12}, -2\kappa_{21}\}.$$

Then $\tau > -2$, and the two boundary inequalities in proposition C.19 hold:

$$\tau + 2\kappa_{12} \ge 0, \qquad \tau + 2\kappa_{21} \ge 0.$$

Because $\mu_1, \mu_2 > 0$ and $\tau \ge 0$, the two drift inequalities also hold:

$$2\mu_1 + \tau\mu_2 > 0, \qquad \tau\mu_1 + 2\mu_2 > 0.$$

Finally, since $\xi \ge \pi/2$, one has $\cos\xi \le 0$, and hence

$$1 + 1 - \tau\cos\xi \ge 2 > 0.$$

Thus all hypotheses of proposition C.19 are satisfied. □

Corollary C.22 (positive inward normal drift in acute wedges)**.** *Assume*

$$\alpha < 2, \qquad 0 < \xi < \frac{\pi}{2}$$

and keep the notation of proposition C.17. Suppose

$$\mu_1 = \mu \cdot \nabla s_1 > 0, \qquad \mu_2 = \mu \cdot \nabla s_2 > 0.$$

If there exists $x > 0$ such that

$$L(x) := \max\left\{0, -2x\kappa_{12}, -\frac{2\kappa_{21}}{x}\right\} \le U_\xi(x) := \frac{1+x^2}{x\cos\xi},$$

then the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. Choose $\tau \in [L(x), U_\xi(x)]$. Since $L(x) \ge 0$, we have $\tau > -2$. The two boundary inequalities in proposition C.19 follow from

$$\tau \ge -2x\kappa_{12}, \qquad \tau \ge -\frac{2\kappa_{21}}{x}.$$

Because $\mu_1, \mu_2 > 0$ and $\tau \ge 0$, the drift inequalities

$$2\mu_1 + \tau x\mu_2 > 0, \qquad \tau\mu_1 + 2x\mu_2 > 0$$

are automatic. Finally, since $0 < \xi < \pi/2$, the Laplacian condition is exactly

$$1 + x^2 - \tau x\cos\xi \ge 0,$$

which follows from $\tau \le U_\xi(x)$. Hence proposition C.19 applies. □

Proposition C.23 (closed-form acute feasibility criterion)**.** *Assume the hypotheses of corollary C.22. Put*

$$c_\xi := \cos\xi > 0, \qquad a_{12} := (-\kappa_{12})_+, \qquad a_{21} := (-\kappa_{21})_+.$$

Define

$$L_* := \max\{0, 2a_{21}c_\xi - 1\}$$

and

$$U_* := \begin{cases} +\infty, & 2a_{12}c_\xi \le 1, \\ (2a_{12}c_\xi - 1)^{-1}, & 2a_{12}c_\xi > 1. \end{cases}$$

Then the scalar feasibility condition in corollary C.22, namely the existence of some $x > 0$ with

$$L(x) := \max\left\{0, -2x\kappa_{12}, -\frac{2\kappa_{21}}{x}\right\} \le U_\xi(x) := \frac{1+x^2}{x\cos\xi},$$

is equivalent to

$$L_* \le U_*.$$

Consequently, whenever $L_* \le U_*$, the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. Since

$$\max\left\{0, -2x\kappa_{12}, -\frac{2\kappa_{21}}{x}\right\} = \max\left\{0, 2a_{12}x, \frac{2a_{21}}{x}\right\},$$

the condition $L(x) \le U_\xi(x)$ is equivalent to the two inequalities

$$2a_{12}x \le \frac{1+x^2}{xc_\xi}, \qquad \frac{2a_{21}}{x} \le \frac{1+x^2}{xc_\xi}.$$

Writing $y = x^2$, these become

$$2a_{12}c_\xi y \le 1 + y, \qquad 2a_{21}c_\xi \le 1 + y.$$

The second inequality is exactly

$$y \ge L_*.$$

For the first inequality, if $2a_{12}c_\xi \le 1$, it imposes no upper restriction on $y > 0$. If $2a_{12}c_\xi > 1$, it is equivalent to

$$y \le (2a_{12}c_\xi - 1)^{-1} = U_*.$$

Thus there exists $x > 0$ satisfying the scalar condition of corollary C.22 if and only if there exists $y = x^2 > 0$ with $L_* \le y \le U_*$, where the upper endpoint is interpreted as $+\infty$ when $2a_{12}c_\xi \le 1$. This is equivalent to $L_* \le U_*$. The final assertion follows directly from corollary C.22. □

Proposition C.24 (the interior reflection cone under the cone-quadratic angle condition)**.** *Assume $0 < \xi < \pi$ and $\alpha < 2$. Work in the boundary-distance coordinates associated with proposition C.17. Assume that*

$$\kappa_{12} < 0, \qquad \kappa_{21} < 0,$$

and put

$$a := -\kappa_{12} > 0, \qquad b := -\kappa_{21} > 0.$$

Assume further that

$$ab > 1 \tag{CQ3a}$$

and that the angle–cross-obliqueness inequality

$$a + b - 2ab\cos\xi > 0 \tag{CQ3b}$$

holds. If the drift lies in the interior of the cone generated by the opposite reflection directions,

$$\mu \in \operatorname{int}\operatorname{cone}\{-v_1, -v_2\},$$

then the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. Since $\mu \in \operatorname{int}\operatorname{cone}\{-v_1, -v_2\}$, there exist $\lambda_1, \lambda_2 > 0$ such that

$$\mu = -\lambda_1 v_1 - \lambda_2 v_2.$$

In the boundary-distance coordinates,

$$D_1 s_1 = 1, \qquad D_1 s_2 = \kappa_{12} = -a, \qquad D_2 s_1 = \kappa_{21} = -b, \qquad D_2 s_2 = 1.$$

Thus the inward-normal drift components appearing in proposition C.19 are

$$\mu_1 := \mu \cdot \nabla s_1 = -\lambda_1 + b\lambda_2, \qquad \mu_2 := \mu \cdot \nabla s_2 = a\lambda_1 - \lambda_2.$$

It remains to verify the two-parameter criterion of proposition C.19.

Introduce the variables

$$y := \tau x, \qquad z := \frac{\tau}{x}.$$

The two boundary inequalities in proposition C.19 are exactly

$$z \ge 2a, \qquad y \ge 2b.$$

Define

$$A_1 := 2\lambda_1(ab-1) > 0, \qquad A_2 := 2\lambda_2(ab-1) > 0,$$
$$c_1 := a\lambda_1 - \lambda_2, \qquad c_2 := -\lambda_1 + b\lambda_2,$$

and

$$F_0 := \frac{a + b - 2ab\cos\xi}{a} > 0.$$

Choose $\varepsilon_1, \varepsilon_2 > 0$ so that

$$\varepsilon_1 < \min\left\{1, \frac{A_1}{2(1+|c_1|)}, \frac{F_0}{4(1/(2a) + |\cos\xi|)}\right\}, \tag{C.1}$$

$$\varepsilon_2 < \min\left\{a, \frac{A_2}{2(1+|c_2|)}, \frac{a^2 F_0}{2b}\right\}, \tag{C.2}$$

where the last bound in (C.2) is omitted when $b = 0$ (here $b > 0$, so it is finite). Put

$$y = 2b + \varepsilon_1, \qquad z = 2a + \varepsilon_2, \qquad x := \sqrt{\frac{y}{z}}, \qquad \tau := \sqrt{yz}.$$

Then $x > 0$, $\tau > 0 > -2$, and $z > 2a$, $y > 2b$, so the two boundary inequalities hold strictly.

The first drift expression is

$$2\mu_1 + y\mu_2 = A_1 + \varepsilon_1 c_1 \geq A_1 - \varepsilon_1 |c_1| > \frac{A_1}{2} > 0,$$

and the second is

$$z\mu_1 + 2\mu_2 = A_2 + \varepsilon_2 c_2 \geq A_2 - \varepsilon_2 |c_2| > \frac{A_2}{2} > 0.$$

It remains to verify the Laplacian inequality. In the variables y, z,

$$F(y,z) := 1 + x^2 - \tau x \cos\xi = 1 + \frac{y}{z} - y\cos\xi.$$

A direct subtraction from the value at $(2b, 2a)$ gives

$$F(2b+\varepsilon_1, 2a+\varepsilon_2) - F_0 = \varepsilon_1 \left(\frac{1}{2a+\varepsilon_2} - \cos\xi\right) - \frac{b\varepsilon_2}{a(2a+\varepsilon_2)}.$$

Since $\varepsilon_2 > 0$,

$$|F(2b+\varepsilon_1, 2a+\varepsilon_2) - F_0| \leq \varepsilon_1 \left(\frac{1}{2a} + |\cos\xi|\right) + \frac{b\varepsilon_2}{2a^2} < \frac{F_0}{2}.$$

Consequently $F(y,z) > F_0/2 > 0$. All hypotheses of proposition C.19 now hold with the explicitly chosen parameters, and the conclusion follows. ☐

Corollary C.25 (automatic full-interior nonexistence in the strict regime). *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Then, for every drift

$$\mu \in \operatorname{int}\operatorname{cone}\{-v_1, -v_2\},$$

the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution. This interior cone-quadratic statement is complemented in the main text by the quadratic Varadhan–Williams closed-cone theorem, which also handles the two boundary rays

$$\{t(-v_1) : t \geq 0\} \cup \{t(-v_2) : t \geq 0\}.$$

Proof. It is enough to verify the hypotheses of proposition C.24. Let $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ be the Lakner–Liu–Reed reflection angles, measured from the inward normals with positive sign toward the vertex, so that

$$\alpha = \frac{\theta_1 + \theta_2}{\xi}.$$

Let $t_1 = u_1$ and $t_2 = u_2$ denote the unit tangents along the two wedge faces pointing away from the vertex, and let $n_1 = \nabla s_1$, $n_2 = \nabla s_2$ be the inward unit normals. With this sign convention,

$$v_i = n_i - (\tan\theta_i) t_i, \qquad i = 1, 2.$$

Hence, using $n_1 \cdot n_2 = -\cos\xi$, $n_2 \cdot u_1 = \sin\xi$, and $n_1 \cdot u_2 = \sin\xi$,

$$\kappa_{12} = D_1 s_2 = n_2 \cdot v_1 = -\cos\xi - \sin\xi\tan\theta_1,$$

$$\kappa_{21} = D_2 s_1 = n_1 \cdot v_2 = -\cos\xi - \sin\xi\tan\theta_2.$$

Thus, with $a = -\kappa_{12}$ and $b = -\kappa_{21}$,

$$a = \cos\xi + \sin\xi\tan\theta_1, \qquad b = \cos\xi + \sin\xi\tan\theta_2.$$

Equivalently,

$$a = \frac{\cos(\xi - \theta_1)}{\cos\theta_1}, \qquad b = \frac{\cos(\xi - \theta_2)}{\cos\theta_2}.$$

Since $\theta_1 + \theta_2 > \xi$ and each $\theta_i < \pi/2$, one has

$$\theta_i > \xi - \frac{\pi}{2}, \qquad i = 1, 2.$$

Together with $\theta_i < \pi/2$ and $0 < \xi < \pi$, this gives

$$-\frac{\pi}{2} < \xi - \theta_i < \frac{\pi}{2}, \qquad i = 1, 2,$$

and therefore $\cos(\xi - \theta_i) > 0$. Since $\cos\theta_i > 0$, it follows that

$$a > 0, \qquad b > 0.$$

We verify both trigonometric identities. Substitution of the formulas for a, b and the product-to-sum identity give

$$\begin{aligned}
&(ab-1)\cos\theta_1\cos\theta_2 \\
&\quad= \cos(\xi-\theta_1)\cos(\xi-\theta_2) - \cos\theta_1\cos\theta_2 \\
&\quad= \frac{1}{2}\big[\cos(2\xi-\theta_1-\theta_2) - \cos(\theta_1+\theta_2)\big] \\
&\quad= \sin\xi\,\sin(\theta_1+\theta_2-\xi).
\end{aligned}$$

Writing $s = \theta_1 + \theta_2$, the second numerator becomes

$$\begin{aligned}
&(a+b-2ab\cos\xi)\cos\theta_1\cos\theta_2 \\
&\quad= \cos(\xi-\theta_1)\cos\theta_2 + \cos(\xi-\theta_2)\cos\theta_1 \\
&\qquad - 2\cos\xi\cos(\xi-\theta_1)\cos(\xi-\theta_2) \\
&\quad= \cos\xi\cos(\theta_1-\theta_2) + \cos(\xi-s) \\
&\qquad - \cos\xi\cos(\theta_1-\theta_2) - \cos\xi\cos(2\xi-s) \\
&\quad= \cos(\xi-s) - \cos\xi\cos(2\xi-s) \\
&\quad= \sin\xi\,\sin(2\xi-s).
\end{aligned}$$

The last equality follows by expanding both sides with the angle-addition formulas. Because $1 < \alpha < 2$, we have

$$\xi < \theta_1 + \theta_2 < 2\xi.$$

Therefore

$$\sin(\theta_1+\theta_2-\xi) > 0, \qquad \sin(2\xi-\theta_1-\theta_2) > 0,$$

since $0 < 2\xi - \theta_1 - \theta_2 < \xi < \pi$. Consequently,

$$ab > 1, \qquad a + b - 2ab\cos\xi > 0.$$

The first three inequalities encode the strict reflection-cone geometry $S \subset \operatorname{cone}\{-v_1, -v_2\}$, and the final inequality is exactly the cone-quadratic angle condition in proposition C.24. That proposition therefore applies to every drift in the interior of $\operatorname{cone}\{-v_1, -v_2\}$. □

Corollary C.26 (the cone-quadratic criterion cannot reach the critical rays)**.** *Assume*

$$0 < \xi < \pi, \qquad 1 < \alpha < 2.$$

Let

$$K_{\mathrm{str}} = \operatorname{cone}\{-v_1, -v_2\}.$$

For each $k \in \{1, 2\}$ and each $t > 0$, set

$$\mu = t(-v_k).$$

Then the two-parameter feasibility system in proposition C.19 has no solution. Equivalently, no cone-quadratic gauge of the form

$$q(z) = As_1(z)^2 + Bs_1(z)s_2(z) + Cs_2(z)^2$$

can satisfy the hypotheses of proposition C.17 for a critical boundary-ray drift.

Proof. We first treat the ray $\mu = t(-v_1)$. In the boundary-distance coordinates,

$$\mu_1 = \mu \cdot \nabla s_1 = -t, \qquad \mu_2 = \mu \cdot \nabla s_2 = -t\kappa_{12}.$$

The trigonometric computation in the proof of corollary C.25 gives

$$\kappa_{12} = -\frac{\cos(\xi - \theta_1)}{\cos\theta_1}, \qquad \kappa_{21} = -\frac{\cos(\xi - \theta_2)}{\cos\theta_2}.$$

Because $\theta_1 + \theta_2 > \xi$ and $\theta_{3-i} < \pi/2$,

$$\theta_i > \xi - \frac{\pi}{2}.$$

Together with $\theta_i < \pi/2$, this gives $-\pi/2 < \xi - \theta_i < \pi/2$, and therefore $\cos(\xi - \theta_i) > 0$. Also $\cos\theta_i > 0$ because $\theta_i \in (-\pi/2, \pi/2)$. Hence

$$\kappa_{12} < 0, \qquad \kappa_{21} < 0.$$

Thus $\mu_2 > 0$.

Suppose, for contradiction, that there exist $x > 0$ and $\tau > -2$ satisfying proposition C.19. The first boundary inequality gives

$$\tau + 2x\kappa_{12} \geq 0.$$

The second drift inequality gives

$$\tau\mu_1 + 2x\mu_2 > 0.$$

Substituting $\mu_1 = -t$ and $\mu_2 = -t\kappa_{12}$ yields

$$-t\big(\tau + 2x\kappa_{12}\big) > 0,$$

so

$$\tau + 2x\kappa_{12} < 0,$$

contradicting the boundary inequality.

For $\mu = t(-v_2)$, one instead has

$$\mu_1 = -t\kappa_{21} > 0, \qquad \mu_2 = -t,$$

and the second boundary inequality

$$\tau x + 2\kappa_{21} \geq 0$$

contradicts the first drift inequality

$$2\mu_1 + \tau x\mu_2 > 0,$$

which becomes

$$-t\big(\tau x + 2\kappa_{21}\big) > 0.$$

Thus the feasibility system has no solution on either critical ray. □

Remark C.27. Corollary C.26 shows that corollary C.25 is sharp within the cone-quadratic class: this class covers the interior of the strict reflection cone but not either critical boundary ray.

Remark C.28. The criterion in proposition C.24 is a two-dimensional nonexistence statement. Corollary C.25 shows that, under the Lakner–Liu–Reed strict nonsemimartingale condition $1 < \alpha < 2$, its cross-obliqueness/angle hypothesis is automatic. In this independent form, it applies to the whole interior of the closed reflection cone whenever the cross-obliqueness constants and the wedge angle satisfy (CQ3b). For right or obtuse wedges the angle term is favorable; for acute wedges the inequality records precisely how strong the inward cross-obliqueness must be for the cone-quadratic criterion to cover the full interior of the reflection cone.

Corollary C.29 (nonnegative cross-obliqueness test)**.** *Assume* $0 < \xi < \pi$ *and* $\alpha < 2$*. If*

$$\kappa_{12} \geq 0, \qquad \kappa_{21} \geq 0, \qquad \mu_1 > 0, \qquad \mu_2 > 0,$$

then the solution to the submartingale problem of definition 2.1 with drift μ *admits no stationary distribution.*

Proof. Use proposition C.19 with $\tau = 0$ and any $x > 0$. The boundary and drift inequalities follow directly, and the Laplacian inequality reduces to $1 + x^2 \geq 0$. □

Remark C.30. Corollaries C.21, C.22 and C.29 extract explicitly checkable nonexistence regions from the cone-quadratic criterion. The right-or-obtuse case requires no sign assumptions on the cross-obliqueness constants κ_{12} and κ_{21}. The acute case first reduces the question to a single scalar feasibility inequality in x, and proposition C.23 rewrites that scalar condition as an explicit comparison between two endpoint quantities L_* and U_*. If the cross-obliqueness constants are already nonnegative, the zero mixed coefficient $\tau = 0$ is sufficient in every wedge angle.

Remark C.31 (comparison of the gauge criteria). The bounded-profile and cone-quadratic criteria below cover the interior of the closed reflection cone but not its two boundary rays. The latter are treated by the two-homogeneous Varadhan–Williams gauge in section 7.

C.4. **A cone-quadratic supersolution in the symmetric strict reflection cone.** Earlier, section C.2 showed that elliptic norms of the form $\rho_Q(z) = (z^T Q z)^{1/2}$ are insufficiently flexible in the symmetric strict model. The next result gives a cone-adapted quadratic criterion inside the reflection cone. The quadratic form used below is positive on the cone S, but it is not positive definite on all of $\mathbb{R}^2$; this is precisely why it escapes the elliptic-norm obstruction of proposition C.15.

Proposition C.32 (a cone-quadratic nonexistence subcone)**.** *Let*

$$S = \mathbb{R}^2_+, \qquad u_1 = e_1, \qquad u_2 = e_2,$$

and

$$v_1 = -\sigma u_1 + u_2, \qquad v_2 = u_1 - \sigma u_2, \qquad \sigma > 1.$$

For this model the corresponding Lakner–Liu–Reed parameter is

$$\alpha = \frac{4\arctan(\sigma)}{\pi} \in (1, 2).$$

If

$$\mu = m_1 u_1 + m_2 u_2, \qquad m_1 > 0, \quad m_2 > 0,$$

then the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. The displayed formula for α follows from the angle convention for the quadrant: the reflection angle on each face is $\arctan(\sigma)$, and $\xi = \pi/2$. Since $\sigma > 1$, this gives $1 < \alpha < 2$. Choose

$$0 < \eta < \frac{1}{2\sigma}$$

and define the cone-quadratic gauge

$$q(x, y) = xy + \eta(x^2 + y^2), \qquad (x, y) \in \mathbb{R}^2_+.$$

Then $q(z) > 0$ for $z \in S \setminus \{0\}$. On the lower boundary $y = 0$,

$$D_1 q = (-\sigma, 1) \cdot (2\eta x + y,\ x + 2\eta y) = (1 - 2\sigma\eta)x \geq 0.$$

On the left boundary $x = 0$,

$$D_2 q = (1, -\sigma) \cdot (y + 2\eta x,\ x + 2\eta y) = (1 - 2\sigma\eta)y \geq 0.$$

Moreover

$$\mathcal{L}q = \mu \cdot \nabla q + \frac{1}{2}\Delta q = m_1(y + 2\eta x) + m_2(x + 2\eta y) + 2\eta.$$

Since $m_1, m_2 > 0$, there is $c_0 > 0$ such that

$$\mathcal{L}q(z) \geq 2\eta + c_0(x + y) \qquad (z = (x, y) \in S).$$

In particular $\mathcal{L}q > 0$ on S.

The following uniform ratio bound will be used. Write $z=(x,y)$ and $|z|=(x^2+y^2)^{1/2}$. Since $q\ge \eta|z|^2$ on S and $x+y\ge |z|$, the previous lower bound gives, for $|z|\ge 1$,

$$\sqrt{1+q(z)}\,\mathcal{L}q(z)\ge \sqrt{q(z)}\,c_0(x+y)\ge c_0\sqrt{\eta}\,|z|^2.$$

On the other hand, $|\nabla q(z)|^2\le C_q|z|^2$ for a constant $C_q<\infty$. Hence the ratio

$$z\longmapsto \frac{|\nabla q(z)|^2}{\sqrt{1+q(z)}\,\mathcal{L}q(z)}$$

is bounded outside the unit ball. On $S\cap\overline{B_1}$, the denominator is strictly positive because $\mathcal{L}q\ge 2\eta$, and the ratio is continuous. Therefore

$$\mathcal{R}_q:=\sup_{z\in S}\frac{|\nabla q(z)|^2}{\sqrt{1+q(z)}\,\mathcal{L}q(z)}<\infty.$$

Choose explicitly

$$\omega:=\frac{1}{1+\mathcal{R}_q}>0.$$

Then

$$1-\frac{\omega\mathcal{R}_q}{2}=1-\frac{\mathcal{R}_q}{2(1+\mathcal{R}_q)}>\frac{1}{2}>0.$$

For each $\delta>0$, choose a smooth function

$$h_\delta:[0,\infty)\to[0,\infty)$$

with the following properties:

$$h_\delta(s)=0\quad(0\le s\le\delta/2),\qquad h_\delta'(s)>0\quad(s>\delta),$$
$$h_\delta'\ge 0,\qquad h_\delta''(s)\ge -\frac{\omega}{\sqrt{1+s}}h_\delta'(s),$$

and h_δ is bounded. One explicit choice is obtained by setting

$$h_\delta'(s)=\psi_\delta(s)e^{-\omega\sqrt{1+s}},$$

where ψ_δ is smooth, nondecreasing, vanishes on $[0,\delta/2]$, is strictly positive on (δ,∞), and is equal to one for large s, and then integrating from zero.

Define

$$f_\delta(z)=h_\delta(q(z)).$$

We verify the global C_b^2 property quantitatively. Since q is a positive quadratic form on the quadrant, there are constants $c_q,C_q,C_1,C_2>0$ such that

$$c_q|z|^2\le q(z)\le C_q|z|^2,\qquad |\nabla q(z)|\le C_1|z|,\qquad \|D^2q(z)\|\le C_2.$$

The plateau of h_δ makes f_δ constant on $\{q<\delta/2\}$. On the compact transition set of ψ_δ, all chain-rule terms are bounded. Outside that set,

$$h_\delta'(s)=e^{-\omega\sqrt{1+s}},\qquad |h_\delta''(s)|=\frac{\omega}{2\sqrt{1+s}}e^{-\omega\sqrt{1+s}}.$$

Hence, for all points outside the compact transition set,

$$|\nabla f_\delta(z)|\le C_1|z|e^{-\omega\sqrt{1+c_q|z|^2}},$$
$$\|D^2f_\delta(z)\|\le \frac{\omega C_1^2|z|^2}{2\sqrt{1+c_q|z|^2}}e^{-\omega\sqrt{1+c_q|z|^2}}+C_2e^{-\omega\sqrt{1+c_q|z|^2}}.$$

Both right-hand sides are bounded and tend to zero at infinity. Together with the transition-set bounds and the vertex plateau, this proves $f_\delta\in C_b^2(S)$. Since $h_\delta'\ge 0$ and $D_iq\ge 0$ on ∂S_i,

$$D_if_\delta=h_\delta'(q)D_iq\ge 0.$$

Furthermore, by the chain rule,

$$\mathcal{L}f_\delta = h'_\delta(q)\mathcal{L}q + \frac{1}{2}h''_\delta(q)|\nabla q|^2.$$

Using the differential inequality for h_δ,

$$\mathcal{L}f_\delta \geq h'_\delta(q)\left(\mathcal{L}q - \frac{\omega}{2\sqrt{1+q}}|\nabla q|^2\right) \geq h'_\delta(q)\mathcal{L}q\left(1 - \frac{\omega\mathcal{R}_q}{2}\right) \geq 0.$$

Moreover, if $q(z) > \delta$, then $h'_\delta(q(z)) > 0$, so the last display is strict.

Thus f_δ is a vanishing-core admissible contradiction family with core function q. Since $q(z) > 0$ for $z \neq 0$ in S, theorem 4.16 gives the nonexistence of a stationary distribution. □

Remark C.33. This gives a nonexistence statement in the closed reflection cone that is not generated by a one-dimensional coordinate and is not covered by the elliptic-norm criteria. In the present symmetric model, the positive quadrant $\{m_1 > 0, m_2 > 0\}$ lies inside $\operatorname{cone}\{-v_1, -v_2\}$, since

$$(m_1, m_2) = \frac{\sigma m_1 + m_2}{\sigma^2 - 1}(\sigma, -1) + \frac{m_1 + \sigma m_2}{\sigma^2 - 1}(-1, \sigma).$$

Thus proposition C.32 gives a two-dimensional nonexistence subcone inside the remaining region.

Proposition C.34 (the full interior reflection cone in the symmetric strict model)**.** *In the symmetric strict quadrant model of proposition C.32, assume*

$$\mu = m_1 u_1 + m_2 u_2 \in \operatorname{int}\operatorname{cone}\{-v_1, -v_2\}.$$

Equivalently,

$$\sigma m_1 + m_2 > 0, \qquad m_1 + \sigma m_2 > 0.$$

Then the reflected Brownian motion specified by definition 2.1 with drift μ admits no stationary distribution.

Proof. We use an asymmetric cone-quadratic gauge. Let

$$q(x, y) = xy + \eta_1 x^2 + \eta_2 y^2, \qquad (x, y) \in \mathbb{R}^2_+,$$

where $\eta_1, \eta_2 > 0$ will be chosen. We first choose them so that

$$\text{(SQ1)} \qquad 0 < \eta_1 < \frac{1}{2\sigma}, \qquad 0 < \eta_2 < \frac{1}{2\sigma},$$

and

$$\text{(SQ2)} \qquad A_x := 2\eta_1 m_1 + m_2 > 0, \qquad A_y := m_1 + 2\eta_2 m_2 > 0.$$

We now construct η_1, η_2 explicitly from the two strict cone inequalities

$$\sigma m_1 + m_2 > 0, \qquad m_1 + \sigma m_2 > 0.$$

For η_1, use the following exhaustive cases. If $m_1 = 0$, then $m_2 > 0$, and set $\eta_1 = 1/(4\sigma)$. If $m_1 > 0$ and $m_2 \geq 0$, use the same value. If $m_1 > 0$ and $m_2 < 0$, put

$$L_1 := -\frac{m_2}{2m_1}.$$

The inequality $m_1 + \sigma m_2 > 0$ gives $0 < L_1 < 1/(2\sigma)$; choose

$$\eta_1 := \frac{1}{2}\left(L_1 + \frac{1}{2\sigma}\right).$$

Then $2\eta_1 m_1 + m_2 = 2m_1(\eta_1 - L_1) > 0$. Finally, if $m_1 < 0$, then $m_2 > 0$, and $\sigma m_1 + m_2 > 0$ yields

$$2\left(\frac{1}{4\sigma}\right)m_1 + m_2 = m_2 + \frac{m_1}{2\sigma} > -\sigma m_1 + \frac{m_1}{2\sigma} > 0;$$

again set $\eta_1 = 1/(4\sigma)$. In every case $0 < \eta_1 < 1/(2\sigma)$ and $A_x > 0$.

We now construct η_2 directly. If $m_2 = 0$, then $m_1 > 0$, and we set $\eta_2 = 1/(4\sigma)$. If $m_2 > 0$ and $m_1 \geq 0$, use the same value. If $m_2 > 0$ and $m_1 < 0$, put

$$L_2 := -\frac{m_1}{2m_2}.$$

The inequality $\sigma m_1 + m_2 > 0$ gives $0 < L_2 < 1/(2\sigma)$; choose

$$\eta_2 := \frac{1}{2}\left(L_2 + \frac{1}{2\sigma}\right).$$

Then $m_1 + 2\eta_2 m_2 = 2m_2(\eta_2 - L_2) > 0$. Finally, if $m_2 < 0$, then $m_1 > 0$, and $m_1 + \sigma m_2 > 0$ yields

$$m_1 + 2\left(\frac{1}{4\sigma}\right)m_2 = m_1 + \frac{m_2}{2\sigma} > -\sigma m_2 + \frac{m_2}{2\sigma} > 0;$$

again set $\eta_2 = 1/(4\sigma)$. Thus in every case $0 < \eta_2 < 1/(2\sigma)$ and $A_y > 0$, proving (SQ1)–(SQ2).

Since $\eta_1, \eta_2 > 0$, the function q is strictly positive on $S \setminus \{0\}$. On the lower boundary $y = 0$,

$$D_1 q = (-\sigma, 1)\cdot(2\eta_1 x + y,\ x + 2\eta_2 y) = (1 - 2\sigma\eta_1)x \geq 0,$$

and on the left boundary $x = 0$,

$$D_2 q = (1, -\sigma)\cdot(y + 2\eta_1 x,\ x + 2\eta_2 y) = (1 - 2\sigma\eta_2)y \geq 0.$$

Moreover,

$$\mathcal{L}q = m_1(y + 2\eta_1 x) + m_2(x + 2\eta_2 y) + \eta_1 + \eta_2 = A_x x + A_y y + \eta_1 + \eta_2.$$

By (SQ2), this is strictly positive and grows at least linearly in $x + y$.

It remains to choose a bounded monotone profile whose concavity is sufficiently controlled to preserve this positive drift. Put $\eta_* := \min\{\eta_1, \eta_2\} > 0$ and $c_* := \min\{A_x, A_y\} > 0$. Then

$$q(z) \geq \eta_* |z|^2, \qquad \mathcal{L}q(z) \geq c_*(x + y) + \eta_1 + \eta_2.$$

For $|z| \geq 1$, since $x + y \geq |z|$, this gives

$$\sqrt{1 + q(z)}\,\mathcal{L}q(z) \geq c_* \sqrt{\eta_*}\,|z|^2.$$

Also $|\nabla q(z)|^2 \leq C_q |z|^2$ for some finite constant C_q. Hence the ratio

$$z \longmapsto \frac{|\nabla q(z)|^2}{\sqrt{1 + q(z)}\,\mathcal{L}q(z)}$$

is bounded outside the unit ball. On $S \cap \overline{B}_1$ it is bounded because $\mathcal{L}q \geq \eta_1 + \eta_2 > 0$ and the denominator is continuous and positive. Thus

$$\mathcal{R}_q := \sup_{z \in S} \frac{|\nabla q(z)|^2}{\sqrt{1 + q(z)}\,\mathcal{L}q(z)} < \infty.$$

Choose explicitly

$$\omega := \frac{2}{1 + \mathcal{R}_q} > 0.$$

Then

$$1 - \frac{\omega \mathcal{R}_q}{4} = 1 - \frac{\mathcal{R}_q}{2(1 + \mathcal{R}_q)} > \frac{1}{2} > 0.$$

For each $\delta > 0$, let $h_\delta \in C^\infty([0, \infty))$ be bounded and satisfy

$$h_\delta(s) = 0 \quad (0 \leq s \leq \delta/2), \qquad h_\delta'(s) > 0 \quad (s > \delta),$$
$$h_\delta' \geq 0, \qquad h_\delta''(s) \geq -\frac{\omega}{2\sqrt{1 + s}} h_\delta'(s).$$

For example, take

$$h_\delta'(s) = \psi_\delta(s) e^{-\omega\sqrt{1+s}},$$

where ψ_δ is smooth, nondecreasing, vanishes on $[0, \delta/2]$, is strictly positive on (δ, ∞), and equals one for all large s, and then integrate from zero. This derivative is integrable at infinity, so h_δ is bounded.

Set

$$f_\delta(z) = h_\delta(q(z)).$$

We verify the global C_b^2 property explicitly. Since q is a positive quadratic form on the quadrant, there are constants $0 < c_q \le C_q < \infty$ and $C_1, C_2 < \infty$ such that

$$c_q|z|^2 \le q(z) \le C_q|z|^2, \qquad |\nabla q(z)| \le C_1|z|, \qquad \|D^2 q(z)\| \le C_2.$$

The plateau of h_δ on $[0, \delta/2]$ makes f_δ constant on the neighborhood $\{q < \delta/2\}$ of the vertex. On the compact transition set on which $\psi_\delta' \neq 0$, all chain-rule terms are bounded. Outside that set, $\psi_\delta = 1$ and

$$h_\delta'(s) = e^{-\omega\sqrt{1+s}}, \qquad |h_\delta''(s)| = \frac{\omega}{2\sqrt{1+s}} e^{-\omega\sqrt{1+s}}.$$

Choose $s_\delta < \infty$ such that $\psi_\delta(s) = 1$ for every $s \ge s_\delta$, and put

$$R_\delta := \sqrt{\frac{s_\delta}{c_q}}.$$

If $|z| \ge R_\delta$, then $q(z) \ge c_q|z|^2 \ge s_\delta$, so $\psi_\delta(q(z)) = 1$. Consequently, for every $|z| \ge R_\delta$,

$$|\nabla f_\delta(z)| \le C_1|z|e^{-\omega\sqrt{1+c_q|z|^2}},$$
$$\|D^2 f_\delta(z)\| \le \frac{\omega C_1^2|z|^2}{2\sqrt{1+c_q|z|^2}} e^{-\omega\sqrt{1+c_q|z|^2}} + C_2 e^{-\omega\sqrt{1+c_q|z|^2}}.$$

Both right-hand sides are bounded and tend to zero at infinity. The chain rule gives bounded first and second derivatives on all of S, with continuous extensions to the open faces. Together with the vertex plateau, this proves $f_\delta \in C_b^2(S)$. The boundary sign follows from $h_\delta' \ge 0$ and $D_i q \ge 0$:

$$D_i f_\delta = h_\delta'(q) D_i q \ge 0.$$

Finally,

$$\mathcal{L} f_\delta = h_\delta'(q)\mathcal{L}q + \frac{1}{2}h_\delta''(q)|\nabla q|^2 \ge h_\delta'(q)\left(\mathcal{L}q - \frac{\omega}{4\sqrt{1+q}}|\nabla q|^2\right) \ge h_\delta'(q)\mathcal{L}q\left(1 - \frac{\omega \mathcal{R}_q}{4}\right) \ge 0.$$

This inequality is strict whenever $q(z) > \delta$, since then $h_\delta'(q(z)) > 0$ and $\mathcal{L}q(z) > 0$. Thus $\{f_\delta\}$ is a vanishing-core admissible contradiction family with core function q. Because $q(z) > 0$ on $S \setminus \{0\}$, theorem 4.16 implies that no stationary distribution exists. ☐

Remark C.35. The preceding result extends proposition C.32: in the symmetric strict quadrant model, the cone-quadratic criterion rules out stationarity throughout the entire interior of the closed reflection cone $K_{\text{str}} = \text{cone}\{-v_1, -v_2\}$. Together with the boundary-ray analysis supplied by theorem 7.12, this yields the symmetric strict phase diagram below.

Theorem C.36 (symmetric strict quadrant phase diagram)**.** *Consider the symmetric strict quadrant model*

$$S = \mathbb{R}_+^2, \qquad v_1 = -\sigma u_1 + u_2, \qquad v_2 = u_1 - \sigma u_2, \qquad \sigma > 1.$$

For every drift $\mu = m_1 u_1 + m_2 u_2$, the reflected Brownian motion specified by definition 2.1 with drift μ admits a stationary distribution if and only if

$$\mu \notin K_{\text{str}}.$$

Equivalently, it admits no stationary distribution if and only if

$$\mu \in K_{\text{str}}.$$

In particular, the two boundary rays of the reflection cone $\{t(-v_1) : t \geq 0\}$ *and* $\{t(-v_2) : t \geq 0\}$*, as well as the zero drift, are on the nonexistence side.*

Proof. In this model the corresponding Lakner–Liu–Reed parameter is $\alpha = 4\arctan(\sigma)/\pi \in (1, 2)$. The strict-regime Lyapunov geometry gives

$$\mathfrak{M}_{\text{Lyap}} = \mathbb{R}^2 \setminus \text{cone}\{-v_1, -v_2\},$$

so every drift outside the closed cone has a stationary distribution by corollary 6.13. On the nonexistence side, proposition C.34 gives a cone-quadratic proof for the interior of $\text{cone}\{-v_1, -v_2\}$. The two boundary rays and the zero drift are covered by the Varadhan–Williams closed-cone theorem, theorem 7.12. Hence all drifts in $\text{cone}\{-v_1, -v_2\}$ are on the nonexistence side, and the displayed equivalence follows. Equivalently, this is the symmetric specialization of the general strict-regime phase diagram corollary 7.13. □

Remark C.37. The cone-quadratic criterion in proposition C.34 gives a bounded-profile proof on the open interior of the closed reflection cone. The two-homogeneous Varadhan–Williams gauge in theorem 7.12 reaches the whole closed cone by a cutoff-and-tightness contradiction rather than by a pointwise vanishing-core supersolution.

Proposition C.38 (a critical-ray obstruction for quadratic composition gauges)**.** *Consider the symmetric strict quadrant model*

$$S = \mathbb{R}^2_+, \qquad v_1 = -\sigma u_1 + u_2, \qquad v_2 = u_1 - \sigma u_2, \qquad \sigma > 1.$$

Let the drift lie on one of the two boundary rays of the reflection cone, for instance

$$\mu = t(-v_1) = t(\sigma u_1 - u_2), \qquad t > 0.$$

Let

$$q(x, y) = Ax^2 + Bxy + Cy^2$$

be a homogeneous quadratic polynomial such that $q > 0$ *on* $\mathbb{R}^2_+ \setminus \{0\}$*. Suppose that, for some nonconstant bounded* $h \in C^2([0, \infty))$ *with* $h' \geq 0$*, the function*

$$f(x, y) = h(q(x, y))$$

satisfies

$$D_i f \geq 0 \quad \text{on } \partial S_i, \qquad \mathcal{L} f \geq 0 \quad \text{in } S^\circ.$$

Then no such pair (q, h) *exists. The same conclusion holds on the other critical ray* $\mu = t(-v_2)$*.*

Proof. We first prove the assertion for $\mu = t(-v_1)$. Since $q(x, 0) = Ax^2$ and $q > 0$ on the positive quadrant away from the origin, we have $A > 0$. Along the lower boundary $y = 0$,

$$\nabla q(x, 0) = (2Ax, Bx),$$

and hence

$$D_1 q(x, 0) = v_1 \cdot \nabla q(x, 0) = (-2\sigma A + B)x.$$

If $B < 2\sigma A$, then the boundary inequality $D_1 f(x, 0) = h'(Ax^2)(B - 2\sigma A)x \geq 0$ for all $x > 0$ would force $h'(Ax^2) = 0$ for all $x > 0$, hence $h' \equiv 0$ on $(0, \infty)$, contrary to nonconstancy. Thus

$$B \geq 2\sigma A. \tag{QG1}$$

Along the left boundary $x = 0$, one has

$$\nabla q(0, y) = (By, 2Cy), \qquad D_2 q(0, y) = v_2 \cdot \nabla q(0, y) = (B - 2\sigma C)y.$$

Positivity of $q(0, y) = Cy^2$ for $y > 0$ gives $C > 0$. If $B < 2\sigma C$, then the boundary inequality

$$D_2 f(0, y) = h'(Cy^2)(B - 2\sigma C)y \geq 0, \qquad y > 0,$$

would imply $h'(Cy^2) = 0$ for every $y > 0$. Since the map $y \mapsto Cy^2$ sends $(0, \infty)$ onto $(0, \infty)$, this would give $h' \equiv 0$ on $(0, \infty)$, again contradicting the nonconstancy of h. Therefore

$$B \geq 2\sigma C. \tag{QG2}$$

Set $p(s) = h'(s)$. Since h is bounded and nondecreasing, $p \geq 0$ and $p \in L^1(0, \infty)$. It remains to restrict the generator inequality to the lower boundary ray in the limiting interior sense. At a point $(x, 0)$, put $s = q(x, 0) = Ax^2$. Direct calculation gives

$$\mathcal{L}q(x,0) = A + C + t(2\sigma A - B)x = A + C + t(2\sigma A - B)\sqrt{s/A},$$

and

$$|\nabla q(x,0)|^2 = (4A^2 + B^2)x^2 = \frac{4A^2 + B^2}{A}s.$$

Therefore the open-wedge inequality, extended continuously to this open face, implies, for every $s > 0$,

$$Ksp'(s) + \Big(A + C + t(2\sigma A - B)\sqrt{s/A}\Big)p(s) \geq 0, \tag{QG3}$$

where

$$K := \frac{4A^2 + B^2}{2A} > 0.$$

Equivalently,

$$p'(s) + R(s)p(s) \geq 0, \qquad R(s) = \frac{A + C + t(2\sigma A - B)\sqrt{s/A}}{Ks}. \tag{QG4}$$

For any $a > 0$, set

$$M_a(s) := \exp\left(\int_a^s R(r)\,dr\right), \qquad s \geq a.$$

Then $M_a > 0$, $M_a(a) = 1$, and

$$(M_a p)' = M_a(p' + Rp) \geq 0 \qquad \text{on } [a, \infty).$$

If $B > 2\sigma A$, define

$$r_0 := \frac{A + C}{K}, \qquad c_0 := \frac{t(B - 2\sigma A)}{K\sqrt{A}} > 0.$$

Then

$$R(s) = \frac{r_0}{s} - \frac{c_0}{\sqrt{s}},$$

so direct integration gives

$$M_a(s) = \left(\frac{s}{a}\right)^{r_0} \exp\{-2c_0(\sqrt{s} - \sqrt{a})\}. \tag{C.3}$$

Therefore

$$M_a(s)^{-1} = a^{r_0} e^{-2c_0\sqrt{a}} s^{-r_0} e^{2c_0\sqrt{s}},$$

which tends to $+\infty$; in particular its integral over $[a, \infty)$ diverges.

If $B = 2\sigma A$, then (QG2) gives $C \leq A$, and

$$R(s) = \frac{A + C}{Ks}, \qquad K = 2A(1 + \sigma^2).$$

In this case

$$M_a(s) = \left(\frac{s}{a}\right)^r, \qquad r = \frac{A + C}{2A(1 + \sigma^2)} \leq \frac{1}{1 + \sigma^2} < 1.$$

Hence

$$\int_a^\infty M_a(s)^{-1}\,ds = a^r \int_a^\infty s^{-r}\,ds = \infty.$$

If $p(s_*) > 0$ for some $s_* > 0$, take $a = s_*$. Since $(M_a p)' \geq 0$, we have

$$p(s) \geq \frac{M_a(s_*)p(s_*)}{M_a(s)}, \qquad s \geq s_*.$$

The divergence of $\int_{s_*}^{\infty} M_a(s)^{-1}\,ds$ contradicts $p \in L^1(0,\infty)$. Hence no such s_* exists and $p \equiv 0$ on $(0,\infty)$. Thus h is constant, contradicting the assumption that h is nonconstant.

It remains to transfer this contradiction to the second critical ray without suppressing any algebra. Let

$$T(x,y) := (y,x), \qquad \widetilde{q} := q \circ T, \qquad \widetilde{f} := h \circ \widetilde{q} = f \circ T.$$

The linear involution T satisfies

$$Tu_1 = u_2, \qquad Tu_2 = u_1, \qquad Tv_1 = v_2, \qquad Tv_2 = v_1,$$

and, because $T = T^{\mathsf{T}} = T^{-1}$,

$$\nabla(g \circ T)(z) = T\nabla g(Tz), \qquad \Delta(g \circ T)(z) = \Delta g(Tz).$$

For any C^2 function g, these identities follow from the chain rule. If $\mu = t(-v_2)$, then

$$T\mu = t(-Tv_2) = t(-v_1).$$

Consequently, for $i = 1,2$,

$$v_i \cdot \nabla \widetilde{f}(z) = (Tv_i) \cdot \nabla f(Tz) = v_{3-i} \cdot \nabla f(Tz),$$

so the two boundary inequalities for f at drift $t(-v_2)$ become the two boundary inequalities for $\widetilde{f}$ after the faces are interchanged. The generators satisfy

$$\left(\frac{1}{2}\Delta + (T\mu)\cdot\nabla\right)\widetilde{f}(z) = \left(\frac{1}{2}\Delta + \mu\cdot\nabla\right)f(Tz).$$

Thus a pair (q,h) for the second critical ray would produce the pair $(\widetilde{q},h)$ for the first critical ray. Moreover, $\widetilde{q}(x,y) = Ay^2 + Bxy + Cx^2 = Cx^2 + Bxy + Ay^2$ is again a homogeneous quadratic polynomial strictly positive on $\mathbb{R}^2_+ \setminus \{0\}$, and h is unchanged. This contradicts the first-ray result. Hence no such pair exists on either critical ray. □

Remark C.39. This proposition identifies a limitation of the bounded-profile quadratic approach. Its role is more precise than the closed-cone theorem: it shows that, even on the boundary rays of the reflection cone, the quadratic-level composition $h(q)$ cannot provide the required admissible supersolution. The boundary-ray nonexistence is therefore supplied instead by the two-homogeneous Varadhan–Williams gauge and the cutoff-and-tightness argument of theorem 7.12.

C.5. **A proper-gauge criterion beyond cone quadratics.** The cone-quadratic gauge above is explicit, but it is not the only possible two-dimensional gauge. The next criterion gives a sufficient logarithmic-gradient condition for the bounded-profile argument.

Proposition C.40 (a proper-gauge admissible supersolution criterion)**.** *Assume $\alpha < 2$ and fix the drift μ in the generator $\mathcal{L} = \mathcal{L}_\mu$. Let $H : S \to [0,\infty)$ be continuous, facewise C^2 and locally C^2-extendable away from the vertex, and proper on S. Assume*

$$H(0) = 0, \qquad H(z) > 0 \quad (z \in S \setminus \{0\}),$$

that the open-face derivatives satisfy

$$D_i H \geq 0 \quad \text{on } \partial S_i \setminus \{0\}, \qquad i = 1,2,$$

and that there exists $\eta_H > 1$ such that, on S° and by continuous extension to the open faces,

$$\mathcal{L}H(z) \geq \frac{\eta_H}{2(1+H(z))}|\nabla H(z)|^2. \tag{C.4}$$

Assume also that $|\nabla H(z)| > 0$ on $S \setminus \{0\}$, again through the open-face extensions. Finally, assume that, for some exponent $p_H \in (0, \eta_H - 1)$,

$$\sup_{z \in S^\circ}(1+H(z))^{-p_H-1}|\nabla H(z)| < \infty, \tag{C.5}$$

$$\sup_{z \in S^\circ}(1+H(z))^{-p_H-1}\|D^2 H(z)\| < \infty, \qquad \sup_{z \in S^\circ}(1+H(z))^{-p_H-2}|\nabla H(z)|^2 < \infty, \tag{C.6}$$

with the same bounds for the continuous open-face extensions. Then the solution to the submartingale problem of definition 2.1 with drift μ admits no stationary distribution.

Proof. Let $p_H \in (0, \eta_H - 1)$ be as in (C.5)– (C.6). For $\delta > 0$, choose $\psi_\delta \in C^\infty([0,\infty))$ such that $0 \le \psi_\delta \le 1$, $\psi'_\delta \ge 0$, $\psi_\delta = 0$ on $[0, \delta/2]$, $\psi_\delta > 0$ on (δ, ∞), and $\psi_\delta = 1$ outside a compact interval. Define

$$h'_\delta(s) = \psi_\delta(s)(1+s)^{-p_H-1}, \qquad h_\delta(s) = \int_0^s h'_\delta(r)\,dr.$$

Because $p_H > 0$, the function $(1+s)^{-p_H-1}$ is integrable at infinity; hence h_δ is bounded. It is constant on $[0, \delta/2]$, belongs to $C_b^2([0,\infty))$, and satisfies

$$h''_\delta(s) = \psi'_\delta(s)(1+s)^{-p_H-1} - (p_H+1)\psi_\delta(s)(1+s)^{-p_H-2} \ge -\frac{p_H+1}{1+s} h'_\delta(s). \tag{C.7}$$

Set $f_\delta = h_\delta \circ H$. Continuity of H and $H(0) = 0$ show that f_δ is constant on a neighborhood of the vertex.

We verify the global C_b^2 regularity rather than referring only to the decay of the profile. Choose R_δ so large that $\psi_\delta = 1$ on $[R_\delta, \infty)$. On the compact set $\{\delta/2 \le H \le R_\delta\}$, local C^2-extendability of H, a finite covering, and the bounded derivatives of h_δ give bounded first and second derivatives of f_δ. On $\{H \ge R_\delta\}$,

$$h'_\delta(H) = (1+H)^{-p_H-1}, \qquad |h''_\delta(H)| = (p_H+1)(1+H)^{-p_H-2}.$$

Consequently,

$$\begin{aligned} |\nabla f_\delta| &\le (1+H)^{-p_H-1}|\nabla H|, \\ \|D^2 f_\delta\| &\le (p_H+1)(1+H)^{-p_H-2}|\nabla H|^2 + (1+H)^{-p_H-1}\|D^2 H\|. \end{aligned}$$

The three bounds in (C.5)– (C.6) make the right-hand sides uniformly bounded and give the corresponding continuous open-face extensions. On $\{H \le \delta/2\}$, the function is constant. These three regions cover S, and therefore $f_\delta \in C_b^2(S)$.

Since $h'_\delta \ge 0$ and $D_i H \ge 0$,

$$D_i f_\delta = h'_\delta(H) D_i H \ge 0 \qquad \text{on } \partial S_i,$$

so f_δ is admissible. In the open wedge the chain rule and (C.7) give

$$\begin{aligned} \mathcal{L} f_\delta &= h'_\delta(H)\mathcal{L}H + \frac{1}{2} h''_\delta(H)|\nabla H|^2 \\ &\ge h'_\delta(H)\left(\mathcal{L}H - \frac{p_H+1}{2(1+H)}|\nabla H|^2\right) \\ &\ge \frac{\eta_H - p_H - 1}{2(1+H)} h'_\delta(H)|\nabla H|^2 \ge 0. \end{aligned}$$

The coefficient $\eta_H - p_H - 1$ is strictly positive. If $H(z) > \delta$, then $\psi_\delta(H(z)) > 0$, hence $h'_\delta(H(z)) > 0$; the assumption $|\nabla H(z)| > 0$ then makes the final display strict. Thus $\{f_\delta\}_{\delta>0}$ is a vanishing-core admissible contradiction family with core function H. The conclusion follows from theorem 4.16. ☐

Remark C.41. The condition (C.4) is a logarithmic-gradient domination condition. If $f = h(H)$ is to be bounded and increasing, then h' must be integrable at infinity, so a negative second derivative of order at least $(1+H)^{-1}h'$ is unavoidable for standard bounded profiles. Thus an admissible gauge of this form must make $\mathcal{L}H$ dominate $|\nabla H|^2/(1+H)$ with margin strictly larger than one. The usual homogeneous quadratic gauges therefore fail at the critical rays whenever they attain or fall below this threshold.

C.6. **Relation of the gauge criteria to the phase diagram.** The criteria in this section have a common form: a gauge is chosen so that, after a bounded monotone profile or a compact localization, it becomes an admissible test whose generator is nonnegative and strictly positive outside a core shrinking to the vertex. The elliptic-norm criterion uses an elliptic distance ρ_Q, the cone-quadratic criterion uses the boundary-distance variables s_1, s_2, and the proper-gauge criterion records the abstract logarithmic-gradient condition under which the same bounded-profile argument works. The finite-dimensional and two-parameter formulations are algebraic ways of checking the displayed sign hypotheses.

The preceding results give several explicit gauge criteria for the weak oblique elliptic system. In the strict regime, bounded cone-quadratic profiles yield nonexistence throughout the interior of the reflection cone, whereas the two critical rays and the zero drift require the Varadhan–Williams gauge of section 7. The symmetric strict quadrant model provides an explicit instance of this distinction.

The preceding results identify the elliptic consequences of the admissible-test inequalities. The drift classification itself follows from the Foster–Lyapunov existence theorem, the borderline projection argument, and the Varadhan–Williams closed-cone nonexistence theorem.

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(Z. Liu) Shanghai Institute for Mathematics and Interdisciplinary Sciences (SIMIS), Shanghai, China, 200433

(Z. Liu) Research Institute of Intelligent Complex Systems, Fudan University, Shanghai 200433, China

Email address: zliu@simis.cn